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Explaining All Three-Alternative Voting Outcomes*

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A theory is developed to explain all possible three-alternative (single-profile) pairwise and positional voting outcomes. This includes all preference aggregation paradoxes, cycles, conflict between the Borda and Condorcet winners, differences among positional outcomes (e.g., the plurality and antiplurality methods), and differences among procedures using these outcomes (e.g., runoffs, Kemeny's rule, and Copeland's method). It is shown how to identify, interpret, and construct all profiles supporting each paradox. Among new conclusions, it is shown why a standard for the field, the Condorcet winner, is seriously flawed. *Journal of Economic Literature* Classification Numbers: D72, D71. © 1999 Academic Press

1. INTRODUCTION

Over the last two centuries considerable attention has focussed on the properties of positional voting procedures. These commonly used approaches assign points to alternatives according to how each voter positions them. The standard plurality method, for instance, assigns one point to a voter's top-ranked candidate and zero to all others, while the Borda count (BC) assigns $n-1$, $n-2$, ..., $n-n=0$ points, respectively, to a voter's first, second, ..., nth ranked candidate. The significance of these procedures derives from their wide usage, but their appeal comes from their mysterious paradoxes (i.e., counterintuitive conclusions) demonstrating highly complex outcomes. By introducing doubt about election outcomes, these paradoxes raise the legitimate concern that, inadvertently, we can choose badly even in sincere elections.

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As these methods serve as prototypes for aggregation procedures, they also identify issues for economics and other areas. This is illustrated by the connection between the manipulation of decision procedures and the subsequent incentive literature. Another example is how the characterization of voting paradoxes (Saari [25]) motivated an extension of the Sonnenschein [38, 39], Mantel [12], and Debreu [7] aggregate excess demand result from their setting of a single set of n commodities to the general setting of all subsets with two or more commodities (Saari [22, 26]). Also, connections between positional and statistical methods have provided new results for nonparametric statistics (Haunsperger [8]).

Positional outcomes also are crucial components of choice procedures. A runoff, for instance, is held among the top-ranked candidates from a first election. An agenda, a tournament, the Copeland method [6, 31, 14], and Kemeny's rule [10, 11, 32] are among the many procedures using pairwise voting outcomes. Other methods, such as the controversial approval voting and the enigmatic rules of figure skating, use positional outcomes in complicated ways.

1.1. *Complexity of Analysis*

Although important, positional procedures have proved to be formidable to analyze. The underlying complexity is indicated by the radical manner in which societal outcomes can change when alternatives are added or dropped. It is further manifested by how these procedures can generate over *84 million different election rankings* with a single 10-candidate profile (Saari [23]). Once the ballots are marked, the voters' opinions remain fixed, but varying the choice of positional methods generates millions of contradictory outcomes where each alternative wins with some procedures but is bottom-ranked with others. Not all of these conflicting outcomes accurately reflect the "voters' opinions," so which is the correct one? Of more importance, what causes these varied outcomes?

A natural reaction to this complexity is a resigned attitude aptly captured by Riker's assertion [19] that "[t]he choice of a positional voting method is subjective." Related comments come from another expert who stressed the importance of *social choice*—which considers only the election winner—over a social ordering. He argues that "[g]iven all the logical barriers that have to be scaled to even come close to making a coherent social choice, demanding a full ordering is a tall order." Indeed, trying to find a full ordering is "something that most of us long ago gave up on as impossible and/or incoherent." His thoughts probably reflect the general sense of the choice community.

In response to these difficulties, a natural first goal is to characterize all possible paradoxes that occur with any single profile. This has been done (for any number of candidates) where the results (see Saari [20, 25] and references therein) prove that positional procedures admit significantly more kinds of problems and complexities than previously suspected. So, the next step is to explain these paradoxes and to construct all possible illustrating profiles. This project now is completed; the three-candidate results, with geometric representations, are reported here. The more abstract case of $n \geqslant 4$ alternatives is in Saari [29]. Fortunately, but unexpected, the answers for these two-century-old challenges are surprisingly natural and simple with the following *profile decomposition*.

1.2. *Profile Decomposition*

To simplify the analysis while addressing the complexities, the approach emphasizes the structure of profile space. (A *profile* lists each voter's ranking of the alternatives.) Namely, the six-dimensional profile space is decomposed into orthogonal subspaces where profiles from each subspace affect only certain classes of procedures.

An ideal starting point is to find the profiles which achieve a major objective of choice theory—they are free from all conflict. Surprisingly, this profile subspace exists (but it does *not* include unanimity profiles). To underscore the central role of these profiles, I call them *Basic profiles*. As shown, *the rankings and even the* (*normalized*) *tallies of all positional methods and pairwise outcomes agree with a Basic profile*. This (two-dimensional) profile subspace, then, allows no conflict or problems among procedures and their derived methods. Basic profiles liberate us from the above difficulties where election outcomes change with the choice of the procedures, or as candidates leave.

Because nothing goes wrong in this space, all election difficulties and conflicts which have driven, motivated, but frustrated this research area are caused by profiles orthogonal to the Basic profile subspace. (As indicated in Section 6.4, this even includes axiomatic mysteries such as Arrow's Impossibility Theorem [1].) Indeed, the (one-dimensional) *Condorcet* space has all of the profiles which cause pairwise voting paradoxes; but they have no effect on positional methods. Profiles in the (two-dimensional) *Reversal* subspace create all differences in positional methods; but they have no effect on pairwise rankings. The last one-dimensional subspace, the *Kernel*, contains the profiles with completely tied pairwise and positional election outcomes.

This description already suggests how to use the decomposition. For instance, to create a profile with a plurality outcome that differs from the pairwise outcomes, start with a Basic profile defining the common $A \succ B \succ C$ for all pairwise and positional rankings. Then add a Reversal profile to change the plurality ranking to $C \succ B \succ A$. (As shown, this is easy to do.) As Reversal profiles do not affect pairwise rankings, we have a desired example. To alter the pairwise outcomes in a specified manner, add an appropriate Condorcet component to the profile. In this elementary manner, we extend the results (Saari [20, 25]) characterizing all election paradoxes to identify all supporting profiles for each paradox. As such, this decomposition resolves a long-standing goal of classical choice theory.

Conversely, simply by decomposing a specified profile into its component parts (Section 8), we can determine its effect on various election methods. To illustrate how this adds to our understanding, I use it to describe new problems with Black's single-peaked condition (Section 8.4), to find different interpretations for historically important examples (Sections 8.1, 8.2), and to refute a conjecture about strategic voting (Section 8.5).

Moreover, by identifying which profiles support each election outcome, we find new explanations for the paradoxes. A surprising one is that certain combinations of profiles weaken the effect of central assumptions such as individual rationality. It can be argued that voters' votes should cancel on these subspaces to cause a complete tie; i.e., they should not contribute toward the societal outcome. But they do with certain procedures, and this causes *all* voting paradoxes.

The properties and peculiarities of derivative methods (e.g., runoffs, approval voting, figure skating) also can be derived from this profile division. This is because, by knowing how each component of a procedure reacts to the different profile subspaces, we obtain a new understanding of the method. To see what else is possible, recall that axiomatic representations identify properties unique to a particular procedure. So, by knowing how a procedure uses and reacts to the different profile subspaces, we can develop new axiomatic characterizations and new proofs for known ones. Similarly, strategic behavior (and related topics such as monotonicity) involve changes in profiles. By knowing how procedures react to profiles from each subspace, we obtain new insights into the manipulability of methods. Thus, the susceptibility of procedures can be compared more deeply than, say, with a measure theoretic approach (Saari [27]). (Because of the importance of these topics, a lengthy analysis is offered elsewhere.) Similarly, answers to historical and contemporary concerns from social choice can be found. This includes the Borda and Condorcet debates of the 1780s which inaugurated the field of social choice.

1.3. *Borda–Condorcet Debates*

For a flavor of the kinds of results which follow, I preview the Section 6.3 conclusions about the Borda–Condorcet debate; a debate which introduced and still shapes social choice. The academic study of voting started in 1770 (e.g., see the books by McLean and coauthors [15, 16]) when Borda [4] constructed a profile (Section 8.2) casting doubt on the wisdom of using the plurality vote. He showed how the BC avoids this difficulty—at least for his profile. In 1785, Condorcet [5] introduced a competing method; his *Condorcet Winner* is the alternative which wins all pairwise elections. Arrow [1], 165 years later, developed his "binary independence" condition and impossibility theorem, which significantly extend Condorcet's notions.

With its natural, intuitive appeal, it is understandable why Condorcet's method has become a widely accepted standard for choice theory. Condorcet distinguished his approach with a profile (Section 8.1) where *all positional methods fail to elect the Condorcet winner*. Thus, a Condorcet winner need not be the BC winner, and this continues to be cited as a fatal BC flaw. But when these historically important examples are examined with the profile decomposition, the surprise is that *for any conflict between the BC and Condorcet rankings, all examples support Borda's approach while raising serious doubts about Condorcet's method—the standard of the field*. The conflict resides in the failings of the pairwise vote—not the BC. This conclusion contradicts what has been accepted for two centuries. In fact, Condorcet's example helps to reverse Condorcet's intended message.

The explanation is that the BC ignores the Condorcet components which affect only pairwise outcomes. This is important because, as shown, *the combination of the pairwise vote with the Condorcet terms loses the crucial fact that voters have transitive preferences*. The same phenomenon explains Arrow's impossibility theorem; by relying upon the Condorcet terms, Arrow's binary independence condition unexpectedly devalues his individual rationality assumption. (See Section 6.4 and Saari [30].) An equally surprising assertion is that rather than being the standard, the Condorcet winner must be held suspect.[1]

1.4. *Removing Paradoxes*

So inconsistencies in election tallies result from profiles orthogonal to the Basic profiles. But as the Basic profiles define only a two-dimensional subspace, these inconsistencies are nearly omnipresent. Indeed (by dimension counting), *a completely tied plurality election* (*a three-dimensional space of profiles*) *is more likely than avoidance of election tallies with inconsistencies.*

Other than with profile restrictions, an alternative way to eliminate the affects of the orthogonal profiles is to use only a profile's Basic component,

[1] In the large literature judging procedures with the Condorcet standard, then, it may be the "standard" (which ignores the rationality of voters) rather than the procedure which is at fault when there is a disagreement.

where all disagreement and conflict disappear. In turn, this underscores an important conclusion of this paper; *the BC is the only positional procedure which ignores all orthogonal components.* As the BC tally of the original profile is what other procedures obtain only after all effects of the other profile components are removed, it follows that using the BC with the original profile is an efficient, pragmatic way to obtain the common Basic outcome.

2. NOTATION AND DIVISION OF PROCEDURES

For the three alternatives $\{A, B, C\}$, let the $3! = 6$ voter types be

Type	Ranking	Type	Ranking
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

(2.1)

2.1. *Terminology and Voting Vectors*

A *profile* specifies the number of voters of each type. Using the labeling of Table 2.1, the integer profile (0, 5, 0, 3, 4, 0) has five voters of type two ($A \succ C \succ B$), three of type four ($C \succ B \succ A$), and four of type-five ($B \succ C \succ A$).

A three-candidate positional election is defined by *voting vector* $\mathbf{w}_s^3 = (w_1, w_2, w_3) = (1, s, 0)$ where s, $0 \leqslant s \leqslant 1$, is a specified value. When a ballot is tallied, w_j points are assigned to the voter's jth ranked alternative, $j = 1, 2, 3$; e.g., the plurality (i.e., $\mathbf{w}_0^3$) outcome of profile (0, 5, 0, 3, 4, 0) is $A \succ B \succ C$ with the 5:4:3 tally. The $\mathbf{w}_s^3$ *vector tally* is $(\tau_s(A), \tau_s(B), \tau_s(C))$, where $\tau_s(K)$ is K's tally; e.g., the ranking defined by the vector tally (70, 20, 90) is $C \succ A \succ B$.

My normalization of voting vectors requires the top-ranked alternative to receive one point. Thus the BC, given by $\mathbf{B}^3 = (2, 1, 0)$, has the normalized form $\mathbf{b}^3 = \frac{1}{2}\mathbf{B}^3 = (1, \frac{1}{2}, 0)$. Similarly, an election tallied by assigning six, five, and zero points, respectively, to a voter's top, second, and bottom ranked candidate has the normalized form $(\frac{6}{6}, \frac{5}{6}, 0)$.

An important relationship (probably due to Borda and known by Nanson [17]) between the pairwise and the BC tallies can be described by computing how a voter with preferences $A \succ B \succ C$ votes in pairwise elections.

Candidates	$\{A\}$	$\{B\}$	$\{C\}$
$\{A, B\}$	1	0	—
$\{A, C\}$	1	—	0
$\{B, C\}$	—	1	0
Total	2	1	0

(2.2)

Thus the sum of points this voter provides a candidate over all pairwise elections equals what he or she assigns the candidate in a BC election. This means (along with neutrality[2] and the fact that each pair is tallied with the same voting vector) that a candidate's BC election tally is the sum of the pairwise tallies for the candidate. (See Saari [27].) Thus the pairwise tallies 31:29 for $A \succ B$, 32:28 for $A \succ C$, and 40:20 for $B \succ C$ define the BC outcome $B \succ A \succ C$ with the BC tally $(40+29):(32+31):(28+20)$. The normalized $\mathbf{b}^3$ vector tally is $\frac{1}{2}(63, 69, 48)$.

