OPTION PRICING UNDER FAST-VARYING LONG-MEMORY STOCHASTIC VOLATILITY

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Abstract. Recent empirical studies suggest that the volatility of an underlying price process may have correlations that decay slowly under certain market conditions. In this paper, the volatility is modeled as a stationary process with long-range correlation properties to capture such a situation and we consider European option pricing. This means that the volatility process is neither a Markov process nor a martingale. However, by exploiting the fact that the price process still is a semimartingale and accordingly using the martingale method, one can get an analytical expression for the option price in the regime where the volatility process is fast mean reverting. The volatility process is modeled as a smooth and bounded function of a fractional Ornstein Uhlenbeck process and we give the expression for the implied volatility which has a fractional term structure.

Key words. Stochastic volatility, Long-range correlation, Mean reversion, Fractional Ornstein-Uhlenbeck process.

AMS subject classifications. 91G80, 60H10, 60G22, 60K37.

1. Introduction.

Stochastic Volatility and the Implied Surface. Under many market scenarios the assumption that the volatility is constant, as in the standard Black-Scholes model, is not realistic. Practically, this reflects itself in an implied volatility that depends on the pricing parameters. This means that, in order to match observed prices, the volatility one needs to use in the Black-Scholes option pricing formula depends on time to maturity and log moneyness, with moneyness being the strike price over the current price of the underlying. The implied volatility is a convenient way to parameterize the price of a financial contract relative to a particular underlying. It gives insight about how the market deviates from the ideal Black-Scholes situation and, after calibration of an implied volatility model to liquid contracts, it can be used for pricing less liquid contracts written on the same underlying. It is therefore of interest to identify a consistent parameterization of the implied volatility that corresponds to an underlying model for stochastic volatility fluctuations. As in Garnier and Solna

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(2015) a main objective of our modeling is to construct a time-consistent scheme so that indeed the volatility model is chosen as a stationary process and we consider general times to maturity. For background on stochastic volatility models we refer to the books and surveys by Fouque et al. (2011); Gatheral (2006); Ghysels et al. (1995); Gulisashvili (2012); Henry-Labordére (2009); Rebonato (2004) (see the references therein). We also refer to our recent paper on fractional stochastic volatility Garnier and Solna (2015) for further references on the recent literature on the class of volatility models we consider here.

Empirical studies suggest that volatility may exhibit a "multi scale" character with long-range correlations as in Bollerslev et al. (2013); Breidt et al. (1998); Chronopoulou and Viens (2012); Cont (2001, 2005); Engle and Patton (2001); Oh et al. (2008). That is, correlations that decay as a power law in offset rather than as an exponential function as in a Markov process. Here we seek to identify parametric forms for the implied volatility consistent with such long-range correlations. In our recent paper Garnier and Solna (2015) we considered this question in the context where the magnitude of the volatility fluctuations is small. Here, we consider the situation where the magnitude of the volatility fluctuations is of the same order as the mean volatility. Indeed empirical studies show that the volatility fluctuations may be quite large: Breidt et al. (1998); Cont (2001); Engle and Patton (2001). While in Garnier and Solna (2015) the volatility fluctuations were small leading to a (regular) perturbative situation, here the situation is different in that it is the fast mean reversion (fast relative to the diffusion time of the underlying) that allows us to push through an asymptotic analysis. However, the presence of long-range correlations in this context gives a novel singular perturbation situation. The analysis becomes significantly more involved. In particular the detailed analysis of the covariation process is an important ingredient. We consider here option pricing, but the approach set forth is general and will be useful in other financial contexts as well.

It follows from our analysis that the form for the implied volatility surface is similar as in the Markovian case. This confirms the robustness of the implied volatility parametric model with respect to the underlying price dynamics. There are, however, central differences. In particular the long-range correlations produce a volatility covariance that is not integrable which in turn gives an implied volatility surface that is a *random field*, whose statistics can be described in detail. Moreover, in the long-range case the implied volatility has a fractional behavior as a function of time to maturity. The empirical study in Fouque et al. (2003) shows that, in order to fit well the implied volatility, it is appropriate to consider a two-time scale model with one slow and one fast volatility factor. In Garnier and Solna (2015) we considered a slow factor, which closely associates with a small fluctuations factor. Here, we consider a fast factor with large fluctuations. Taken together we have a generalization of the two-factor model of Fouque et al. (2003, 2011) to the case of processes with long-range correlations. This leads to a fractional term structure of the implied volatility and it was shown in Fouque et al. (2004) that such a term structure may be useful to fit the implied volatility under certain market conditions.

Long Memory and Fast Mean Reversion. As mentioned above the asymptotic regime considered in this paper is the situation where the volatility is fast mean reverting. We denote its time scale by ε and this is the small parameter in our model. The volatility then decorrelates on the time scale ε.

Stochastic volatility models are most often posed with a volatility driving process that has mean zero and mixing properties. This means that the random values of the volatility driving process at times t and $t+\Delta t$, that are Z_t^ε and $Z_{t+\Delta t}^\varepsilon$, become rapidly uncorrelated when $\Delta t \to \infty$, i.e., the autocovariance function $\mathcal{C}^\varepsilon(\Delta t) = \mathbb{E}[Z_t^\varepsilon Z_{t+\Delta t}^\varepsilon]$ decays rapidly to zero as $\Delta t \to \infty$. More precisely we say that the volatility driving process is mixing if its autocovariance function decays fast enough at infinity so that it is absolutely integrable:

$$\int_0^\infty |\mathcal{C}^\varepsilon(t)| dt < \infty \,. \tag{1.1}$$

In this case we may associate the process with the finite correlation time $t_c = 2\int_0^\infty \mathcal{C}^\varepsilon(t)dt/\mathcal{C}^\varepsilon(0)$, which is of order ε.

Stochastic volatility models with long-range correlation properties have recently attracted a lot of attention, as more and more data collected under various situations confirm that this situation can be encountered in many different markets. Qualitatively, the long-range correlation property means that the random process has long memory (in contrast with a mixing process). This means that the correlation between the random values Z_t^ε and $Z_{t+\Delta t}^\varepsilon$ taken at two times separated by Δt is not completely negligible even for large Δt. More precisely we say that the random process Z_t^ε has the H-long-range correlation property if its autocovariance function satisfies:

$$\mathcal{C}^{\varepsilon}(t) \overset{|t|\to\infty}{\simeq} r_H \left|\frac{t}{\varepsilon}\right|^{2H-2}, \tag{1.2}$$

where $r_H > 0$ and $H \in (1/2, 1)$. We refer to H as the Hurst exponent. Here the correlation time ε is the critical time scale beyond which the power law behavior (1.2) is valid. Note that the autocovariance function is not integrable as $2H - 2 \in (-1, 0)$, which means that a random process with the H-long-range correlation property is not mixing. As we describe in more detail below a common approach for modeling long-range dependence is via using fractional Brownian motion (fBm) processes as introduced in Mandelbrot and Van Ness (1968).