The three-candidate division of voting vectors is simple. It consists of the $(1, 0)$ methods used to tally pairwise elections which, as described above, define the $\mathbf{b}^3$ tally. All remaining $\mathbf{w}_s^3$ methods are represented as a sum of $\mathbf{b}^3$ and the *derived* vector $\mathbf{d}^3 = (0, 1, 0)$. (Sieberg [37] uses $\mathbf{d}^3$ to capture a statistical "variance" in election outcomes.)

THEOREM 2.1. *All three-candidate voting vectors can be expressed as*

$$\mathbf{w}_s^3 = (1, s, 0) = \mathbf{b}^3 + (s - \tfrac{1}{2})\, \mathbf{d}^3 \quad 0 \leqslant s \leqslant 1. \tag{2.3}$$

Proof. This is a simple algebraic relationship. ∎

Let $F(\mathbf{p}, \mathbf{w}_s^3)$ be the $\mathbf{w}_s^3$ election tally for profile $\mathbf{p}$. To motivate the use of the linearity of F in the $\mathbf{w}_s^3$ variable, suppose the $\mathbf{B}^3 = (2, 1, 0)$ tally of an election is $(20, 40, 30)$ and the plurality tally is $(9, 8, 13)$. Because $(7, 2, 0) = 2\mathbf{B}^3 + 3(1, 0, 0)$, the 2 and 3 multiples require the voters' $(7, 2, 0)$ tally to be $2(20, 40, 30) + 3(9, 8, 13) = (67, 104, 99)$ with a $B \succ C \succ A$ ranking. The following extends this observation to normalized voting vectors. (The proof is an immediate consequence of Eq. (2.3) and the linearity of F.)

THEOREM 2.2. *The $\mathbf{w}_s^3$ election tally can be expressed as*

$$F(\mathbf{p}, \mathbf{w}_s^3) = F(\mathbf{p}, \mathbf{b}^3) + (s - \tfrac{1}{2})\, F(\mathbf{p}, \mathbf{d}^3), \qquad 0 \leqslant s \leqslant 1. \tag{2.4}$$

[2] Neutrality is where interchanging the names of the candidates similarly interchanges the election tallies.

The line of election outcomes defined by Eq. (2.4) is called the *procedure line* (Saari [24, 27]).

2.2. *Geometry*

For a geometric representation of rankings and profiles, assign each candidate a vertex of an equilateral triangle (Saari [24, 27]). The *ordinal ranking* of a point in the triangle comes from its distances to the vertices where "closer is better." Points equidistant between two vertices represent indifference. In this manner, the "representation triangle" is divided into "ranking regions." (The numbers in the left triangle of Fig. 1 identify the region's Eq. (2.1) voter type.) Represent a profile by placing the number of voters of each type in its ranking region as illustrated on the right in Fig. 1.

The representation triangle geometry simplifies computing plurality, BC, pairwise, $\mathbf{d}^3$, and (with Eq. (2.4)) $\mathbf{w}_s^3$ tallies. To tally pairwise elections notice that the central vertical line is equidistant between the A and B vertices; it is the $A \sim B$ indifference line. Thus, the $\{A, B\}$ pairwise tallies are the sums of profile entries on each side of the line; e.g., in the left triangle of Fig. 1, B's tally is the sum of the entries in the darker shaded region. With the Fig. 1 profile on the right, A beats B by $25 + 33 = 58$ to $17 + 14 + 25 = 56$. The tallies for the other pairs, listed by the appropriate edge of the triangle, crown A as the Condorcet winner.

A candidate's BC tally is the sum of her pairwise tallies, so the $\mathbf{b}^3$ vector tally (58, 64, 49) defines the BC ranking $B \succ A \succ C$; it conflicts with the pairwise rankings as A, the Condorcet winner, is not BC top-ranked. A candidate's plurality tally is the number of voters who have her top-ranked, so it is the sum of the profile entries in the two ranking regions sharing the candidate's vertex. In the left triangle of Fig. 1, A's tally is the sum of entries in the lightly shaded region. These tallies for the profile in the right triangle, given by the bracketed numbers near the vertices, define the plurality ranking $C \succ B \succ A$. *For this profile A is the Condorcet winner,*

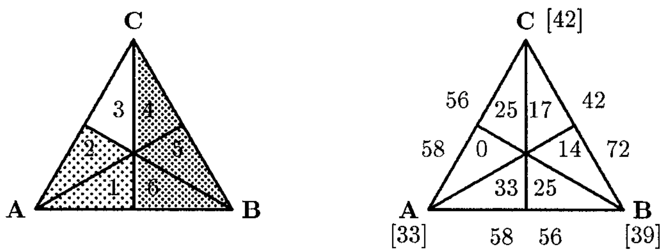

FIG. 1. Representation triangle.

B is the BC winner, and C is the plurality winner. Which alternative is the voters' true top choice?

Finally, *A*'s $\mathbf{d}^3$ tally is the sum of the entries of the two ranking regions midway from the *A* vertex to the opposing edge; it is the sum of the entries in regions 3 and 6. Thus the *A*, *B*, *C* tallies with $\mathbf{d}^3$ are, respectively, $25+25=50$, $17+33=50$, $0+14=14$. According to Eq. (2.4) and the $\mathbf{b}^3$ tally, the $\mathbf{w}_s^3$ election tally is

$$\begin{aligned} F(\mathbf{p}, \mathbf{w}_s^3) &= (58, 64, 49) + (s - \tfrac{1}{2})(50, 50, 14) \\ &= (33 + 50s, 39 + 50s, 42 + 14s), \qquad s \in [0, 1]. \end{aligned} \tag{2.5}$$

Incidentally, algebra proves that with this profile *no* $\mathbf{w}_s$ method elects the Condorcet winner *A*.

3. PROFILE DECOMPOSITION

To quickly analyze three-candidate profiles, I recommend the approach in Saari [27]. But that approach is an approximation, so it cannot address certain issues. The following provides an accurate analysis for all profiles.

3.1. *Profile Differential*

My approach involves adding profiles, so the analysis is simplified by using profile differentials—the difference between two profiles with the same number of voters—rather than profiles.

DEFINITION 1. A *profile differential* is the difference between two profiles involving the same number of voters. Equivalently, a listing of the number of voters of each type is a profile differential if and only if the sum of the entries is zero.

Profile differentials define the basis for different profile subspaces. To simplify the use, for two subspaces I specify three choices even though any two suffice. As a profile differential involves negative numbers of voters, they are converted into an "actual" profile (with a non-negative number of voters of each type) by adding a "neutral" profile. To illustrate with profile differential $\mathbf{p_d} = (1, 0, -2, 0, 1, 0)$, by adding $(2, 2, 2, 2, 2, 2)$ (a profile forcing completely tied elections) we obtain the profile $(3, 2, 0, 2, 3, 2)$.

3.2. *Decomposition*

The profile decomposition has four components. The *Kernel* has no effect on any procedure. The *Basic* portion is where all procedures agree. The

Condorcet portion affects only pairwise votes; e.g., it explains cycles and all differences between the pairwise and BC outcomes. The *Reversal* portion causes all differences in positional outcomes.

DEFINITION 2. The *three-candidate profile decomposition* is defined by the following basis vectors for the different subspaces.

1. The *Kernel* is spanned by the Kernel vector $\mathbf{K} = (1, 1, ..., 1)$.

2. The *Basic vector* for candidate X, $X = A, B, C$, is the profile differential with one voter for each type where X is top-ranked and -1 voters where X is bottom-ranked. A basis for the two-dimensional Basic subspace is any two of

$$\begin{aligned} \mathbf{B}_A &= (1, 1, 0, -1, -1, 0), \\ \mathbf{B}_B &= (0, -1, -1, 0, 1, 1), \\ \mathbf{B}_C &= (-1, 0, 1, 1, 0, -1) \end{aligned} \tag{3.1}$$

3. The Condorcet space is defined by $\mathbf{C}^3 = (1, -1, 1, -1, 1, -1)$.

4. The Reversal vector for candidate X, $X = A, B, C$, is the profile differential with one voter for each type where X is top-ranked, one voter for each type where X is bottom-ranked, and -2 voters for the remaining two voter types (where X is middle-ranked). A basis for the two-dimensional Reversal subspace is any two of

$$\begin{aligned} \mathbf{R}_A &= (1, 1, -2, 1, 1, -2), \\ \mathbf{R}_B &= (-2, 1, 1, -2, 1, 1), \\ \mathbf{R}_C &= (1, -2, 1, 1, -2, 1). \end{aligned} \tag{3.2}$$

The symmetry of these differentials is apparent from Fig. 2.

3.3. *Impact of Decomposition*

The value of the decomposition derives from how procedures react to the different subspaces.

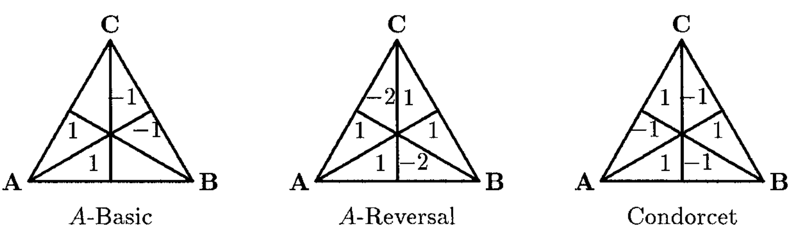

FIG. 2. Profile decomposition.

THEOREM 3.3. *All profiles can be expressed as*

$$\mathbf{p} = \mathbf{p}_K + \mathbf{p}_B + \mathbf{p}_C + \mathbf{p}_R,$$

where the profile differentials on the right-hand side come from, respectively, the Kernel, the Basic, the Condorcet, and the Reversal subspaces. The four subspaces are mutually orthogonal.

1. *All pairwise and positional rankings of* $\mathbf{K}$ *are complete ties. The tallies differ.*

2. *All normalized positional methods have an identical tally for a Basic profile. The common tally for* $a_B\mathbf{B}_A + b_B\mathbf{B}_B + c_B\mathbf{B}_C$ *is*

$$(2a_B - b_B - c_B,\ 2b_B - a_B - c_B,\ 2c_B - a_B - b_B). \tag{3.3}$$

The pairwise rankings of a Basic profile always are transitive and agree with the common ranking of the positional methods. For $a_B\mathbf{B}_A + b_B\mathbf{B}_B + c_B\mathbf{B}_C$, *the* $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ *tallies are, respectively,*

$$(2a_B - 2b_B)(1, -1), \qquad (2b_B - 2c_B)(1, -1), \quad (2a_B - 2c_B)(1, -1). \tag{3.4}$$

3. *For* $\mathbf{C}^3$, *all positional methods assign a zero tally to each candidate but the pairwise outcomes define the cycle* $A \succ B$, $B \succ C$, $C \succ A$ *with identical* $1 : -1$ *tallies.*

4. *Each candidate's pairwise and BC tally for a Reversal profile is zero. All non-BC positional procedures have a non-zero tally for each basis profile. The* $\mathbf{w}_s^3$ *tally for* $a_R\mathbf{R}_A + b_R\mathbf{R}_B + c_R\mathbf{R}_C$ *is*

$$(1 - 2s)(2a_R - b_R - c_R,\ 2b_R - a_R - c_R,\ 2c_R - a_R - b_R). \tag{3.5}$$

Proof. An elementary computation proves the subspaces are mutally orthogonal. The $\mathbf{K}$ assertion is obvious. Using the geometric approach with Fig. 2, the $(1, s, 0)$ tally of $\mathbf{B}_A$ is $(2, -1, -1)$. Thus, by linearity, the outcome for the specified profile is $a_B(2, -1, -1) + b_B(-1, 2, -1) + c_B(-1, -1, 2)$, or the Eq. (3.3) outcome. The pairwise vote and Part 3 are direct computations. For Part 4, as the $(1, s, 0)$ tally of $\mathbf{R}_A$ is $(2 - 4s, -1 + 2s, -1 + 2s) = (1 - 2s)(2, -1, -1)$, the result follows.

These differentials are easy to use because the algebraic rankings of coefficients define the rankings. To illustrate, I construct a profile with conflicting BC, plurality, and pairwise outcomes. A "$C \succ A \succ B$" BC ranking is determined (Theorem 3, part 3.3) if the Basic vector coefficients satisfy $c_B > a_B > b_B$; e.g., $c_B = 2$, $a_B = 1$, $b_B = 0$ define the profile differential $\mathbf{p}_B = (-1, 1, 2, 1, -1, -2)$ where (Eq. (3.3)) all positional procedures have the identical $(0, -3, 3)$ tally. To make B (the BC bottom ranked candidate) the plurality winner, add $\mathbf{p}_R = 3\mathbf{R}_B$ to obtain $(-7, 4, 5, -5, 2, 1)$. As $3\mathbf{R}_B$ adds $(-3, 6, -3)$ to the Basic profile plurality tally (Eq. (3.5)), B is the plurality winner. But (part 4) the BC and pairwise rankings remain untouched. To generate conflicting pairwise rankings, notice how the Condorcet portion introduces a cyclic effect (part 3) without changing positional rankings. Adding $\mathbf{p}_C = -2\mathbf{C}^3$ creates a cyclic effect helping B over A and A over C; the resulting $\mathbf{p}_B + \mathbf{p}_R + \mathbf{p}_C = (-9, 6, 3, -3, 0, 3)$ requires the previously middle-ranked A to tie each candidate in pairwise elections. The profile differential has a -9 term, so convert it into a profile by adding $\mathbf{p}_K = 9\mathbf{K}$ to obtain $(0, 15, 12, 6, 9, 12)$. By construction, this profile has

- the BC outcome $C \succ A \succ B$,
- the plurality outcome $B \succ C \succ A$, and
- the pairwise outcome $A \sim B$, $A \sim C$, $C \succ B$, which, while not transitive, is not cyclic.

3.4. *Choice of Coefficients*

Non-negative coefficients simplify the analysis; this choice always is possible. An alternative choice, which I often use, is to require the coefficient for candidate C to be zero with no sign restrictions on the remaining two coefficients.

COROLLARY 1. *The Basic and Reversal vectors satisfy*

$$\mathbf{B}_A + \mathbf{B}_B + \mathbf{B}_C = \mathbf{R}_A + \mathbf{R}_B + \mathbf{R}_C = \mathbf{0}. \tag{3.6}$$

Consequently, vectors in the Basic and Reversal subspaces always can be represented with two non-negative coefficients and a zero one, or with a zero coefficient for a designated candidate and no sign restriction on the coefficients for the other two.

To illustrate, because Eq. (3.6) requires $-\mathbf{B}_C = \mathbf{B}_A + \mathbf{B}_B$, the Basic vector $\mathbf{p}_B = -6\mathbf{B}_A - 9\mathbf{B}_C = -6\mathbf{B}_A + 9(\mathbf{B}_A + \mathbf{B}_B) = 3\mathbf{B}_A + 9\mathbf{B}_B$ can be described with $b_B = 9$, $a_B = 3$, $c_B = 0$ defining the $B \succ A \succ C$ outcome for all pairs and positional methods. Similarly, $2\mathbf{R}_A + \mathbf{R}_C = \mathbf{R}_A - \mathbf{R}_B$ defines $a_R = 1$, $c_R = 0$, $b_R = -1$ and the $A \succ C \succ B$ Basic outcome.

4. GEOMETRY OF PAIRWISE VOTING

Theorem 3.3 identifies the Basic and Condorcet vectors, $\mathbf{p}_B + \mathbf{p}_C$, as the *only* profile portions which affect pairwise rankings. Consequently, all BC and pairwise differences, all properties of the Condorcet winner, agendas, cycles, etc. are completely and quickly determined by these differentials. Tools to assist this analysis are developed next. Combined with the Section 7 discussion which exploits the Theorem 3 assertion that only the $\mathbf{p}_B + \mathbf{p}_R$ portion of a profile effects positional rankings, we create a straightforward way to understand why different procedures have different societal rankings and what they are.

The large pairwise voting literature (e.g., for a sample, see Austen-Smith & Banks [2], Kelly [9], McKelvey [13], Richards [18], Sen [36], Zwicker [42]) considers cycles, properties of the Condorcet winner and loser, properties of procedures based on pairwise election outcomes, the Borda–Condorcet conflict, etc. Theorem 3 provides an easy way to find stronger results. Indeed, as all properties of the pairwise rankings and tallies are strictly due to the Basic and Condorcet profile differentials, only these portions need be considered. This analysis is further simplified with the following geometric description.

4.1. *Additive Transitivity*

An unexpected but valuable property of the Basic profiles is that their pairwise election rankings go beyond defining ordinal transitive rankings to have the tallies satisfy a highly idealized *additive transitivity*[3] property. Namely, as asserted next, the tallies mimic the additive properties of points x, y, z on the line where $(x-y)+(y-z)=(x-z)$.

COROLLARY 2. *The pairwise rankings of a Basic profile are transitive, and the tallies from any two pairwise elections uniquely determine the tally for the remaining pairwise election. More specifically, if $\tau_B(X, Y)$ denotes the difference between X's and Y's Basic pairwise tallies, then for candidates X, Y, Z*

$$\tau_B(X, Y) + \tau_B(Y, Z) = \tau_B(X, Z). \tag{4.1}$$

Equation 4.1 captures what a novice might believe about voting. For instance, such a naive individual might expect the election rankings $A \succ B$ and $B \succ C$ to imply that $A \succ C$, or that A has a larger victory tally over C than over B. These assertions are false in general because we cannot even ensure ordinal transitivity. This idealized setting, however, holds for Basic profiles because the $\tau_B(A, B)$ and $\tau_B(B, C)$ sum equals the $\tau_B(A, C)$ difference.

So, going beyond ensuring transitive pairwise rankings, the point totals for Basic profiles capture the intuitive sense that a wider point

[3] This term was suggested to me by Duncan Luce.

spread—even for just one pairwise election—signals a stronger candidate. But, in general, non-transitive results occur. As Corollary 2 proves that such behavior cannot be attributed to the Basic profile, it follows from Theorem 3 that all blame for the failure of transitivity must be assigned to the Condorcet portion of a profile. The Basic profile tallies satisfy additive transitivity (Eq. (4.1)); the Condorcet portion disrupts both additive and ordinal transitivity.

4.2. *Geometry and the Transitivity Plane*

To geometrically compare the Basic and Condorcet differentials, I use the *representation cube* $\mathcal{RC}$ introduced in Saari [24, 27]. Here the difference between pairwise tallies (not just the Basic terms) $\tau(X, Y)$ is replaced with a fraction. Namely, with v voters, let

$$x_{X,Y} = \tau(X, Y)/v, \tag{4.2}$$

which requires

$$-1 \leqslant x_{X,Y} \leqslant 1, \qquad x_{X,Y} = -x_{Y,X}.$$

For instance, the three choices of $x_{A,B} = 1, 0, -1$ mean, respectively, that A wins all votes, is tied, and does not receive a single vote when compared with B.

Point $(x_{A,B}, x_{B,C}, x_{C,A})$ in R^3 defines the normalized difference between outcomes for all pairs. The $-1 \leqslant x_{X,Y} \leqslant 1$ restriction forces these values into a cube, the *orthogonal cube*, centered at the origin. Six of the eight cube vertices correspond to unanimity profiles. The last two cannot; if they did, then the "cyclic vertices" would require all voters to have cyclic preferences. For instance, when all voters have the preferences $A \succ C \succ B$, the unanimity outcomes $x_{A,B} = -x_{B,C} = -x_{C,A} = 1$ define the type 2 unanimity vertex $(1, -1, -1)$. If cyclic vertex $(1, 1, 1)$, were an unanimity vertex, then all voters would have the cyclic preferences $A \succ B$, $B \succ C$, $C \succ A$.

Election outcomes are represented by the convex sum of the unanimity vertices. Namely, if p_j is the fraction of all voters with the jth preference, and $\mathbf{E}_j$ is the unanimity vertex with this ranking, then the election outcome is

$$\mathbf{q} = \sum_{j=1}^{6} p_j \mathbf{E}_j.$$

Because election outcomes are given by this convex sum, they are in the convex hull of the six unanimity vertices; this is *the representation cube* $\mathcal{RC}$.

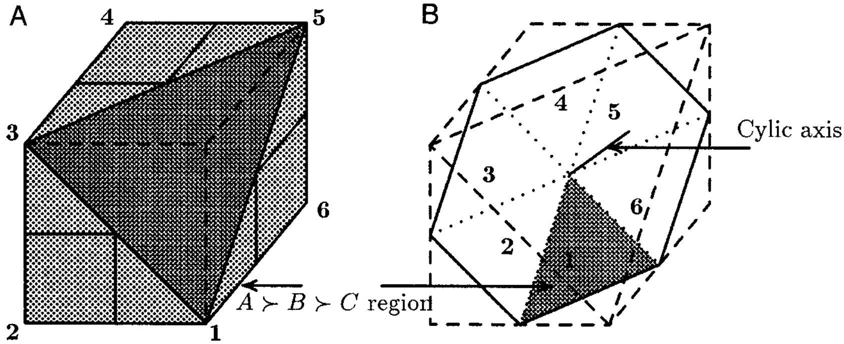

FIG. 3. Pairwise outcomes. (A) Representation cube. (B) Transitivie plane.

This $\mathscr{RC}$, depicted by the shaded region in Fig. 3A, is the set of points in the orthogonal cube satisfying

$$-1 \leqslant x_{A,B} + x_{B,C} + x_{C,A} \leqslant 1. \tag{4.3}$$

The importance of $\mathscr{RC}$ is that each (rational) point is the pairwise election outcome for some profile. Conversely, each pairwise election outcome defines a rational point in this cube (Saari [24, 27]).

A plane parallel to these faces, the *transitivity plane*, passes through the origin so it is given by

$$x_{A,B} + x_{B,C} + x_{C,A} = 0. \tag{4.4}$$

According to Eq. (4.1), this plane, represented in Fig. 3B, contains all Basic pairwise outcomes. Perpendicular to the plane is the axis connecting the cyclic rankings of the two vertices $\pm(1, 1, 1)$; call this the *cyclic axis*. The $\mathbf{C}^3$ pairwise election outcomes are in this direction.

PROPOSITION 1. *A Basic profile pairwise outcome is in the transitivity plane; a Condorcet profile pairwise outcome is on the cyclic axis. The outcome of any profile has an orthogonal decomposition described by the distance of the point along the cyclic axis and the remaining component in the transitivity plane. Geometrically, point* $\mathbf{q} \in \mathscr{RC}$ *has the unique decomposition*

$$\mathbf{q} = \mathbf{q}_T + \mu(1, 1, 1), \tag{4.5}$$

where $\mathbf{q}_T$ *is a point in the transitivity plane and* μ *is a scalar. This is called the* transitivity plane coordinate representation.

Proof. A computation proves that the pairwise outcome of a Condorcet term is along the cyclic axis. Another computation proves that the pairwise outcomes for $\mathbf{B}_A$ and $\mathbf{B}_B$ are, respectively, $(1, 0, -1)$ and $(-1, 1, 0)$. The span of these two vectors is the transitivity plane.

To prove uniqueness, suppose

$$\mathbf{q} = \mathbf{q}_T + \mu(1, 1, 1) = \mathbf{q}_T^* + \mu^*(1, 1, 1). \tag{4.6}$$

Thus, $\mathbf{q}_T - \mathbf{q}_T^* = (\mu^* - \mu)(1, 1, 1)$ where each term is in a different orthogonal subspace. As this requires each term to be zero, uniqueness is proved. The rest of the assertion follows from the decomposition.

Algebra and Theorem 3 connect the geometry with the profile decomposition.

PROPOSITION 2. *Profile $a_B\mathbf{B}_A + b_B\mathbf{B}_B + \gamma\mathbf{C}^3 + k\mathbf{K}$ defines the point*

$$\frac{1}{3k}[2(a_B - b_B, b_B, -a_B) + \gamma(1, 1, 1)], \tag{4.7}$$

where the first vector is the transitive plane component and the second is the cyclic axis component. Conversely, the $\mathscr{RC}$ point $(q_1^T, q_2^T, q_3^T) + \mu(1, 1, 1)$ (so $q_1^T + q_2^T + q_3^T = 0$), has the profile description

$$a_B = -\lambda q_3^T/2, \qquad b_B = \lambda q_2^T/2, \quad \gamma = \lambda\mu, \quad k = \lambda/3 \tag{4.8}$$

for any $\lambda > 0$.

Proof. The verification of Eq. (4.7) follows directly from Theorem 3. Eq. (4.8) is obtained by a direct algebraic computation. ∎

Equations (4.7) and (4.8) illustrate important points. First, because a profile differential involves "zero voters," it defines a *direction* rather than a $\mathscr{RC}$ point; e.g., this direction is the term in the brackets of Eq. (4.7). To convert the direction into a point, we need the magnitude, or, as indicated by the scalar multiple of the brackets in Eq. (4.7), a term measuring the total number of voters. Thus, for instance,

$$[3\mathbf{B}_A + 2\mathbf{B}_C] - 4\mathbf{C}^3$$

defines the direction $2(12, -8, -4) - 4(1, 1, 1) = (20, -20, -12)$. Once we know the number of voters, this direction is converted into a point by using Eq. (4.7). So, when points from the cube are expressed in terms of transitivity plane coordinates, the outcome defines a direction capturing the relative magnitudes of the cyclic and transitive plane coordinates. The actual values, which are complicated by the Reversal terms, are given by Eq. (8.2).

5. GEOMETRY OF PAIRWISE PROCEDURES

The Introduction includes a claim that this approach helps to address the major (and difficult) theme of analyzing and comparing procedures which use pairwise and positional outcomes. I illustrate how to do this, along with identifying other decomposition properties, by comparing procedures dependent upon pairwise election outcomes.

A natural way to compare methods is to characterize and then analyze all profiles where they agree and differ. Because of the complexity of the analysis, however, very few published works attempt to do this. Instead, comparisons are made by constructing illustrating but essentially isolated examples. But the profile decomposition and its associated geometry allow us to realize the original, more general objective; we can identify *all* profiles where the outcomes of specified methods differ. For instance, a major mystery is to understand when and why the BC and pairwise rankings differ; I characterize *all* such profiles.