Long-memory stochastic volatility models are indeed easy to pose, however, their analysis is quite challenging. This is largely due to the fact that the volatility process is then neither a Markov process nor a semimartingale. It is, however, important to notice that the price process is still a semimartingale and the problem formulation does not entail arbitrage (Mendes et al. (2015)), as has been argued for some models whose price process itself is driven by fractional processes as in Bjork and Hult (2005); Rogers (1997); Shiryaev (1998). A main motivation for long-memory is to be able to fit observed implied volatilities. One classic challenge regarding fitting of implied volatility surfaces is to capture a strong moneyness dependence for short time to maturity without creating artificial behavior for long time to maturity. Another one is to retain a strong parametric dependence for long maturities despite averaging effects that occur in this regime, as discussed in Bollerslev and Mikkelsen (1999); Bollerslev et al. (2013); Comte et al. (2012); Sundarsen et al. (2000). We remark that models involving jumps have been promoted as one approach to meet these challenges by Carr and Wu (2003); Mijatovic and Tankov (2016). Recent works show that stochastic volatility models with long-range dependence also provide a promising framework for meeting such challenges. Approaches based on using fractional noises in the description of the stochastic volatility process were used by Comte and Renault (1998); Comte et al. (2012). This provides an approach for endowing the volatility process with high persistence in the long run (long memory with $H > 1/2$) in order to capture the steepness of long term volatility smiles without overemphasizing the short run persistence. In order to get explicit results for the implied volatility a number of asymptotic regimes have been considered. Chief among them has been the regime of short time to maturity. The model presented in Comte et al. (2012) was recently revisited in Guennoun et al. (2014) where short and long time to maturity asymptotics are analyzed using large deviations principles. In Alòs et al. (2007) the authors use Malliavin calculus to decompose option prices as the sum of the classic Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due to the leverage effect (i.e., the correlation between the underlying return and its changes of volatility) and a term due to the volatility of the volatility. Their model is a fractional version of the Bates model (Bates (1996)). They find that the implied volatility flattens in the long-range dependent case in the limit of small time to maturity. In Forde and Zhang (2015) the authors use large deviation principles to compute the short time to maturity asymptotic form of the implied volatility. They consider the leverage effect and obtain results that are consistent with those in Alòs et al. (2007). They consider stochastic volatility models driven by fBms which are analyzed using rough path theory. They also consider large time asymptotics for some fractional processes. Small time to maturity asymptotic results were recently also presented in Gulisashvili et al. (2015) in a context of long-range processes. In Fukasawa (2011) the author discusses the asymptotic regime with small volatility fluctuations and long-range dependence impact on the implied volatility as an application of the general theory he sets forth. In this paper as well as in Alòs et al. (2007) the authors use a modeling where the time 0 plays a special role and hence the modeling is not completely satisfactory because it leads to a non-stationary volatility model. This is also the case in Bayer et al. (2016) where the authors consider the so-called rough Bergomi, or "rBergomi", model. In this paper and in Garnier and Solna (2015) which deals with small volatility fluctuations we use a formulation with a stationary model. This is also the case in the recent paper by Fukasawa (2017) which considers small time asymptotics in the rough volatility case, with $H < 1/2$. This distinction is important: If the volatility factor is a fBm emanating from the origin, then the implied volatility surface is identified conditioned on the present value of the volatility factor only. Below with a stationary model the implied volatility surface depends on the path of the volatility factor until the present, reflecting the non-Markovian nature of fBm. We discuss in detail in Section 6 the consequences of this for the interpretation of the implied volatlity surface as a random field. Recently, pricing approximations in the regime of small fractional volatility fluctuations were presented in Alòs and Yang (2017). In terms of computation of prices for general maturities and fractional volatility fluctuations, so far mainly numerical approximations have been available. However, here we present an asymptotic regime based on fast mean reversion which gives explicit price approximations in this context. Taken together the results of Garnier and Solna (2015) and this paper allow to construct a fractional two-time scale stochastic volatility model and flexibility to fit both the short and long time to maturity parts of the implied volatility surface.

We remark that we here consider the case with $H > 1/2$ and long-range correlation only as opposed to the case with rough volatility and $H < 1/2$ corresponding to sharp decay of the correlations at the origin. Indeed both regimes have been identified from the empirical perspective. We refer to for instance Gatheral et al. (2016) for observations of rough volatility, while in Chronopoulou and Viens (2012) cases of long-range volatility are reported. A persistent or long-range mean reverting volatility situation is reported in Jensen (2016) in a discrete modeling framework. Long-range volatility situations are also reported for currencies in Walther et al. (2017), for commodities in Charfeddine (2014), and for equity index in Chia et al. (2015), while analysis of electricity markets data typically gives $H < 1/2$ as in Simonsen (2002); Rypdal and Lovsletten (2013); Bennedsen (2015). We believe that both the rough and the long-range cases are important and can be seen depending on the specific market and regime. Even though the "rough" case with $H < 1/2$ may be the most common situation, it may be of particular importance to understand the situation where $H > 1/2$ and the ramification of this for pricing and hedging. In this paper we only consider the analytic aspects of our model. The fitting with respect to specific data is beyond the scope of this paper and will be presented elsewhere.

The fractional model we set forth here produces typical "stylized facts", like heavy tails of returns, volatility clustering, mean reversion, and long memory or volatility persistence. Additionally, we here incorporate the leverage effect. A term coined by Black et al. (1976) referring to stock price movements which are correlated (typically negatively) with volatility, as falling stock prices may imply more uncertainty and hence volatility. Note, however, that the model for the implied volatility surface derived below is linear in log moneyness. This may seem somewhat restrictive from the point of view of fitting because in many cases a strong skew in log moneyness may be observed in certain markets. This has particularly been the case for stock markets, but relatively less so in other markets like fixed income markets. However, if one considers higher order approximations, then this generates also skew effects. A number of modeling issues not considered here, like transaction costs, bid-ask spreads and liquidity, may also affect the skew shape. Note also that for simplicity we do not incorporate a non-zero interest rate, nor do we incorporate market price of risk aspects.

Rapid-Clustering, Long-Memory and the Implied Surface. We summarize next the main result of the paper from the point of view of calibration. That is, the form of the implied volatility surface in the context of a stochastic volatility modeled by a fast process with long-range correlation properties. We summarize first some aspects of the modeling.

We consider a continuous time stochastic volatility model that is a smooth function of a Gaussian long-range process. Explicitly, we model the fractional stochastic volatility (fSV) as a smooth function of a fractional Ornstein-Uhlenbeck (fOU) process. The fOU process is a classic model for a stationary process with a fractional long-range correlation structure. This process can be expressed in terms of an integral of a fractional Brownian motion (fBm) process. The distribution of a fBm process is characterized in terms of the Hurst exponent $H \in (0,1)$. The fBm process is locally Hölder continuous of exponent H' for all $H' < H$ and this property is inherited by the fOU process. The fBm process, W_t^H, is also self-similar in that

$$\left\{W_{\alpha t}^H, t \in \mathbb{R}\right\} \overset{dist.}{=} \left\{\alpha^H W_t^H, t \in \mathbb{R}\right\} \text{ for all } \alpha > 0. \tag{1.3}$$

The self-similarity property is inherited approximately by the fOU process on scales smaller than the mean reversion time of the fOU process which we denote by ε below. In this sense we may refer to the fOU process as a multiscale process on short scales. The case $H \in (1/2, 1)$ that we address in this paper gives a fOU process that is a long-range process. This regime corresponds to a persistent process where consecutive increments of the fBm are positively correlated. The stronger positive correlation for the consecutive increments of the associated fBm process with increasing H values gives a smoother process whose autocovariance function decay slowly. For more details regarding the fBm and fOU processes we refer repectively to Biagini et al. (2008); Coutin (2007); Doukhan et al. (2003); Mandelbrot and Van Ness (1968) and Cheridito et al. (2003); Kaarakka and Salminen (2011).

The volatility driving process is the ε-scaled fractional Ornstein-Uhlenbeck process (fOU) defined by:

$$Z_t^\varepsilon = \varepsilon^{-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H. \tag{1.4}$$

It is a zero-mean, stationary Gaussian process, that exhibits long-range correlations for the Hurst exponent $H \in (1/2, 1)$. It is important to note that this is a process whose "natural time scale" is ε, this in the sense that the mean reversion time or time before the process reaches its equilibrium distribution scales like ε. It is also important to note that the decay of the correlations (on the ε time scale) is polynomial rather than exponential as in the standard Ornstein-Uhlenbeck process. Explicitly, the correlation of the process between times t and $t + \Delta t$ decays as $(\Delta t/\varepsilon)^{2H-2}$, while the variance of the process is independent of ε.

In this paper we consider a stochastic volatility model that is a smooth function of the rapidly varying fractional Ornstein-Uhlenbeck process with Hurst coefficient $H \in (1/2, 1)$, it is given by

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{1.5}$$

where F is a smooth, positive, one-to-one, bounded function with bounded derivatives and with an additional technical condition that is given in Eq. (3.5). The process σ_t^ε inherits the long-range correlation properties of the fOU Z_t^ε.

The main result we set forth in Section 5 is an expression for the implied volatility of the European Call Option for strike K, maturity T, and current time t:

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds \big| \mathcal{F}_t\Big]^{1/2} + \overline{\sigma} a_F \left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right)\right]. \tag{1.6}$$

Here

$$a_F = \varepsilon^{1-H} \frac{\widetilde{\sigma}\sigma_{\mathrm{ou}}\rho \left\langle FF' \right\rangle \bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+3/2)}, \tag{1.7}$$

$\tau = T - t$ is time to maturity, ρ the correlation between the Brownian motion driving the fBM and the Brownian motion driving the underlying, and

$$\bar{\tau} = \frac{2}{\overline{\sigma}^2} \tag{1.8}$$

is the characteristic diffusion time. Furthermore, we have with $\sigma_{\mathrm{ou}}^2 = 1/(2\sin(\pi H))$:

$$\begin{aligned}
\overline{\sigma}^2 = \left\langle F^2 \right\rangle &= \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z)^2 p(z) dz, \\
\widetilde{\sigma} = \left\langle F \right\rangle &= \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z) p(z) dz, \\
\left\langle FF' \right\rangle &= \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z) F'(\sigma_{\mathrm{ou}} z) p(z) dz,
\end{aligned}$$

with $p(z)$ the pdf of the standard normal distribution. That is, we form moments of the volatility function averaged with respect to the invariant distribution of the fOU process Z_t^ε.

The first term in Eq. (1.6) is indeed the expected effective volatility until maturity conditioned on the present. The second term is a skewness term that is non-zero only when the volatility process and the underlying are correlated so that ρ is non-zero. Note that the exponent of the fractional term structure depends on the Hurst exponent which determines the smoothness and the decorrelation rate of the volatility driving process Z_t^ε. The smoother the process the larger the implied volatility for large times to maturity.

In the fast case presented here with large and fast volatility fluctuations the implied volatility explodes in the regime of short time to maturity. Indeed, short time to maturity means time to maturity smaller than the diffusion time (1.8), but larger than the mean reversion time ε. Therefore short time to maturity involves large volatility fluctuations over a short maturity horizon resulting in a moneyness correction that explodes and dominates the pure maturity term. In the context of short or long times to maturity the conditional expected effective volatility gives a small contribution and we have for short times to maturity and $K \neq X_t$:

$$I_t \sim a_F \Big[\Big(\frac{\tau}{\bar{\tau}} \Big)^{H-3/2} \log \Big(\frac{K}{X_t} \Big) \Big], \tag{1.9}$$

and respectively in the regime of long times to maturity:

$$I_t \sim a_F \Big(\frac{\tau}{\bar{\tau}} \Big)^{H-1/2}. \tag{1.10}$$

We remark here that the fractional scaling in the skewness term in Eq. (1.6) is exactly the fractional scaling that corresponds to the case of large time to maturity and small volatility fluctuations given in Garnier and Solna (2015). That is, with large times to maturity there we have a situation reminiscent of the one we have here with rapid volatility fluctuations, however, here the volatility fluctuations are large as compared to the small volatility fluctuations in Garnier and Solna (2015).