All procedures agree on the Basic portion of a profile, so all differences are caused by the Condorcet portion. Thus the previous technically difficult analysis of cycles, agendas, Kemeny's rule, Copeland's method, the Borda and Condorcet debate, etc., reduce to determining how each procedure treats the Condorcet profiles. In what follows, I emphasize this effect. By combing this information with an interpretation for the Condorcet component (Section 6), we obtain new interpretations for differences among procedures.

5.1. *Agendas*

An *agenda* $\langle X, Y, Z \rangle$ is where the majority winner of the first two candidates, X and Y, is advanced to a majority vote comparison with the last listed candidate Z. As it is known, if the pairwise outcome $\mathbf{q}$ is transitive, then the Condorcet winner wins with any agenda. If the rankings are cyclic, the last listed candidate of an agenda, Z, always wins. Thus, when $\mathbf{q}$ defines cyclic rankings the outcome depends upon the choice of the agenda. Even stronger, we know (e.g., see [27]) that the complexities and difficulties associated with agendas[4] are caused by the cyclic pairwise outcomes.

According to the Eq. (4.5) profile decomposition, when $\mathbf{q} = \mathbf{q}_T$, the pairwise rankings satisfy additive transitivity. Thus, with Basic profiles, agendas are spared these difficulties; e.g., all agendas yield the same out-

[4] For instance, if two subcommittees have identical winners with the same agenda, they could have a different sincere outcome with the same agenda when they vote as a single committee.

come. So, the Condorcet portion is totally responsible for all troubles with agendas as well as conflicts with other procedures.

5.2. *The Borda Count*

As the BC tally for each candidate equals the sum of her tallies from her two pairwise elections, the $\mathbf{b}^3$ tally of a normalized profile with pairwise outcome $\mathbf{q}$ is obtained by adding appropriate $\mathbf{q}=(x_{A,B}, x_{B,C}, x_{C,A})$ components. For instance, A's tally is $x_{A,B}+x_{A,C}=x_{A,B}-x_{C,A}$. With transitivity plane coordinates $\mathbf{q}=\mathbf{q}_T+\mu(1, 1, 1)$, find the BC outcome by separately summing components of $\mathbf{q}_T$ and of $\mu(1, 1, 1)$, and then adding them together. The $\mu(1, 1, 1)$ term requires each candidate to win and lose one competition with the same μ difference, so the BC tallies on the cyclic axis cancel. Only the $\mathbf{q}_T$ term in the transitivity plane affects the BC tally; this component, which comes from the Basic profile term, completely determines the BC outcome. This leads to the following result.

THEOREM 4. *Let $\mathbf{q}=\mathbf{q}_T+\mu(1, 1, 1)$ be an election point in $\mathscr{RC}$. The BC election ranking for $\mathbf{q}$ is the ranking assigned to $\mathbf{q}_T$. The $\mathbf{b}^3$ vector tally is*

$$\alpha(x_{A,B}-x_{C,A}, x_{B,C}-x_{A,B}, x_{C,A}-x_{B,C})+\beta(1, 1, 1) \tag{5.1}$$

for scalars $\alpha>0$, β.

Proof. As the BC ranking is strictly determined by $\mathbf{q}_T$ from the Basic profile, the pairwise rankings defined by $\mathbf{q}_T$ are transitive and agree with the BC ranking (Theorem 3). To prove the assertion about the tallies, recall from Eq. (4.2) that with v voters $vx_{X,Y}=\tau(X, Y)$. But (Theorem 3) $\tau(A, B)+\tau(A, C)=\tau_B(A, B)+\tau_B(A, C)$, so the assertion follows by summation and the fact (Theorem 3) that all other profile portions add a fixed amount to each voter's BC tally. ∎

5.2.1. *BC Sphere*

To "see" all differences between the pairwise and BC election outcomes, use the Theorem 4 result that $\mathbf{q}_T$ is the closest transitivity plane point to $\mathbf{q}$. All points equal distance from $\mathbf{q}$ are on a sphere centered at $\mathbf{q}$, so center a sphere at $\mathbf{q}$; call it the *BC sphere*. Next, treat this sphere as a balloon and blow it up (Fig. 4a) until it first touches the transitivity plane at $\mathbf{q}_T$—this defines the BC ranking and tally. Clearly, if $\mathbf{q}$ is in the transitivity plane, then the BC and pairwise outcomes agree. Thus, as asserted next, all differences in BC and pairwise outcomes are due to the Condorcet portion of a profile.

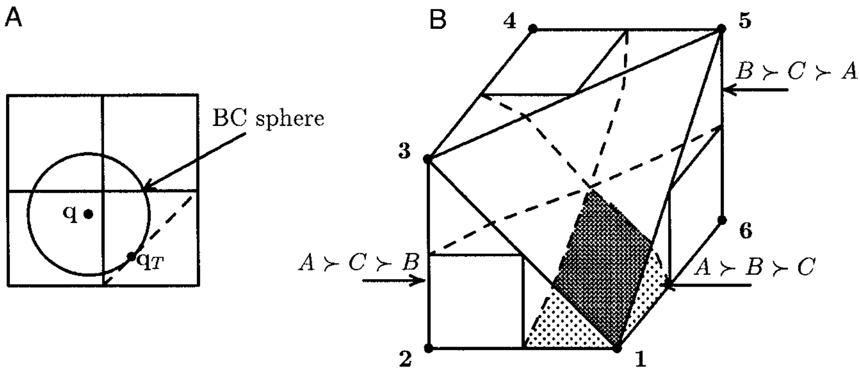

FIG. 4. Comparing the BC and pairwise outcomes. (A) Conflicting BC and pairwise outcomes. (B) BC outcomes.

PROPOSITION 3. *All differences between the BC and pairwise rankings are due to the Condorcet portion of a profile; the BC ignores this term while the pairwise rankings rely upon it.*

By combining the BC sphere with the tilt of the transitivity plane, it becomes easy to find profiles where the pairwise and BC rankings differ. As illustrated in Fig. 4a, choose **q** close enough to a $x_{i,j}=0$ coordinate plane so that the BC sphere hits the transitivity plane in a different ranking region; to find an associated profile, use Proposition 2. Indeed, we can identify all **q**'s with different pairwise and BC outcomes. As the BC ranking is completely determined by $\mathbf{q}_T$, the same BC ranking holds for $\mathbf{q}_T+\mu(1, 1, 1)$ for all μ values. Therefore, to find all possible **q** with a specified BC outcome, say $A \succ B \succ C$, trace a line parallel to $(1, 1, 1)$ along the boundary of the $A \succ B \succ C$ ranking region in the transitivity plane. (See Fig. 3B.) These lines define a boundary surface for the desired pairwise outcomes as given by the dashed lines in Fig. 4B. (The shaded area depicts all **q**'s with a $A \succ B \succ C$ BC outcome.)

Another way to find this geometry is to to cut $\mathscr{RC}$ into six pieces according to the template provided by Fig. 3B. Each ranking region of the transitivity plane is defined by the dotted boundaries in Fig. 3B. So, cut $\mathscr{RC}$ where the slices are perpendicular to the transitivity plane and pass through these Fig. 3B dotted boundaries. This defines the six wedges—one for each strict ranking—depicted by the dashed line boundaries in Fig. 4B. Because the origin of $\mathscr{RC}$, $\mathbf{0}=(0, 0, 0)$, is on the boundary of all regions, so are the two cyclic vertices $\pm(1, 1, 1)$.

This geometry identifies all profiles that exhibit a variety of behaviors; this includes all profiles where the BC does not elect the Condorcet winner. For instance, according to Fig. 4B, all **q**'s with transitive rankings which allow a $A \succ C \succ B$ BC outcome are in a type 1, 2, or 3 pairwise ranking region; the commonality of these three rankings is that A is strictly ranked above B. This (and neutrality) provides a geometric proof of the known fact that *the BC ranks the Condorcet winner above the Condorcet loser; e.g., the Condorcet winner cannot be BC bottom-ranked.*

THEOREM 5. *The set of pairwise outcomes defining a $A \succ B \succ C$ BC outcome is given by the convex hull of the vertices*

$$\{\pm(1,1,1), (1,0,-1), (0,1,-1), (\tfrac{1}{3}, 1, -\tfrac{1}{3}),$$
$$(1, \tfrac{1}{3}, -\tfrac{1}{3}), (-\tfrac{1}{3}, \tfrac{1}{3}, -1), (\tfrac{1}{3}, -\tfrac{1}{3}, -1)\}$$

Proof. This follows directly from the geometry. ∎

5.2.2. *Profile Conditions*

I am unaware of any description of necessary and sufficient conditions for profiles which describe when the Condorcet winner is *not* BC top-ranked, or when the BC and pairwise rankings differ, etc. To develop them, assume the BC ranking is $A \succ B \succ C$ which occurs (Theorem 3) if and only if the Basic profile coefficients satisfy $a_B > b_B > c_B = 0$ to create the $\mathbf{b}^3$ tally $(2a_B - b_B,\ 2b_B - a_B,\ -(a_B + b_B))$. Here the pairwise and BC rankings agree and the $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ pairwise tallies are, respectively,

$$2(a_B - b_B) : -2(a_B - b_B), \qquad 2b_B : -2b_B, \quad 2a_B : -2a_B \tag{5.2}$$

The $\gamma \mathbf{C}^3$ Condorcet term (Theorem 3) changes the $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ tallies to become, respectively,

$$\begin{gathered} 2(a_B - b_B) + \gamma : -2(a_B - b_B) - \gamma, \qquad 2b_B + \gamma : -2b_B - \gamma, \\ 2a_B - \gamma : -2a_B + \gamma. \end{gathered} \tag{5.3}$$

Now by using elementary algebra with Eq. (5.3), we can completely characterize how the Condorcet term forces differences between pairwise and BC (or Basic) rankings. For instance, the $A \succ B$ ranking persists if $2(a_B - b_B) + \gamma > -2(a_B - b_B) - \gamma$, or as long as $\gamma > -2(a_B - b_B)$. But if $\gamma < -2(a_B - b_B)$, then this pairwise ranking becomes $B \succ A$. Similarly, the $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ pairwise rankings change, respectively, when γ passes through the values $-2(a_B - b_B)$, $-b_B$, a_B. Plotting these values on a "γ number line" charts all possible pairwise ranking changes. The two number lines of Fig. 5 correspond to whether $(a_B - b_B)$ or b_B has the larger value. The rankings defined by each γ interval are listed next to the appropriate line; if the same ranking holds for each line, it is listed in the

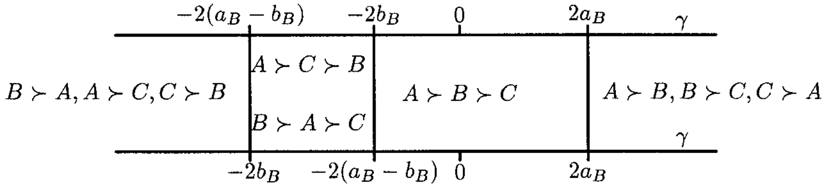

FIG. 5. How γ changes pairwise outcomes.

middle. (This figure is *not* in scale. As the $a_B - b_B = b_B$ case just eliminates a second strict transitive ranking, it is not represented. The easy $b_B = 0$ case is left to the reader.)

As Fig. 5 identifies all possible pairwise rankings associated with a specified Basic profile, it describes all possible profiles where the BC and pairwise rankings agree or differ. For instance, the γ interval defining the pairwise ranking $A \succ B \succ C$ has the upper value of $\gamma = 2a_B$ but the lower value depends upon which line is relevant for the Basic coefficients. As the choice depends upon whether $2b_B$ or $2(a_B - b_B)$ has the smaller value, the relevant γ values are given by Eq. (5.4). In the same way, all results specified in Theorem 6 are found.

THEOREM 6. *Assume the BC ranking is $A \succ B \succ C$ ranking (so $a_B > b_B > 0$). A necessary and sufficient condition for the BC and pairwise rankings to agree is that the coefficient of the Condorcet term $\gamma \mathbf{C}^3$ satisfies*

$$-2\,\mathrm{Min}((a_B - b_B), b_B) < \gamma < 2a_B. \tag{5.4}$$

A necessary and sufficient condition for the BC ranking to differ from a transitive pairwise ranking is

$$-2\,\mathrm{Max}((b_B, (a_B - b_B)) \leqslant \gamma \leqslant -2\,\mathrm{Min}(b_B, (a_B - b_B)). \tag{5.5}$$

A necessary and sufficient condition for the BC ranking to have a candidate other than the BC winner as the Condorcet winner (B) *is that*

$$-2b_B < \gamma < -2(a_B - b_B). \tag{5.6}$$

A necessary and sufficient condition for the BC loser not to be the Condorcet loser is

$$-2(a_B - b_B) < \gamma < -2b_B. \tag{5.7}$$

A necessary and sufficient condition that the BC ranking is accompanied by a pairwise cycle is that γ satisfies one of the inequalities

$$2a_B < \gamma, \qquad \gamma < -2\,\mathrm{Max}((a_B - b_B), b_B)). \tag{5.8}$$

The first inequality defines the cycle with $A \succ B$; the second defines the cycle with $B \succ A$.[5]

It is instructive to translate these conditions into the transitive plane geometry of $\mathcal{RC}$. In this manner we discover, for instance, how the tilt of the transitivity plane mandates negative μ and γ values to make B the Condorcet winner. (According to Fig. 4B, positive μ and γ values are needed if the BC ranking is, say, $A \succ C \succ B$.) These negative γ values also reflect the need for the cycle to give B an advantage in its tally over A. The specific choices of $a_B = \frac{5}{3}$, $b_B = \frac{4}{3}$, $\gamma = -\frac{5}{3}$ define the profile differential (0, 2, −3, 0, −2, 3) or (by adding $3\mathbf{K}^3$) the illustrating profile (3, 5, 0, 3, 1, 6).

To illustrate the Theorem 6 assertion about pairwise cycles, choose $a_B = 2$, $b_B = 1$, $c_B = 0$ to define the basic profile (2, 1, −1, −2, −1, 1) with its universal $A \succ B \succ C$ election ranking. The accompanying $A \succ B$, $B \succ C$, $C \succ A$ pairwise cycle occurs with $\gamma = 5 > 2a_B = 4$ to define the profile differential (7, −4, 4, −7, 4, −4) or a profile (14, 3, 11, 0, 11, 3). To construct an example where the BC ranking is accompanied with a $B \succ A$, $A \succ C$, $C \succ B$ cycle, choose $\gamma = -3 < -2\,\text{Max}((a_B - b_B), b_B) = -2$ to obtain the profile differential (−1, 4, −4, 1, −4, 4) or the profile (3, 8, 0, 5, 0, 8).

5.2.3. *Simplifying a Condorcet Result*

In Section 1.3, I mention Condorcet's assertion that there exist profiles where no positional method elects the Condorcet winner. The complexity of creating profiles, however, severely limited finding illustrating examples. Also, it was not known why this phenomenon occurs. But as the profile decomposition allows us to create examples illustrating *any* admissible behavior, we can design many supporting profiles. One approach combines Theorem 6 with Theorem 3.

COROLLARY 3. *If profile* $\mathbf{p}$ *does not have a Reversal component and the Condorcet winner is not BC top-ranked, then no positional method has the Condorcet winner top-ranked. Similarly, if a profile has no Reversal component and if the BC does not have the Condorcet loser bottom-ranked, then no positional method has the Condorcet loser bottom-ranked.*

Proof. With no Reversal portion, all positional rankings agree with the BC. ▮

By construction, the above profile (3, 5, 0, 3, 1, 6) has no Reversal component and A, rather than the Condorcet winner B, is BC top-ranked.

[5] While I was revising this paper, Zwicker called my attention to his paper [42] where he extends Sen's [36] conditions about cycles. As such, his results explaining when cycles occur may be related to Eq. (5.8). It appears he found his result by factoring the profiles giving binary outcomes by the Condorcet profile.

So, according to Corollary 3, all positional methods share the $A \succ B \succ C$ ranking where the Condorcet winner B is not top-ranked. Other examples come from Theorem 6 (or the geometric approach) which provides a wide selection of profiles where the common positional method outcome conflicts with the pairwise outcomes. According to Corollary 3 and Eq. (5.7), a profile differential where the Condorcet loser is not bottom-ranked with any positional method is $3\mathbf{B}_A + \mathbf{B}_B - 3\mathbf{C}^3$ or $(0, 5, -4, 0, -5, 4)$; a supporting profile is $(5, 10, 1, 5, 0, 9)$. Clearly, these effects are due to the profile's Condorcet portion; a deeper explanation is in Section 6.

5.2.4. *Tallies*

Relationships between the BC and pairwise rankings have been known since Borda, but the reasons for the conclusions have not been well understood. The earlier results use the Eq. (2.2) condition that a candidate's BC tally is the sum of her pairwise tallies. As we now know from Theorem 3, this summation cancels the tallies from the Condorcet portion of the profile leaving only the Basic profile terms to influence the BC outcome. By using this observation, we obtain extensions and new proofs of known statements.

THEOREM 7. *Assume there are $n = 3$ candidates.*

1. *For any profile, there exists a unique γ value so that by removing γ points from each of $\tau(A, B)$, $\tau(B, C)$, $\tau(C, A)$, the reduced tallies satisfy the additive transitivity condition of Eq.* (4.1). *The removed cyclic terms from the tally correspond to the Condorcet portion of the profile, while the reduced tally is due to the Basic portion. The ranking from the reduced, or transitive plane tally, agrees with the BC ranking.*

2. *If all pairwise tallies have a complete tie, then the BC outcome is a complete tie. If the BC outcome is a complete tie, then either all pairwise elections are tied, or they define a cycle with the same victory margin in each pairwise election.*

3. *The Condorcet winner cannot be BC bottom-ranked. The Condorcet loser cannot be BC top-ranked. The Condorcet winner is BC strictly ranked above the Condorcet loser.*

Proof. The proof of part 1 follows from Theorem 3. Because the first part of part 2 requires the pairwise tallies to satisfy additive transitivity, the profile has no Condorcet portion. (The point is in the transitive plane.) Thus each pairwise outcome for the Basic profile is zero, so, according to Theorem 3, the BC ranking also is a tie. Conversely, a BC complete tie requires a zero Basic portion for the profile. Consequently, as the pairwise vote is strictly determined by the Condorcet portion, the outcome is cyclic.

To prove part 3, notice that a Condorcet winner and/or Condorcet loser requires (from part 1) a nonzero Basic portion. On the Basic portion, the pairwise and BC rankings agree, and the pairwise tallies satisfy the additive transitivity condition Eq. (4.1). The $\tau(X, Y)$ outcomes are

$$\tau(A, B) = \tau_B(A, B) + \gamma, \qquad \tau(B, C) = \tau_B(B, C) + \gamma, \quad \tau(A, C) = \tau_B(A, C) - \gamma \tag{5.9}$$

for any γ. The assertion now follows from simple algebra. ▮

While Theorem 7 part 3 is well known, I know of no statement even speculating whether the converse is true. The reason is clear; to find how a Condorcet winner fares within a BC ranking, we add the pairwise tallies to *cancel* Condorcet terms. But to prove a statement about how the BC winners or losers fare within the pairwise rankings, we *introduce* Condorcet terms to the Basic profile. As such, finding relationships requires determining the relationship between the Basic and Condorcet components. As this is done in Fig. 5, the following is a direct consequences of this figure. (The result holds for all $n \geqslant 3$ alternatives.)

THEOREM 8. *If a profile has a transitive pairwise ranking which is not a complete tie, then it strictly ranks the BC winner above the BC loser.*

Proof. The profile for a transitive ranking which is not a complete tie has a non-zero Basic component where the BC winner (or winners as there may be a tie) ranked above the BC loser (or losers). The rest of the proof follows from Fig. 5 which starts with any Basic rankings and finds all corresponding pairwise rankings. ▮

5.3. *Copeland Method*

The Copeland method (CM) (see Copeland [6], Saari and Merlin [31], Merlin and Saari [14]) often is used to rank sports teams. This is where the winning team receives one point, the losing team -1 and, if there is a tie, both receive zero points. A team's ranking is determined by the sum of received points.

For a geometric description of CM, replace each non-zero component of $(x_{A,B}, x_{B,C}, x_{C,A})$ with the nearest of ± 1. With no pairwise ties, this process defines a vertex $\mathbf{V}^{CM}$ of the orthogonal cube with the transitivity plane representation

$$\mathbf{V}^{CM} = \mathbf{V}_T^{CM} + \mu^{CM}(1, 1, 1).$$

Each candidate's CM score is found from $\mathbf{V}^{CM}$ by adding the points she receives in the two elections. As true with the BC, this summation forces a cancellation in the cyclic direction, so the CM election is determined by the ranking of the transitivity plane coordinate $\mathbf{V}_T^{CM}$. Thus the CM outcome also can be described with an expanding balloon but centered at $\mathbf{V}^{CM}$ (rather than $\mathbf{q}$). This balloon, or *CM sphere*, is blown up until it first touches the transitivity plane; this first point of contact $\mathbf{V}_T^{CM}$ determines the CM ranking.

According to this description, the CM and BC differ only by the chosen center for the expanding spheres; all BC and CM differences are due to this translation. To identify all profiles with different BC and CM outcomes, select $\mathbf{q}_T$ in a particular ranking region and vary the μ value for $\mathbf{q} = \mathbf{q}_T + \mu(1, 1, 1)$ so that the associated $\mathbf{V}^{CM}$ vertex defines conflicting CM and BC outcomes. (This occurs when $\mathbf{q}$ and $\mathbf{q}_T$ are in different transitive ranking regions, or when $\mathbf{q}$ is in a cyclic region.) Thus, in a subtle but crucial manner, the CM depends upon the Condorcet portion of a profile.

5.4. *Kemeny's Rule*

If we ignore pairwise ties, Kemeny's rule (KR) (Kemeny [10], Saari, Merlin [32]) for three candidates can be described in the following manner.[6]

- When the pairwise rankings define a transitive ranking, that is the KR ranking.
- When the pairwise rankings define a cycle, reverse the ranking of the pair with the smallest difference between the tallies of the candidates. (That is, the ranking with the smallest $x_{X,\,Y}$ value. If this can be done in more than one way, the KR ranking consists of all possibilities.)

So, if $\mathbf{q} = \mathbf{q}_T + \mu(1, 1, 1)$ defines a transitive ranking, this is the KR ranking. As we now know, the Condorcet term can force different transitive rankings to be associated with $\mathbf{q}$ and $\mathbf{q}_T$. Thus, the Condorcet term affects KR outcomes even when $\mathbf{q}$ is transitive. This KR dependency becomes more dramatic when $\mathbf{q}$ defines a cycle. Here, the KR divides each cyclic region into three equal parts (rather than six as with the BC) with the cyclic axis on the boundary of each part. The boundaries for this region are where some two coordinates, say $x_{A,\,B}$ and $x_{B,\,C}$, agree.

[6] For $n \geqslant 4$ candidates, this description does not hold. For more details, see Saari and Merlin [32].

5.5. *Comparisons*

Combining the geometry depicted by Figs. 4 and 5 with Theorem 6, we now can identify all profiles where any two specified methods differ.

THEOREM 9. *If* $\mathbf{q}=\mathbf{q}_T+\mu(1, 1, 1)$ *is in transitivity plane (so* $\mu=0$*), the BC, CM, KR have the same ranking where the top-ranked candidate is the Condorcet winner and she wins with any agenda. When* μ *is such that* $\mathbf{q}$ *defines a transitive ranking which differs from that for* $\mathbf{q}_T$*, the CM and KR have the same ranking where the top-ranked candidate is the Condorcet winner and she wins with any agenda; the BC ranking, however, differs as it is defined by* $\mathbf{q}_T$.

When μ *is such that* $\mathbf{q}$ *defines a cyclic ranking, there is no Condorcet winner, the BC outcome agrees with that of* $\mathbf{q}_T$*, the CM outcome is the complete tie* $A\sim B\sim C$*, and the agenda winner is the candidate last listed in the agenda. If* μ *is positive, the KR outcome is a ranking from the Condorcet triplet* $\{A\succ B\succ C, B\succ C\succ A, C\succ A\succ B\}$ *which has the BC winner ranked above the BC loser. If* μ *is negative, then the same condition holds, but for the reversed Condorcet cycle.*

Proof. Only the assertion about the KR winner with a cyclic outcome needs to be justified. Notice that if the profile consists of the cyclic term alone, then each term in the appropriate Condorcet cycle satisfies the described KR selection process; thus all three are in the KR outcome. Now consider the effect of the $\mathbf{q}_T$ term which we can assume defines the $A\succ B\succ C$ ranking. This $\mathbf{q}_T$ term changes the tallies according to Eq. (5.3) where the biggest differential in the new tally occurs in the $\{A, C\}$ election. The conclusion follows. (An extension is in Saari and Merlin [32].) ∎

Thus, all differences among these procedures must be attributed to the Condorcet portion. An interpretation of $\mathbf{p}_C$ is given next.

6. THE CONDORCET PORTION

Any analysis of $\mathbf{C}^3$ must explain the cyclic outcomes. I do this by showing that even though all voters have transitive preferences, *the pairwise vote applied to* $\mathbf{p}_C$ *loses the crucial assumption of individual rationality*. For the mathematically inclined reader, let me briefly suggest the mathematical structure which generates this surprising conclusion. The argument uses the natural symmetries of voting which, for a pairwise vote, is captured by neutrality; if each voter interchanges the names of the two specified alternatives, the outcome is similarly exchanged. (This is the Z_2 orbit acting on particular set of alternatives.) Doing this for all pairs of a specific ranking, say $A \succ B \succ C$, we end up not with six, but with eight distinct rankings—two are cyclic. This indicates that the natural domain for pairwise voting is where any complete ranking is admitted as long the pairwise rankings are strict. The following represents this algebraic structure.

6.1. *Geometry*

The pairwise vote can be used by any voter capable of ranking each pair of candidates *whether the rankings are transitive or cyclic*. As it is irrelevant for the procedure whether a voter's rankings are transitive, the transitivity assumption is a *profile restriction*. Indeed, the pairwise vote for all voters, even cyclic ones, is given by a point in the orthogonal cube where the cyclic vertices now represent unanimity vertices; e.g., profile $(1-\lambda)(1, 1, -1)+\lambda(1, 1, 1)$ defines where λ of the voters have the cyclic $A \succ B$, $B \succ C$, $C \succ A$ preferences. Transitive preferences *restrict* the orthogonal cube to $\mathscr{RC}$.