We remark also that the case with a mixing volatility, and hence integrable correlation function for the volatility fluctuations, would correspond to $H \searrow 1/2$. Note, however, that our derivation is valid only for $H \in (1/2, 1)$. If we consider the formula (4.10) for σ_ϕ that determines the variance of the first term in Eq. (1.6), we can observe that it vanishes when $H \searrow 1/2$, which shows that the first term in Eq. (1.6) becomes to leading order deterministic. In the mixing case the implied volatility is deterministic to leading correction order, while the non-integrability of the volatility covariance function makes it a stochastic process in the general long-range case with a variance that goes to zero as $H \searrow 1/2$. Indeed in the limit case $H \searrow 1/2$ we get a result as in (Fouque et al. , 2000, Section 5.2.5) that deals with the mixing case. Explicitly, consider the mixing case where the volatility driving process is an ordinary Ornstein-Uhlenbeck process, moreover, the interest rate and market price of volatility risk are zero as we consider here. Then (Fouque et al. , 2000, Eq. (5.55)) gives the implied volatility in terms of a coefficient V_3 defined in (Fouque et al. , 2000, Section 5.2.5):

$$I_t = \overline{\sigma} - V_3 \left[\frac{1}{2\overline{\sigma}} + \frac{1}{\overline{\sigma}^3 \tau} \log\left(\frac{K}{X_t} \right) \right], \tag{1.11}$$

that has the same form as the formal limit of (1.6) as $H \searrow 1/2$. However the averaging expression giving the coefficient V_3 does not correspond to the interpretation we arrive at here by the formal limit $H \searrow 1/2$. This is because the singular perturbation situation we consider in fact is "singular" at $H = 1/2$ and ordering of important terms becomes different. Nevertheless it is important from the calibration point of view that we have continuity of the implied volatility parameterization and its form at $H = 1/2$, providing robustness to the asymptotic framework.

In Section 6 we give the complete statistical description of the stochastic correction coefficient which determines the random component of the price correction and the implied volatility (the first term in Eq. (1.6)). It is a random function of the maturity T and the current time t with Gaussian statistics and with a covariance function that we describe in detail. This covariance function has interesting and non-trivial self-similar properties and it is important in order to construct and characterize estimators of the implied volatility surface.

Outline. The outline of the paper is as follows. In Section 2 we describe the fractional Ornstein-Uhlenbeck process and derive some fundamental a priori bounds. In Section 3 we describe the stochastic volatility model. In Section 4 we derive the expression for the price in the fast mean reverting fractional case. The derivation is based on the martingale method. That is, we make an ansatz for the price as a process that has the correct payoff and to leading order is a martingale. Then indeed this process is the leading order expression for the price with an error that is of the order of the non-martingale part. This approach involves introducing correctors so that the non-martingale part is pushed to a small term and we give the resulting decomposition in Section 4. Based on the expression for the price we derive the associated implied volatility in Section 5 and present finally some concluding remarks in Section 7. We give a convenient Hermite decomposition of the volatility in Appendix A. A number of the technical lemmas are proved in Appendix B.

2. The Rapid Fractional Ornstein-Uhlenbeck Process. We use a rapid fractional Ornstein-Uhlenbeck (fOU) process as the volatility factor and describe here how this process can be represented in terms of a fractional Brownian motion. Since fractional Brownian motion can be expressed in terms of ordinary Brownian motion we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \tag{2.1}$$

where σ_H is a positive constant. We use the following moving-average stochastic integral representation of the fBM (Mandelbrot and Van Ness (1968)):

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \tag{2.2}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion over $\mathbb{R}$. Then indeed $(W_t^H)_{t\in\mathbb{R}}$ is a zero-mean Gaussian process with the covariance (2.1) and where we now have

$$\begin{aligned}\sigma_H^2 &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\Big[\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\Big]\\ &= \frac{1}{\Gamma(2H+1)\sin(\pi H)}.\end{aligned} \tag{2.3}$$

We introduce the ε-scaled fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^\varepsilon = \varepsilon^{-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H = \varepsilon^{-H} W_t^H - \varepsilon^{-1-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} W_s^H ds. \tag{2.4}$$

Thus, the fractional OU process is in fact a fractional Brownian motion with a restoring force towards zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_{\rm ou}^2, \text{ with } \sigma_{\rm ou}^2 = \frac{1}{2}\Gamma(2H+1)\sigma_H^2 = \frac{1}{2\sin(\pi H)}, \tag{2.5}$$

that is independent of ε, and covariance:

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_{\rm ou}^2 \mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big),$$

that is a function of s/ε only, with

$$\begin{aligned}\mathcal{C}_Z(s) &= \frac{1}{\Gamma(2H+1)}\Big[\frac{1}{2}\int_{\mathbb{R}} e^{-|v|}|s+v|^{2H}dv - |s|^{2H}\Big]\\ &= \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(sx)\frac{x^{1-2H}}{1+x^2}dx.\end{aligned}$$

This shows that ε is the natural scale of variation of the fOU Z_t^ε. Note that the random process Z_t^ε is not a martingale, neither a Markov process. For $H \in (1/2, 1)$ it possesses long-range correlation properties:

$$\mathcal{C}_Z(s) = \frac{1}{\Gamma(2H-1)}s^{2H-2} + o\big(s^{2H-2}\big), \qquad s \gg 1. \tag{2.6}$$

This shows that the correlation function is non-integrable at infinity. In this paper we focus on the case $H \in (1/2, 1)$.

We remark that if $H = 1/2$, then the standard OU process (synthesized with a standard Brownian motion) is a stationary Gaussian Markov process with an exponential correlation and hence a mixing process. It is possible to simulate paths of the fractional OU process using the Cholesky method (see Figure 2.1) or other well-known methods described in Omre et al. (1993); Bardet et al. (2003).

Using Eqs. (2.2) and (2.4) we arrive at the moving-average integral representation of the scaled fOU as:

$$Z_t^\varepsilon = \sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(t-s)dW_s, \tag{2.7}$$

where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}\Big(\frac{t}{\varepsilon}\Big), \qquad \mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}}e^{-s}ds\Big]. \tag{2.8}$$

The main properties of the kernel $\mathcal{K}$ in our context are the following ones (valid for any $H \in (1/2, 1)$):
- $\mathcal{K}$ is nonnegative-valued, $\mathcal{K} \in L^2(0,\infty)$ with $\int_0^\infty \mathcal{K}^2(u)du = 1$, but $\mathcal{K} \notin L^1(0,\infty)$.
- for small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big(t^{H-\frac{1}{2}} + O\big(t^{H+\frac{1}{2}}\big)\Big), \tag{2.9}$$

- for large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\Big(t^{H-\frac{3}{2}} + O\big(t^{H-\frac{5}{2}}\big)\Big), \tag{2.10}$$

and in particular $\mathcal{K}(t) - \frac{1}{\Gamma(H-\frac{1}{2})} t^{H-\frac{3}{2}} \in L^1(0,\infty)$.

3. The Stochastic Volatility Model. The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*, \tag{3.1}$$

where the stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{3.2}$$

and with Z_t^ε being the scaled fOU introduced in the previous section which is adapted to the Brownian motion W_t. Moreover, W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t, \tag{3.3}$$

where the Brownian motion B_t is independent of W_t.

The function F is assumed to be one-to-one, positive-valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration $\mathcal{F}_t$ generated by (B_t, W_t) is also the one generated by X_t. Indeed, it is equivalent to the one generated by (W_t^*, W_t), or (W_t^*, Z_t^ε). Since F is one-to-one, it is equivalent to the one generated by (W_t^*, σ_t). Since F is positive-valued, it is equivalent to the one generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or X_t.

We denote the Hermite coefficients of the volatility function F with respect to the invariant distribution of the fOU process by C_k :

$$C_k = \int_{\mathbb{R}} H_k(z) F^2(\sigma_{\rm ou} z) p(z) dz, \qquad H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}, \tag{3.4}$$

with $p(z) = \exp(-z^2/2)/\sqrt{2\pi}$. We use these in Appendix A to derive some technical lemmas. Indeed, for a technical reason, we require that F satisfies the following condition: there exists some $\alpha > 2$ such that

$$\sum_{k=0}^\infty \frac{\alpha^k C_k^2}{k!} < \infty. \tag{3.5}$$

As we have discussed above, the volatility driving process Z_t^ε possesses long-range correlation properties. As we now show the volatility process σ_t^ε itself inherits this property.

LEMMA 3.1. *We denote, for $j = 1, 2$:*

$$\left\langle F^j \right\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^j p(z) dz, \qquad \left\langle F'^j \right\rangle = \int_{\mathbb{R}} F'(\sigma_{\rm ou} z)^j p(z) dz, \tag{3.6}$$

where $p(z)$ is the pdf of the standard normal distribution.

1. *The process σ_t^ε is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] = \langle F \rangle$ and variance ${\rm Var}(\sigma_t^\varepsilon) = \left\langle F^2 \right\rangle - \langle F \rangle^2$, independently of ε.*
2. *The covariance function of the process σ_t^ε is of the form*

$$\mathrm{Cov}\big(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon\big) = \big(\left\langle F^2 \right\rangle - \langle F \rangle^2 \big) \mathcal{C}_\sigma\Big(\frac{s}{\varepsilon}\Big), \tag{3.7}$$

where the correlation function $\mathcal{C}_\sigma$ satisfies $\mathcal{C}_\sigma(0) = 1$ and

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_{\rm ou}^2 \langle F' \rangle^2}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H-2} + o\big(s^{2H-2}\big), \qquad \textit{for } s \gg 1. \tag{3.8}$$

Consequently, the process σ_t^ε possesses long-range correlation properties (i.e. its correlation function is not integrable at infinity).

Proof. The fact that σ_t^ε is a stationary random process with mean $\langle F \rangle$ is straightforward in view of the definition (3.2) of σ_t^ε.

For any t, s, the vector $\sigma_{\rm ou}^{-1}(Z_t^\varepsilon, Z_{t+s}^\varepsilon)$ is a Gaussian random vector with mean $(0, 0)$ and 2×2 covariance matrix:

$$\mathbf{C}^\varepsilon = \begin{pmatrix} 1 & \mathcal{C}_Z(s/\varepsilon) \\ \mathcal{C}_Z(s/\varepsilon) & 1 \end{pmatrix}.$$

Therefore, denoting $F_c(z) = F(\sigma_{\rm ou} z) - \langle F \rangle$, the covariance function of the process σ_t^ε is

$$\begin{aligned} \mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) &= \mathbb{E}\big[F_c(\sigma_{\rm ou}^{-1} Z_t^\varepsilon) F_c(\sigma_{\rm ou}^{-1} Z_{t+s}^\varepsilon)\big] \\ &= \frac{1}{2\pi \sqrt{\det \mathbf{C}^\varepsilon}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big(- \frac{(z_1, z_2) {\mathbf{C}^\varepsilon}^{-1} (z_1, z_2)^T}{2} \Big) dz_1 dz_2 \\ &= \Psi\Big(\mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big)\Big), \end{aligned}$$

with

$$\Psi(C) = \frac{1}{2\pi\sqrt{1 - C^2}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big(- \frac{z_1^2 + z_2^2 - 2C z_1 z_2}{2(1 - C^2)} \Big) dz_1 dz_2 \,.$$

This shows that $\mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon)$ is a function of s/ε only. Moreover, the function Ψ can be expanded in powers of C for small C:

$$\begin{aligned}\Psi(C) = &\frac{1}{2\pi}\iint_{\mathbb{R}^2} F_c(z_1)F_c(z_2)\exp\Big(-\frac{z_1^2+z_2^2}{2}\Big)dz_1dz_2 \\ &+C\frac{1}{2\pi}\iint_{\mathbb{R}^2} z_1z_2F_c(z_1)F_c(z_2)\exp\Big(-\frac{z_1^2+z_2^2}{2}\Big)dz_1dz_2 + O(C^2), \qquad C \ll 1,\end{aligned}$$

which gives with (2.6) the form (3.8) of the correlation function for σ_t^ε. $\square$

4. The Option Price. We aim at computing the option price defined as the martingale

$$M_t = \mathbb{E}\big[h(X_T)|\mathcal{F}_t\big], \tag{4.1}$$

where h is a smooth function. In fact weaker assumptions are possible for h, as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h.

We introduce the operator

$$\mathcal{L}_{\mathrm{BS}}(\sigma) = \partial_t + \frac{1}{2}\sigma^2x^2\partial_x^2, \tag{4.2}$$

that is, the standard Black-Scholes operator at zero interest rate and (constant) volatility σ.