To interpret $\mathbf{C}^3$, notice that its pairwise vote tally defines a direction toward the cyclic vertex $(1, 1, 1)$. As far as the pairwise vote is concerned, then, the Condorcet portion can be identified as representing the unanimity preferences of irrational voters with the cyclic preferences $A \succ B$, $B \succ C$, $C \succ A$ rather than the actual preferences of transitive voters. Indeed, $\mathbf{q}_T + \mu(1, 1, 1)$ admits the interpretation that a fraction (represented by μ) of the voters have irrational, cyclic preferences. Consequently, the geometry indicates that applying the pairwise vote to the Condorcet portion of a profile has the effect of dropping the assumption of individual rationality.

6.2. *Lost Information*

By emphasizing different profile information, procedures produce different outcomes. So, to analyze the pairwise vote, I identify what profile information it retains, and what it devalues with the traditional three-voter Condorcet profile

$$A \succ B \succ C, \qquad B \succ C \succ A, \quad C \succ A \succ B \tag{6.1}$$

given by $\frac{1}{2}(\mathbf{K}+\mathbf{C}^3)$. As each candidate is in first, second, and last place exactly once, it is easy to argue (using neutrality and anonymity) that no candidate has an advantage; in particular, these voters' votes should cancel. This complete tie outcome does occur for all positional methods, but a pairwise vote yields the $A \succ B$, $B \succ CC \succ A$ cycle.

To explain this cycle, suppose we know only that a voter prefers $A \succ C$ from $\{A, B, C\}$. With this limited information, it is impossible to determine whether his full preferences are rational or irrational. This is because transitivity involves specific sequencing conditions on all three pairwise rankings. Similarly, if a procedure ignores this sequencing data, then it discards information about the individual rationality of voters. This occurs with the pairwise vote as it solely concentrates on how voters rank a particular pair when determining that pair's societal ranking. All information about the relative rankings of other pairs—information vital to determine whether preferences are rational—is ignored.

To illustrate, notice that the irrational voters with the cyclic preferences

Number	Pairwise Rankings	
2	$A \succ B,\ B \succ C,\ C \succ A$	(6.2)
1	$B \succ A,\ C \succ B,\ A \succ C$	

cannot vote in a $\mathbf{w}_s^3$ election. (To use a $\mathbf{w}_s^3$ procedure, voters need a transitive ranking (for $0 < s < 1$), or at least a top (for $s = 0$) or a bottom-ranked (for $s = 1$) candidate, but cyclic voters fail these minimal conditions.) But as a pairwise vote ignores information about individual rationality, the irrational voters of Eq. (6.2) can use the pairwise vote; when they do, they obtain the expected $A \succ B$, $B \succ C$, $C \succ A$ cycle with 2:1 tallies.

The point is that with a sufficiently heterogeneous society, the pairwise vote cannot distinguish whether the profile involves voters with transitive or irrational preferences. Theorem 3 identifies $\mathbf{C}^3$ as the precise heterogeneity needed for this confusion to occur. Indeed, the combination of anonymity[7] and the ignored sequencing information makes it impossible for the pairwise vote to distinguish between the Eq. (6.1) and Eq. (6.2) profiles because the only relevant data is the number of voters with each pairwise ranking, and these numbers agree for both profiles. Consequently, the pairwise vote cannot distinguish between the Condorcet triplet of Eq. (6.1) and all irrational ways voters rank pairs that generate the same tallies.

A computation proves there are *four ways* to combine the pairs from Eq. (6.1) to define profiles which differ from the Condorcet triplet. Three have two voters with transitive rankings that reverse each other (so, their votes cancel) while the tie is broken by the third with cyclic preferences $A \succ B$, $B \succ C$, $C \succ A$. Thus, the cyclic ranking is a natural conclusion. The final profile is Eq. (6.2), where, again, the cyclic outcome is most reasonable.

Thus by ignoring information about the transitivity of preferences, the pairwise vote cannot distinguish among the actual Condorcet profile of transitive preferences (where the arguable outcome is a complete tie), and

[7] Here, anonymity means that the procedure does not check the names of the voters for any input; it only determines the number of voters with each ranking.

four other profiles involving irrational voters where the cyclic outcome is the "correct" one. The cyclic pairwise outcome merely manifests the pairwise vote's attempt to reflect most ways profiles can be defined which create this particular arrangement of pairwise tallies. By doing so, the emphasis is on the beliefs of potential (but non-existent) irrational voters. Stated in another way, the properties of the pairwise vote are such that when applied to $\mathbf{C}^3$, the assumption that individual preferences are transitive is lost. Thus all non-transitive arrangements of pairwise outcomes—quasi-transitive rankings, acyclic rankings, cyclic rankings, tallies violating additive transitivity (Eq. (4.1))—are due to this $\mathbf{C}^3$ portion of a profile because this is where the pairwise vote loses the assumption of individual rationality.

To see why a Basic profile avoids these difficulties, notice that $\mathbf{B}_A$ has one voter with $A \succ B \succ C$ and another with $A \succ C \succ B$ causing the $\{B, C\}$ comparisons to cancel. (The same cancellation holds for the $\mathbf{B}_A$ rankings associated with negative numbers of voters.) This cancellation accentuates A's role while treating equally all other candidates with a tie vote. Consequently, the pairwise ranking of a general Basic profile $a_B\mathbf{B}_A + b_B\mathbf{B}_B + c_B\mathbf{B}_C$ strictly manifests the ordering properties of the a_B, b_B, c_B coefficients. Rather than reflecting desirable properties of the procedure, the transitivity of the pairwise vote in this setting is preserved by the nature of the Basic profiles.

6.3. *Borda–Condorcet Comparison*

As shown in Section 5.2, *all* differences among the BC, pairwise rankings (even tallies), and the Condorcet winner and loser are due to the $\mathbf{p}_C$ term; the BC ignores $\mathbf{p}_C$ while the pairwise vote and Condorcet winner crucially depend upon it. But because the pairwise vote depends on the Condorcet portion of a profile, we now know that *whenever there is a difference between the BC and Condorcet rankings, it is due to the Condorcet's partial dismissal of the crucial assumption of the individual rationality of voters*. For instance, Section 5.2.3 has examples illustrating Condorcet's result that the Condorcet winner need not be top-ranked by any positional method. The construction of these examples must rely upon the $\mathbf{p}_C$ portion, so Condorcet's assertion is strictly due to the pairwise vote ignoring the rationality of voters. Thus, rather than supporting the Condorcet winner, these examples expose a flaw.

It also follows that rather than reflecting poorly on the BC, any difference in Condorcet and BC outcomes demonstrates a BC strength while indicating a serious failing of the Condorcet approach. This, of course, contradicts a general choice theory belief of more than two centuries. The same assertion extends to all differences between the BC and other methods using pairwise rankings. As shown in Section 5, the non-BC procedures depend upon the Condorcet portion of a profile. Thus one can argue that the procedure's outcomes are compromised because it involves the partial dismissal of the crucial assumption of the rationality of voters. This charge applies to agendas, the Copeland Method, Kemeny's rule, and many others. In fact, combining the profile decomposition and the interpretation of the Condorcet term leads to the following more general assertion.

THEOREM 10. *Assume that a voting method $\mathscr{M}$ (either a social choice or welfare function) depends upon pairwise tallies and/or rankings. Assume that with a Basic profile, the outcome of $\mathscr{M}$ is consistent with the Condorcet ranking. If there exists a profile where $\mathscr{M}$ disagrees with the BC ranking, then the difference is because $\mathscr{M}$ has partially dismissed the assumption of the individual rationality of the voters.*

I now explore how Theorem10 extends and sheds light on Arrow's Theorem.

6.4. *Arrow's Theorem*

This discussion directly counters standard beliefs from choice theory. For instance, it is easy to find criticisms arguing that although the BC has desirable properties, it "violates the binary independence axiom $\cdots$ it is not rationalizable and violates choice theoretic conditions." (Schofield, p. 12 [35]) But instead of being a BC fault, the real flaw is that the binary independence condition unintentionally drops the crucial assumption of individual rationality when applied to $\mathbf{p_C}$. An easy proof is to note that by dropping the Condorcet portion of the profile, the BC satisfies binary independence. (This is immediate from Theorem 3.) More generally, *by removing the $\mathbf{p_C}$ portion of a profile before applying IIA, Arrow's theorem is replaced with a positive assertion.*

THEOREM 11. *Let $\mathscr{N}\mathscr{C}$ be the set of all profiles with no Condorcet portion. The set of procedures with transitive outcomes that satisfy anonymity, weak Pareto, and binary independence on $\mathscr{N}\mathscr{C}$ include the BC, the CM, the usual pairwise ranking, KR, and any other pairwise method $\mathscr{M}$ which agrees with the BC on the Basic profiles.*

All non-BC positional methods rely upon the reversal portion of a profile, so they cannot satisfy the binary independence conditions even when the Condorcet portion is removed. Therefore, the BC is the only positional method satisfying Theorem 11. Incidentally, $\mathscr{N}\mathscr{C}$ defines a larger dimensional space of profiles than Black's single peaked condition [3].

Thus, from the perspective of probability or dimensions, $\mathcal{NC}$ is less restrictive than Black's condition.

Proof. These procedures satisfy all conditions with the possible exception of binary independence. Without the Condorcet portion, the rankings of these procedures are strictly determined by the Basic portion. The conclusion follows from Theorem 3. ∎

Theorem 11 provides insight into other ways advanced to avoid Arrow's assertion. (See, for example, the papers of Weymark and his coauthors, e.g., [41].) One approach (Saari [27]) is to modify the binary independence condition so that the procedure must use transitive preferences; again, we end up with a conclusion similar to Theorem 11. Indeed, it is possible to show how many (if not all) extensions or profile restrictions of Arrow's Theorem that avoid a dictator can be viewed as finding ways to minimize or counter the effects of the Condorcet portion of a profile. In other words, as Arrow's impossibility theorem is completely due to the $\mathbf{p_C}$ portion of a profile, it underscores the importance and utility of the profile decomposition.

6.5. *Summary*

This analysis compromises arguments advanced to support procedures, such as the Condorcet, Copeland, Kemeny, agendas, and other procedures based on pairwise outcomes; i.e., it is difficult to justify the Condorcet bias manifesting a violation of individual rationality. A natural way to correct this difficulty is to remove the $\mathbf{p}_C$ portion so that the pairwise vote is determined only by the Basic portion of the profile. This approach, which also changes Arrow's assertion, removes all flaws and faults of the pairwise vote so that procedures regain their merits.

According to Theorem 3, the BC and pairwise outcomes agree on Basic profiles. So, once the source of the flaws of pairwise voting are removed, the virtues identified with any procedure apply to the BC. Thus a pragmatic way to correct the pairwise vote—and all reasonable procedures based on the pairwise vote—is to use the BC.

7. POSITIONAL METHODS AND REVERSAL TERMS

Theorem 3 ensures that all ranking and choice difficulties caused by positional methods can be completely analyzed with just the $\mathbf{p}_B$ and $\mathbf{p}_R$ portions of a profile. Because all positional methods agree on the Basic portion, all conflict in societal tallies, rankings, and choice must be attributed to the Reversal portion for non-BC positional procedures. Thus all of the perplexing three-candidate difficulties—problems central to choice theory—admit a simple yet complete analysis.

7.1. *Reversal Symmetry*

A useful way to think of elections is to pair voters with directly opposing opinions. Presumably, the votes from each pair define a tie which is then broken by the preferences of the remaining voters. For instance, if 20 voters prefer $A \succ B$ and 18 prefer $B \succ A$, then the tie created by the 18 pairs of opposing voters is broken by the last two who prefer A; i.e., with the earlier notation $\tau(A, B) = 2$.

"Neutrality" is where vote tallies change with the candidates' names; e.g., if all voters thought A was B and B was A, then the new outcome assigns the correct name to a tally—now B beats A by 20 to 18. Similarly, if each voter reverses his ranking, we might expect a reversed election outcome. The following definition extends this reversal behavior to any number of candidates. Let $\rho(r)$ be the reversal of ranking r and $\rho(\mathbf{p})$ the profile where each voter's ranking is reversed.

DEFINITION 3. A ranking procedure f satisfies *Reversal Symmetry* if $f(\rho(\mathbf{p})) = \rho(f(\mathbf{p}))$ for all profiles $\mathbf{p}$.

While Reversal Symmetry appears to be an innocuous, reasonable condition that should be expected from all election procedures, it probably is possible to find some argument against it. However, I adopt the pragmatic stance of emphasizing the central role of this condition in explaining all positional voting paradoxes and behavior.

To demonstrate this condition, suppose all voters erred by marking their ballots opposite to what was intended. To find the correct outcome, we might follow the lead of the neutrality example by reversing the election outcome. For instance, consider the profile

Number	Ranking	Number	Ranking	
5	$A \succ C \succ B$	5	$B \succ C \succ A$	(7.1)
3	$A \succ B \succ C$	3	$C \succ B \succ A$	

and its $A \succ B \succ C$ plurality ranking with tally $8:5:3$. When each voter reverses his ranking, it is reasonable to expect the new outcome to be the reversed $C \succ B \succ A$. It is not; the election outcome remains $A \succ B \succ C$ with the identical $8:5:3$ tally.

To explain, notice that all positional and pairwise procedures satisfy Reversal Symmetry on the $\mathbf{p}_B, \mathbf{p}_C, \mathbf{p}_K$ components; only the Reversal portion $\mathbf{p}_R$ remains. But $\rho(\mathbf{p}_R)=\mathbf{p}_R$; e.g., for $X=A, B, C$, the differential $\mathbf{R}_X$ remains the same after each voter reverses his ranking. This explains Table 7.1 because each voter's preferences are reversed by another voter; i.e., this example consists of Reversal terms; it is $\frac{1}{3}[5\mathbf{R}_A+2\mathbf{R}_B+8\mathbf{K}]$. The "votes with opposite preferences should cancel" belief suggests that the votes for each Table 7.1 pair should cancel causing the $A\sim B\sim C$ election outcome. This is true for the BC and pairwise votes, but, as asserted next, it fails for all other $\mathbf{w}_s^3$ choices.

So, to check whether a procedure satisfies Reversal Symmetry, we need only examine it with $\mathbf{p}_R$ components. If a procedure satisfies Reversal Symmetry on this subspace where $\rho(\mathbf{p}_R)=\mathbf{p}_R$, then it must be that $f(\mathbf{p}_R)=\rho(f(\mathbf{p}_R))$; that is, the outcome is a complete tie.

THEOREM 12. *The pairwise vote and the BC satisfies Reversal Symmetry. All positional procedures satisfy Reversal Symmetry on the Condorcet and Basic portions of a profile. On the Reversal profile* $\mathbf{R}_X$, $X=A, B, C$, *the* $\mathbf{w}_s^3$ *tally gives* $2(1-2s)$ *points to X and* $-(1-2s)$ *points to each of the other two candidates; only the BC gives a completely tied outcome.*

Proof. This is a simple computation. ▮

7.2. *Symmetry Breaking*

Theorem 12 captures the mathematical concept of "symmetry breaking." To illustrate, place a plastic stirring stick on a plane along the x-axis and then squeeze the ends inwards. Initially, when the stick remains straight, there is a rotation symmetry; i.e., rotating the plane about the endpoints defines the same configuration. With certain pressure, the stick "breaks" this symmetry by bending either upwards (the $y>0$ region) or downwards (the $y<0$ region). But rather than being "broken," the symmetry is transferred so that it now relates the two bending possibilities; i.e., rotating this plane about the endpoints (or multiplying the y coordinates by -1) maps each "bending" scenario into the other.

To see this symmetry breaking with positional voting, rewrite the Theorem 12 tallies for $\mathbf{R}_A$ as

$$(4(\tfrac{1}{2}-s),\ -2(\tfrac{1}{2}-s),\ -2(\tfrac{1}{2}-s)).$$

The tallies for the two s choices with a fixed $|\frac{1}{2}-s|$ value differ only by sign. So, multiplying by -1, a rotation, converts one tally into the other. For instance, the $\mathbf{w}_{1/4}^3$ tally of $\mathbf{R}_A$ is $(1, -\frac{1}{2}, -\frac{1}{2})$, so symmetry requires the $\mathbf{w}_{3/4}^3$ tally to be $-(1, -\frac{1}{2}, -\frac{1}{2})$. Because the $\mathbf{w}_0^3$ plurality tally of $\mathbf{R}_A$ is $(2, -1, -1)$, the $\mathbf{w}_1^3$ antiplurality tally is $-(2, -1, -1)$. This "Reversal symmetry breaking" explains the following result.

THEOREM 13 (Saari [27]). *Let ρ be the reversal of a ranking, and $\rho(\mathbf{p})$ be the profile which reverses each voter's preferences. The $\mathbf{w}_s^3$ ranking of profile $\mathbf{p}$ is the reversal of the $\mathbf{w}_{1-s}^3$ ranking of $\rho(\mathbf{p})$.*

Proof. This is a simple computation using the above. ∎

This theorem and the $\mathbf{w}_s^3$ and $\mathbf{w}_{1-s}^3$ symmetry relative to the BC simplifies our analysis of positional procedures by identifying the central symmetry role of $\mathbf{b}^3$.

7.3. *Cancellation of Opposites*

Reversal Symmetry is equivalent to accepting that the $\mathbf{p}_R$ components should have no effect on the societal ranking. When the effect of this term do cancel, only the Basic term remains where all positional procedures agree. Conversely, as the cancellation effect separates the BC from other positional methods (Theorems 3, 12), arguments countering this "cancellation of opposites" belief are required to justify positional procedures other than the BC. So, in spite of several axiomatic representations for different procedures, it turns out that *all differences causing all positional paradoxes, behavior, and axiomatic representations for all positional methods are strictly due to how a procedure handles the Reversal terms.* In turn, all explanations and support for non-BC procedures require justifying a rejection of the cancellation argument.

To reject the cancellation property, first determine whether the outcome for profile $\{A \succ B \succ C, C \succ B \succ A\}$ should be $A \sim C \succ B$ (to support $\mathbf{w}_s^3$ procedures where $s < \frac{1}{2}$) or $B \succ A \sim C$ (to support $\mathbf{w}_s^3$ procedures where $s > \frac{1}{2}$). Next determine the difference in the tallies; arguing for greater differences between the $A \sim C$ and B tallies supports larger $|s - \frac{1}{2}|$ values. This introduces a different, but simple and focussed way to compare positional methods.

7.4. *Differences in Outcomes*

As shown in Sects. 4–6, all pairwise voting difficulties represent the reliance of the pairwise vote on the Condorcet portion of a profile and its devaluation of the individual rationality assumption. Similarly, *the level to which a non-BC $\mathbf{w}_s^3$ procedure devalues Reversal Symmetry explains all differences in positional election outcomes.* Activating the $\mathbf{p}_R$ component is the $\mathbf{d}^3 = (0, 1, 0)$ portion of a voting vector (Theorem 2) which recognizes voters' second-ranked candidates.

To illustrate the Reversal profile with Eq. (2.5) and the Fig. 1 profile, Theorem 3 ensures that the $\mathbf{b}^3$ ranking of $B \succ A \succ C$ and (58, 64, 49) tally hold for all positional procedures on the $\mathbf{p}_B$ portion. All differences generating the *profile line* $(s-\frac{1}{2})(50, 50, 14)$ term of Eq. (2.5) come from the $\mathbf{p}_R$ portion. The different outcomes measure how much a $\mathbf{w}_s^3$ procedure violates Reversal Symmetry where a larger $|s-\frac{1}{2}|$ value creates a larger deviance from the Basic profile outcome; i.e., this magnitude determines the level a $\mathbf{w}_s^3$ procedure devalues Reversal Symmetry and its difference with the BC outcome. To show how $\mathbf{p}_R$ can change rankings, solving the obvious election inequalities proves that this Fig. 1 profile admits the following *five* different rankings.

s values	Ranking	s values	Ranking
$0 \leqslant s < \frac{1}{12}$	$C \succ B \succ A$	$s = \frac{1}{12}$	$C \sim B \succ A$
$\frac{1}{12} < s < \frac{1}{4}$	$B \succ C \succ A$	$s = \frac{1}{4}$	$B \succ C \sim A$
$\frac{1}{4} < s = 1$	$B \succ A \succ C$		

7.5. *Constructing Examples*

Something similar to the representation cube does not exist for positional voting because a four-dimensional space is needed to capture the $\mathbf{p}_B + \mathbf{p}_R$ consequences. (Both components are in two-dimensional spaces.) The approach developed here describes how $\mathbf{p}_R$ terms affect the outcome for a specified $\mathbf{p}_B$. To do this, plot the $\mathbf{p}_B$ outcome and then consider all ways the $\mathbf{p}_R$ term can alter this outcome with various $\mathbf{w}_s^3$ methods. As Theorem 3 asserts, this is captured by the procedure line which can be displayed in the representation triangle of Saari [27].

View the equilateral triangle (Fig. 6) as the election tallies normalized to unity; i.e., the set $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1,\ x_i \geqslant 0\}$. (The normalized version of the vector tally of (60, 10, 30) is (0.6, 0.1, 0.3).) The $\mathbf{p}_B$ (and BC) outcome defines the center of a coordinate system: it is the dot in Fig. 6. The dashed arrows indicate how each $\mathbf{R}_X$ term moves a $\mathbf{w}_s^3$, $0 \leqslant s < \frac{1}{2}$ outcome from the dot; i.e., adding $\mathbf{R}_A$ moves the plurality outcome in the indicated southwest direction from the BC outcome. The $\mathbf{w}_s^3$ effects for $\frac{1}{2} < s \leqslant 1$ go in the opposite direction (Theorem 12).

The *procedure line* starts from the plurality outcome of the $\mathbf{p}_B + \mathbf{p}_R$ differential, passes through the dot (the BC outcome) and ends half again as far away; this line, which captures all possible $\mathbf{w}_s^3$ outcomes, is illustrated by the slanted line in Fig. 6. (As differentials define *directions* rather than points, increasing the Reversal term moves the reference Basic point toward the point of complete indifference. Also, the representation triangle distorts relationships among $\mathbf{w}_s^3$ rankings as it uses the normalized voting vector $(w_1, w_2, 0)$ where $w_1 + w_2 = 1$ rather than $w_1 = 1$. See Saari [27] for more details.)

Figure 6 shows how to create examples exhibiting a host of different positional method behavior associated with the $A \succ B \succ C$ BC outcome.

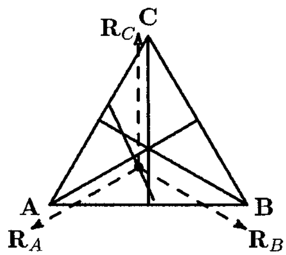

FIG. 6. Reversal behavior.

To change the plurality ranking to $B \succ C \succ A$, for instance, the dashed arrows require adding $\mathbf{R}_B$ and $\mathbf{R}_C$ components to $\mathbf{p}_B$. As the positional line passes through the $\mathbf{p}_B$ (or BC) outcome, the antiplurality outcome must be $A \succ B \succ C$. Each procedure line represents a profile, so the procedure line in Fig. 6 illustrates a profile where *each candidate* wins with appropriate $\mathbf{w}_s^3$ procedures.

I now show how to find all possible profiles defining each possible outcome. To do so, notice that the procedure line is determined by its endpoints (the plurality and antiplurality outcomes). So, this goal can be

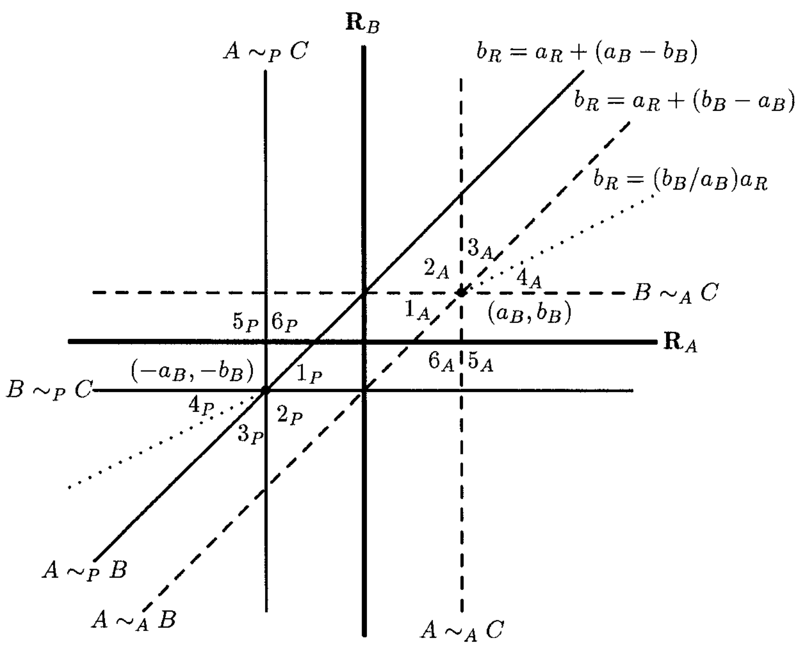

FIG. 7. Effects of Reversal profiles.

accomplished by determining the (a_R, b_R) coefficients of $\mathbf{p}_R = a_R\mathbf{R}_A + b_R\mathbf{R}_B$ which define different endpoint rankings. (Assume that $\mathbf{p}_B = a_B\mathbf{B}_A + b_B\mathbf{B}_B$ defines the Basic profile ranking $A \succ B \succ C$.) To do so, notice that the $\mathbf{p}_B + \mathbf{p}_R$ plurality tally is

$$(2a_B - b_B + 2a_R - b_R,\ 2b_B - a_B + 2b_R - a_R,\ -a_B - b_B - a_R - b_R).$$

To find where rankings change, set appropriate components equal and use algebra to determine all (a_R, b_R) values defining, say, a $A \sim_P B$ relative plurality indifference outcome. (These profiles cause a plurality ranking change.) The results are in Fig. 7 where the numbers with subscripts P indicate (a_R, b_R) regions (see Fig. 1A) defining associated plurality rankings; the solid lines depict profiles causing plurality indifference between the indicated pairs. Namely, *this portion of the figure displays all possible profiles which define all possible plurality outcomes which accompany the specified BC and Basic ranking.*

Similarly, by using the antiplurality tally

$$(2a_B - b_B - 2a_R + b_R,\ 2b_B - a_B - 2b_R + a_R,\ -a_B - b_B + a_R + b_R),$$

we can determine the (a_R, b_R) values causing all antiplurality outcomes; they are denoted with subscript A and the dashed lines denote antiplurality indifference between the indicated alternatives.