We next exploit the fact that the price process is a martingale to obtain an approximation, via constructing an explicit function $Q_t^\varepsilon(x)$ so that $Q_T^\varepsilon(x) = h(x)$ and so that $Q_t^\varepsilon(X_t)$ is a martingale to first-order corrected terms. Then, indeed $Q_t^\varepsilon(X_t)$ gives the approximation for M_t to this order.

The following proposition gives the first-order correction to the expression for the martingale M_t in the regime of ε small.

PROPOSITION 4.1. *When ε is small, we have*

$$M_t = Q_t^\varepsilon(X_t) + o(\varepsilon^{1-H}), \tag{4.3}$$

where

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \big(x^2\partial_x^2Q_t^{(0)}(x)\big)\phi_t^\varepsilon + \varepsilon^{1-H}\widetilde{\sigma}\rho Q_t^{(1)}(x), \tag{4.4}$$

$Q_t^{(0)}(x)$ is deterministic and given by the Black-Scholes formula with constant volatility $\overline{\sigma}$,

$$\mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x), \tag{4.5}$$

with

$$\overline{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z)^2 p(z) dz, \qquad \widetilde{\sigma} = \langle F \rangle = \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z) p(z) dz, \tag{4.6}$$

$p(z)$ the pdf of the standard normal distribution, ϕ_t^ε is the random component

$$\phi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_t^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big) ds \big| \mathcal{F}_t\Big], \tag{4.7}$$

and $Q_t^{(1)}(x)$ is the deterministic correction

$$Q_t^{(1)}(x) = x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big) D_t, \tag{4.8}$$

with D_t defined by

$$D_t = \overline{D}(T-t)^{H+\frac{1}{2}}, \qquad \overline{D} = \frac{\sigma_{\mathrm{ou}} \langle FF' \rangle}{\Gamma(H+\frac{3}{2})} = \frac{\sigma_{\mathrm{ou}}}{\Gamma(H+\frac{3}{2})} \int_{\mathbb{R}} FF'(\sigma_{\mathrm{ou}} z) p(z) dz. \tag{4.9}$$

As shown in Lemma B.3 (first item), as $\varepsilon \to 0$, the zero-mean random variable $\varepsilon^{H-1}\phi_t^\varepsilon$ has a variance that converges to $\sigma_\phi^2 (T-t)^{2H}$, with

$$\sigma_\phi^2 = \sigma_{\mathrm{ou}}^2 \langle FF' \rangle^2 \Big(\frac{1}{\Gamma(2H+1)\sin(\pi H)} - \frac{1}{2H\Gamma(H+\frac{1}{2})^2}\Big), \tag{4.10}$$

moreover, it converges in distribution to a Gaussian random variable with mean zero and variance $\sigma_\phi^2 (T-t)^{2H}$. This shows that the two corrective terms in (4.4) are of the same order ε^{1-H}, but the first one is random, zero-mean and approximately Gaussian distributed, while the second one is deterministic.

Proof. For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t) dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^* + \frac{1}{2}\big(x^2\partial_x^2 q_t\big)(X_t)(\sigma_t^\varepsilon)^2 dt \\ &= \mathcal{L}_{\mathrm{BS}}(\sigma_t^\varepsilon) q_t(X_t) dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^*, \end{aligned}$$

the last term being a martingale. Therefore, by (4.5), we have

$$dQ_t^{(0)}(X_t) = \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big) Q_t^{(0)}(X_t) dt + dN_t^{(0)}, \tag{4.11}$$

with $N_t^{(0)}$ a martingale:

$$dN_t^{(0)} = \big(x\partial_x\big) Q_t^{(0)}(X_t)\sigma_t^\varepsilon dW_t^*.$$

Note also that in Eq. (4.11) (and below) we use the notation

$$\big(x^2\partial_x^2\big) Q_t^{(0)}(X_t) = \Big(\big(x^2\partial_x^2\big) Q_t^{(0)}(x)\Big)\big|_{x=X_t}.$$

Let ϕ_t^ε be defined by (4.7). We have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds,$$

where the martingale ψ_t^ε is defined by

$$\psi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_0^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds\big|\mathcal{F}_t\Big]. \tag{4.12}$$

We can write

$$\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt = \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon - \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon.$$

By Itô's formula:

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* \\ & + \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\phi^\varepsilon, W^*\right\rangle_t. \end{aligned}$$

Since $\mathcal{L}_{\rm BS}(\sigma_t^\varepsilon) = \mathcal{L}_{\rm BS}(\overline{\sigma}) + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)$ and $\mathcal{L}_{\rm BS}(\overline{\sigma})\big(x^2\partial_x^2\big)Q_t^{(0)}(x) = 0$, this gives

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\ & + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\phi^\varepsilon, W^*\right\rangle_t \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon. \end{aligned}$$

We have $\left\langle\phi^\varepsilon, W^*\right\rangle_t = \left\langle\psi^\varepsilon, W^*\right\rangle_t = \rho\left\langle\psi^\varepsilon, W\right\rangle_t$ and therefore

$$\begin{aligned} d\big[(\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\ & + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\psi^\varepsilon, W\right\rangle_t \\ & + dN_t^{(1)}, \end{aligned}$$

where $N_t^{(1)}$ is a martingale,

$$dN_t^{(1)} = \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon.$$

Therefore:

$$\begin{aligned} d\big[Q_t^{(0)}(X_t) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\psi^\varepsilon, W\right\rangle_t \\ & + dN_t^{(0)} + dN_t^{(1)}. \end{aligned} \tag{4.13}$$

The deterministic function $Q_t^{(1)}$ defined by (4.8) satisfies

$$\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(1)}(x) = -\big(x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\big)\theta_t, \qquad Q_T^{(1)}(x)=0,$$

where $\theta_t = -dD_t/dt$ is such that

$$d\left\langle \psi^\varepsilon, W\right\rangle_t = \big(\varepsilon^{1-H}\theta_t + \widetilde{\theta}_t^\varepsilon\big)dt,$$

as shown in Lemmas B.1-B.2 with $\widetilde{\theta}_t^\varepsilon$ characterized in Eq. (B.9). Applying Itô's formula

$$\begin{aligned}
dQ_t^{(1)}(X_t) &= \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)Q_t^{(1)}(X_t)dt + \big(x\partial_x\big)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(1)}(X_t)dt + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt \\
&\quad + \big(x\partial_x\big)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt - \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\theta_t dt + dN_t^{(2)},
\end{aligned}$$

where $N_t^{(2)}$ is a martingale,

$$dN_t^{(2)} = \big(x\partial_x\big)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^*.$$