To illustrate a use of Fig. 7, notice that a point to the right of the vertical dashed line and between the solid and dashed slanted lines has 1_P and 3_A rankings; i.e., these coefficients identify profiles which define type 1 plurality and type 3 antiplurality outcomes. Thus, this outcome occurs if and only if $b_R > b_B$ (vertical dashed line condition) and $|a_R - b_R| < (a_B - b_B)$ (slanted lines condition). Next, use the end points to draw a positional line in Fig. 6. Independent of the values, the line crosses the type 2 region and two regions involving a tie. Thus, the profile defines five different $\mathbf{w}_s^3$ outcomes with the indicated plurality and antiplurality rankings.

To explain the Fig. 7 dotted lines, select a point in the 4_A region. The (a_R, b_R) coefficients, which define a $C \succ B \succ A$ antiplurality and $A \succ B \succ C$ plurality rankings, create ambiguity in the associated Fig. 6 procedure line as it can pass on either side of the complete indifference outcome. One situation has each candidate winning with an appropriate $\mathbf{w}_s^3$, while the other has each candidate losing with some $\mathbf{w}_s^3$ and only A and C are winners. The division, where the line passes through the indifference point, is given by the dotted line. (Just solve the algebraic equation requiring this condition.) As points in this 4_A region below the dotted line are closer to a 5_A outcome, these are the (a_R, b_R) values where each candidate wins with some $\mathbf{w}_s^3$.

Figures 6 and 7 do for positional methods what Fig. 5 does for pairwise rankings; they provide all possible positional properties along with all possible supporting profiles. While a theorem paralleling Theorem 6 describing all positional outcomes is possible, it involves so many possibilities that only a sampler is offered.

THEOREM 14. *Assume the Basic profile ranking for* $\mathbf{p}_B = a_B\mathbf{B}_A + b_B\mathbf{B}_B$ *is* $A \succ B \succ C$ *and that the Reversal profile is* $\mathbf{p}_R = a_R\mathbf{R}_A + b_B\mathbf{R}_B$. *A necessary and sufficient condition that a profile requires different plurality, BC, and antiplurality winners is that either* $b_R - a_R < b_B - a_B$ *and* $a_R < -a_B$, *or* $b_R - a_R > a_B - b_B$ *and* $a_R > a_B$.

A necessary and sufficient condition that a profile allows seven different $\mathbf{w}_s^3$ *rankings where each candidate is the winner with some s is that either the above conditions are satisfied, or* $b_B < b_R < (b_B/a_B)\, a_R$, *or* $(b_B/a_B)\, a_R < b_R < -b_B$.

A necessary and sufficient condition for a profile to have all $\mathbf{w}_s^3$ *procedures with the same* $A \succ B \succ C$ *ranking is that* $|b_R| < b_B$ *and* $|a_R - b_R| < (a_B - b_B)$.

Proof. For each identified procedures to have a different winner, the procedure line in Fig. 6A must pass through region 1 (for the BC outcome) with endpoints in regions 3 and 6. Similarly, the only other way all candidates can be winners is if the endpoints are in regions 1 and 4 where the procedure line is on the correct side of complete indifference. The stated conditions follow from Fig. 7. Finally, if all outcomes agree, then the plurality and antiplurality outcomes must have a type 1 ranking. Again, the conditions follow from Fig. 7. ∎

To use Theorem 2 to construct a profile where *A*, *B*, and *C* are, respectively, the BC, plurality, and antiplurality winners, select a point from the 6_P and 3_A region. With $a_B = 2, b_B = 1$, such a point is $a_R = 3, b_R = 5$ to define the profile differential $(-5, 9, -2, -9, 7, 0)$. Thus the associated profile (4, 18, 7, 0, 16, 9) has the desired properties.

7.6. *Combinations*

The profile decomposition is a powerful, easily used tool because the $\mathbf{p}_C$ and $\mathbf{p}_R$ portions, respectively, have no effect on the positional and the pairwise rankings. Consequently the effects of these profile portions can be separately considered. Thus, a recipe to create all possible examples is to start with a Basic profile $\mathbf{p}_B$; say, with the $A \succ B \succ C$ outcome. Use Fig. 5 to select a Condorcet coefficient to force the desired admissible pairwise rankings; use Figs. 6 and 7 to select the desired behavior for the positional methods.

To illustrate by designing a profile with the BC outcome $A \succ B \succ C$, Condorcet and antiplurality winner B, and plurality winner C, the Condorcet choice imposes (Theorem 6) a constraint on a_B and b_B; e.g., the $a_B = \frac{5}{3}$, $b_B = \frac{4}{3}$, $\gamma = -\frac{5}{3}$ choices suffice to define the profile differential (0, 2, −3, 0, −2, 3) with the desired pairwise behavior. To satisfy the positional requirements, (a_R, b_R) must come from the 3_P and 6_A regions of Fig. 7; e.g., $a_R = b_R = -2$ suffice. This defines the profile differential (2, −2, −1, 2, −6, 5), or profile (8, 4, 5, 8, 0, 11).

Results of the following type now are immediate.

THEOREM 15 (Saari [20]). *Choose any ranking of the three candidates and any rankings for the pairs. If $\mathbf{w}_s^3 \neq \mathbf{b}^3$, there is a profile where the pairwise and the $\mathbf{w}_s^3$ rankings are as described. Only the BC ranking must be related to the pairwise rankings.*

THEOREM 16. *Choose any two rankings of the three candidates. Next, choose a ranking procedure, say the pairwise vote, Kemeny's rule or the Copeland method, which depends upon the pairwise tallies, and choose a $\mathbf{w}_s^3 \neq \mathbf{b}^3$ method. There exists a profile where the first selected ranking is the outcome for the pairwise procedure and the second is the $\mathbf{w}_s^3$ outcome.*

8. CONVERSION OF PROFILES

It remains to convert profiles between the Eq. (2.1) and profile decomposition descriptions. To change a profile decomposition into the standard representation, use

$$\mathbf{p} = a_B \mathbf{B}_A + b_B \mathbf{B}_B + a_R \mathbf{R}_A + b_R \mathbf{R}_B + \gamma \mathbf{C}^3 + k\mathbf{K}. \tag{8.1}$$

Expressing Eq. (8.1) in a matrix representation $\mathbf{p} = \mathscr{A}(\mathbf{v})$, we have that matrix $T = \mathscr{A}^{-1}$ converts a standard profile $\mathbf{p}$ into its profile decomposition format.

$$T = \frac{1}{6} \begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{8.5}$$

The effects of profile $\mathbf{p}$ are determined by $T(\mathbf{p})$.

8.1. *Condorcet's Example*

To illustrate T, an historically important profile is Condorcet's $\mathbf{p} = (30, 1, 10, 1, 10, 29)$ intended to discredit the BC because $\mathbf{p}$ allows no positional procedure to elect the Condorcet winner. The first two terms of $T(\mathbf{p}) = \frac{1}{6}(68, 76, -28, -20, 19, 81)$ require a $B \succ A \succ C$ Basic ranking. The next two terms (equivalent to $a_R = 0, b_R = \frac{8}{6}, c_R = \frac{28}{6}$) capture a weak Reversal effect favoring B and C which alters only the antiplurality ranking to $A \sim B \succ C$. The real impact is the $\gamma \mathbf{C}^3$ coefficient $\gamma = \frac{19}{6}$ and its cyclic distortion which changes the Condorcet winner from B to A. As this cyclic effect reflects the loss of the assumption of individual rationality, rather than supporting the Condorcet winner, this profile identifies a flaw of Condorcet's procedure by demonstrating its susceptibility to the distorting $\mathbf{C}^3$ portion of a profile.

8.2. *Borda's Example*

Another historically important profile is $\mathbf{p} = (0, 5, 0, 3, 4, 0)$ used by Borda [4] to show that the pairwise and BC rankings can radically disagree with the plurality ranking. His example, which initiated the mathematical investigation of voting procedures, has the decomposition $T(\mathbf{p}) = \frac{1}{6}(-5, -4, 9, 6, -4, 12)$. The Basic portion $a_B = -\frac{5}{6}$, $b_B = -\frac{4}{6}$, $c_B = 0$ supports the $C \succ B \succ A$ election ranking. The effectiveness of the profile for Borda's purposes are the $a_R = \frac{9}{6}$, $b_R = 1$ terms indicating a strong Reversal effect favoring A and helping B to create the conflicting $A \succ B \succ C$ plurality ranking. The cyclic coefficient $\gamma = -\frac{4}{6}$, which favors B in the $\{A, B\}$ pairwise election, sharpens the pairwise conflict with the plurality ranking.

8.3. *Unanimity Profile*

An instructive example is the *unanimity profile* where all voters have the same $A \succ B \succ C$ preference. Intuition suggests that nothing surprising can occur, but this is not the case. The Basic terms of $T(\mathbf{p}) = \frac{1}{6}(2, 1, 0, -1, 1, 1)$ capture the $A \succ B \succ C$ ranking, but unexpected are the $\mathbf{p}_R$ and $\mathbf{p}_C$ terms. The Reversal terms $a_R = c_R = \frac{1}{6}$ capture the conflict between plurality $A \succ B \sim C$ outcome and the unanimity preference; it is caused because the plurality method fails Reversal Symmetry. While the pairwise outcomes agree with the unanimity ranking, the tallies fail to reflect A's distinct favored status. Compare this with the respective $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ Basic pairwise outcomes of $(\frac{2}{6}, -\frac{2}{6})$, $(\frac{2}{6}, -\frac{2}{6})$, $(\frac{4}{6}, -\frac{4}{6})$ which provide A an healthier spread over C than over B. This diminished respect for A in the usual election comes from the Condorcet coefficient $\gamma = \frac{1}{6}$ which introduces enough rotation in C's favor to reduce A's victory margin in their pairwise election. Cyclic effects even influence unanimity outcomes; only the BC captures the essence of the unanimity profile.

8.4. *Black's Method*

Black's single-peaked restriction [3] (where some candidate never is bottom-ranked) avoids cycles because his condition tempers (but does not eliminate) the $\mathbf{p}_C$ term. As the Condorcet term can change the transitive ranking (Fig. 5), *Black's restriction need not preserve the Basic portion's pairwise rankings*; *it still admits Condorcet effects*. Indeed, Black's condition is satisfied if six voters prefer $A \succ B \succ C$, six prefer $C \succ A \succ B$ and one prefers $B \succ A \succ C$. The profile decomposition $a_B = \frac{11}{6}$, $b_B = 0$, $c_B = \frac{5}{6}$ defines the $A \succ C \succ B$ Basic ranking which conflicts with the pairwise $A \succ B \succ C$ outcome caused by the $\mathbf{C}^3$ coefficient of $\gamma = \frac{11}{6}$.

8.5. *Strategic Behavior*

To conclude, the Gibbard–Satterthwaite Theorem asserts that with more than two alternatives, a non-dictatorial procedure admits settings which can be manipulated. But, what are these settings? When is a procedure susceptible to strategic behavior? (A technique which characterizes all such settings for any specified procedure is developed in Saari [27] and then applied to the Copeland method (Merlin and Saari [14]).) With the plurality vote, for instance, if over half of the voters prefer a particular candidate, then the system is free from successful manipulation. B. Grofman and E. Niou called my attention to a reasonable conjecture that the plurality system remains free from strategic action when the plurality winner also is the Condorcet winner.

The reason this conjecture is false is that $\mathbf{p}_R$ and $\mathbf{p}_C$ have independent affects on the plurality and pairwise outcomes. So, to construct counterexamples, add appropriate $\mathbf{p}_R$ terms to a Basic profile to barely satisfy the plurality ranking condition, and a $\mathbf{p}_C$ term to make A the Condorcet winner. Add an appropriate Kernel term to have enough voters to create manipulative opportunities. For instance, the 14-voter profile $(2, 0, 0, 0, 1, 0) + 2\mathbf{K}$ has A as the (one-vote) plurality and Condorcet winners. However, the two $C \succ B \succ A$ voters prefer B to A, so voting for B instead of C strategically changes the plurality outcome to B.

The $\mathbf{p} = (2, 0, 0, 0, 1, 0)$ decomposition $\frac{1}{6}[3\{\mathbf{B}_A + \mathbf{B}_B\} + \{\mathbf{R}_A - \mathbf{R}_B\} + 3\mathbf{C}^3 + 3\mathbf{K}]$ illustrates the description. The $\mathbf{p}_B$ term requires the BC and Basic $A \sim B \succ C$ outcome. But the $\mathbf{p}_R$ term breaks this tie to make A a one vote plurality winner over B while the $\mathbf{p}_C$ term ensures A is the Condorcet winner. The strategic behavior keeps A as the Condorcet winner; but it acts on $2\mathbf{K}$ (not $\mathbf{p}$) to generate $\frac{1}{6}[\{2\mathbf{B}_A + 4\mathbf{B}_B\} + \{2\mathbf{R}_A + 4\mathbf{R}_B\}]$ terms making B the new plurality and BC winner.

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