Therefore

$$\begin{aligned}
& d\big[Q_t^{(0)}(X_t) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(X_t)\big] \\
&= \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt + \frac{\varepsilon^{1-H}}{2}\rho\widetilde{\sigma}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt \\
&\quad + \varepsilon^{1-H}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)(\sigma_t^\varepsilon - \widetilde{\sigma})\theta_t dt + \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\widetilde{\theta}_t^\varepsilon dt \\
&\quad + dN_t^{(0)} + dN_t^{(1)} + \varepsilon^{1-H}\rho\widetilde{\sigma}dN_t^{(2)}. \qquad (4.14)
\end{aligned}$$

We next show that the first four terms of the right-hand side are of small order ε^{1-H}. We introduce for any $t \in [0,T]$:

$$R_{t,T}^{(1)} = \int_t^T \frac{1}{2}\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)\big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)\phi_s^\varepsilon ds, \qquad (4.15)$$

$$R_{t,T}^{(2)} = \int_t^T \frac{\varepsilon^{1-H}}{2}\rho\widetilde{\sigma}\big(x^2\partial_x^2\big)Q_s^{(1)}(X_s)\big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds, \qquad (4.16)$$

$$R_{t,T}^{(3)} = \int_t^T \varepsilon^{1-H}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)\theta_s(\sigma_s^\varepsilon - \widetilde{\sigma})ds, \qquad (4.17)$$

$$R_{t,T}^{(4)} = \int_t^T \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)\sigma_s^\varepsilon\widetilde{\theta}_s^\varepsilon ds. \qquad (4.18)$$

We show that, for $j = 1,2,3,4$,

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R_{t,T}^{(j)})^2\big]^{1/2} = 0. \qquad (4.19)$$

Step 1: Proof of (4.19) for $j = 1$.

We denote

$$Y_s^{(1)} = \big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_s^{(0)}(X_s)$$

and

$$\gamma_t^\varepsilon = \frac{1}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)\phi_s^\varepsilon ds, \tag{4.20}$$

so that

$$R_{t,T}^{(1)} = \int_t^T Y_s^{(1)}\frac{d\gamma_s^\varepsilon}{ds}ds.$$

Note that $Y_s^{(1)}$ is a bounded semimartingale with bounded quadratic variations, so that its mean square increments $\mathbb{E}[(Y_s^{(1)} - Y_{s'}^{(1)})^2]$ are uniformly bounded by $K|s-s'|$. Let N be a positive integer. We denote $t_k = t + (T-t)k/N$. We have

$$\begin{aligned}
R_{t,T}^{(1)} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y_s^{(1)}\frac{d\gamma_s^\varepsilon}{ds}ds = R_{t,T}^{(1,a)} + R_{t,T}^{(1,b)},\\
R_{t,T}^{(1,a)} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y_{t_k}^{(1)}\frac{d\gamma_s^\varepsilon}{ds}ds = \sum_{k=0}^{N-1} Y_{t_k}^{(1)}\big(\gamma_{t_{k+1}}^\varepsilon - \gamma_{t_k}^\varepsilon\big),\\
R_{t,T}^{(1,b)} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \big(Y_s^{(1)} - Y_{t_k}^{(1)}\big)\frac{d\gamma_s^\varepsilon}{ds}ds.
\end{aligned}$$

Note that we have by Minkowski's inequality:

$$\mathbb{E}\big[(R_{t,T}^{(1,a)})^2\big]^{1/2} \le 2\sum_{k=0}^{N}\|Y^{(1)}\|_\infty\mathbb{E}[(\gamma_{t_k}^\varepsilon)^2]^{1/2} \le 2(N+1)\|Y^{(1)}\|_\infty\sup_{s\in[0,T]}\mathbb{E}[(\gamma_s^\varepsilon)^2]^{1/2},$$

so that, by Lemma B.4, for any fixed N:

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R_{t,T}^{(1,a)})^2\big]^{1/2} = 0.$$

On the other hand

$$\begin{aligned}
\mathbb{E}\big[(R_{t,T}^{(1,b)})^2\big]^{1/2} &\le \|F\|_\infty^2\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\mathbb{E}[\big(Y_s^{(1)} - Y_{t_k}^{(1)}\big)^4]^{1/4}\mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4}ds\\
&\le K\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(s-t_k)^{1/2}ds\sup_{s\in[0,T]}\mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4}\\
&\le \frac{K'}{\sqrt{N}}\sup_{s\in[0,T]}\mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4}.
\end{aligned}$$

Therefore, by Lemma B.3 (fourth item), we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(1)}_{t,T})^2\big]^{1/2}\le \limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(1,b)}_{t,T})^2\big]^{1/2}\le\frac{K'}{\sqrt{N}}.$$

Since this is true for any N, we get the desired result.

Step 2: Proof of (4.19) for $j=2$.

We denote

$$Y^{(2)}_s=\rho\widetilde{\sigma}\big(x^2\partial_x^2\big)Q^{(1)}_s(X_s)$$

and

$$\kappa^\varepsilon_t=\frac{\varepsilon^{1-H}}{2}\int_0^t\big((\sigma^\varepsilon_s)^2-\overline{\sigma}^2\big)ds, \tag{4.21}$$

so that

$$R^{(2)}_{t,T}=\int_t^T Y^{(2)}_s\frac{d\kappa^\varepsilon_s}{ds}ds.$$

Note that $Y^{(2)}_s$ is a bounded semimartingale with bounded quadratic variations. Let N be a positive integer. We denote as above $t_k=t+(T-t)k/N$. We then have

$$\begin{aligned}
R^{(2)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}Y^{(2)}_s\frac{d\kappa^\varepsilon_s}{ds}ds=R^{(2,a)}_{t,T}+R^{(2,b)}_{t,T},\\
R^{(2,a)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}Y^{(2)}_{t_k}\frac{d\kappa^\varepsilon_s}{ds}ds=\sum_{k=0}^{N-1}Y^{(2)}_{t_k}\big(\kappa^\varepsilon_{t_{k+1}}-\kappa^\varepsilon_{t_k}\big),\\
R^{(2,b)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\big(Y^{(2)}_s-Y^{(2)}_{t_k}\big)\frac{d\kappa^\varepsilon_s}{ds}ds.
\end{aligned}$$

Then, on the one hand

$$\mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2}\le 2\sum_{k=0}^{N}\|Y^{(2)}\|_\infty\mathbb{E}[(\kappa^\varepsilon_{t_k})^2]^{1/2}\le 2(N+1)\|Y^{(2)}\|_\infty\sup_{s\in[0,T]}\mathbb{E}[(\kappa^\varepsilon_s)^2]^{1/2},$$

so that, by Lemma B.6,

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2}=0.$$

On the other hand

$$\begin{aligned}
\mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2}&\le\varepsilon^{1-H}\|F\|_\infty^2\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\mathbb{E}[\big(Y^{(2)}_s-Y^{(2)}_{t_k}\big)^2]^{1/2}ds\\
&\le K\varepsilon^{1-H}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(s-t_k)^{1/2}ds\\
&\le\frac{K'\varepsilon^{1-H}}{\sqrt{N}}.
\end{aligned}$$

Therefore, we get

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(R^{(2)}_{t,T})^2\big]^{1/2} \leq \limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2} \leq \frac{K'}{\sqrt{N}}.$$

Since this is true for any N, we get the desired result.

Step 3: Proof of (4.19) for $j = 3$.

This proof follows the same lines as the proof of Step 2 with

$$\eta^\varepsilon_t = \varepsilon^{1-H} \int_0^t \big(\sigma^\varepsilon_s - \widetilde{\sigma}\big) ds, \tag{4.22}$$

instead of κ^ε_t, and using that θ_t is bounded. We then get the desired result by Lemma B.5.

Step 4: Proof of (4.19) for $j = 4$.

We have

$$\mathbb{E}\big[(R^{(4)}_{t,T})^2\big]^{1/2} \leq K \int_t^T \mathbb{E}\big[(\widetilde{\theta}^\varepsilon_s)^2\big]^{1/2} ds \leq K' \sup_{s\in[0,T]} \mathbb{E}\big[(\widetilde{\theta}^\varepsilon_s)^2\big]^{1/2}.$$

By Lemma B.2,

$$\lim_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(R^{(4)}_{t,T})^2\big]^{1/2} = 0.$$

We can now complete the proof of Proposition 4.1. In (4.4) we introduced the approximation:

$$Q^\varepsilon_t(x) = Q^{(0)}_t(x) + \phi^\varepsilon_t\big(x^2\partial_x^2\big)Q^{(0)}_t(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q^{(1)}_t(x).$$

We then have

$$Q^\varepsilon_T(x) = h(x),$$

because $Q^{(0)}_T(x) = h(x)$, $\phi^\varepsilon_T = 0$, and $Q^{(1)}_T(x) = 0$. Let us denote

$$R_{t,T} = R^{(1)}_{t,T} + R^{(2)}_{t,T} + R^{(3)}_{t,T} + R^{(4)}_{t,T}, \tag{4.23}$$

$$N_t = \int_0^t dN^{(0)}_s + dN^{(1)}_s + \varepsilon^{1-H}\rho\widetilde{\sigma}dN^{(2)}_s. \tag{4.24}$$

By (4.14) we have

$$Q^\varepsilon_T(X_t) - Q^\varepsilon_t(X_t) = R_{t,T} + N_T - N_t.$$

Therefore

$$
\begin{aligned}
M_t &= \mathbb{E}\big[h(X_T)|\mathcal{F}_t\big] = \mathbb{E}\big[Q_T^\varepsilon(X_T)|\mathcal{F}_t\big] = Q_t^\varepsilon(X_t) + \mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big] + \mathbb{E}\big[N_T - N_t|\mathcal{F}_t\big] \\
&= Q_t^\varepsilon(X_t) + \mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big],
\end{aligned} \tag{4.25}
$$

which gives the desired result because $\mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big]$ is of order $o(\varepsilon^{1-H})$ in L^2. □

5. Call Price Correction and Implied Volatility. We denote the Black-Scholes call price, with current time t, maturity T, strike K, underlying value x, and volatility σ, by $C_{\rm BS}(t, x; K, T; \sigma)$, so that $Q_t^{(0)}$ in Eq. (4.5) is

$$
Q_t^{(0)}(x) = C_{\rm BS}(t, x; K, T; \overline{\sigma}).
$$

Indeed, $C_{\rm BS}$ gives an explicit formula for the price in the case with constant volatility. In the situation with a stochastic volatility as considered here no explicit pricing formula exists. However, as shown in Eq. (4.4) we can get an asymptotic expression for the price in the case with the stochastic volatility (1.5) as a correction to $Q_t^{(0)}(x)$, the Black-Scholes price evaluated at the effective or "homogenized" volatility $\bar{\sigma}$. Here, we show that this corrected price takes on a rather simple generic form in the two parameters, relative time to maturity and moneyness. This representation then leads to a simple representation for the implied volatility as we show below. The long-range character of the volatility fluctuations indeed has a strong impact on the form of the implied volatility and this observation is important in a calibration context.

We denote the time to maturity by $\tau = T - t$ and we introduce the characteristic diffusion time $\bar{\tau} = 2/\overline{\sigma}^2$ and the dimensionless effective skewness factor:

$$
a_F = \varepsilon^{1-H} \frac{\rho \widetilde{\sigma} \overline{D} \bar{\tau}^H}{2^{3/2} \overline{\sigma}} = \varepsilon^{1-H} \frac{\widetilde{\sigma} \sigma_{\rm ou} \rho \left\langle F F' \right\rangle \bar{\tau}^H}{2^{3/2} \overline{\sigma} \Gamma(H + 3/2)}, \tag{5.1}
$$

with $\overline{\sigma}$, $\widetilde{\sigma}$ and $\overline{D}$ given in Proposition 4.1 and the correlation ρ introduced in Eq. (3.3).

Lemma 5.1. *The price correction in Eq. (4.4), normalized by the strike K, can be written in the form*

$$
\begin{aligned}
&\frac{1}{K}\Big(\phi_t^\varepsilon(x^2\partial_x^2) Q_t^{(0)}(x) + \varepsilon^{1-H} \rho \widetilde{\sigma} Q_t^{(1)}(x)\Big) \\
&= \left(\frac{e^{-d_1^2/2} \frac{x}{K}}{\sqrt{\pi}}\right) \left\{ \frac{\phi_t^\varepsilon}{2} \left(\frac{\tau}{\bar{\tau}}\right)^{-1/2} + a_F \left[\left(\frac{\tau}{\bar{\tau}}\right)^H + \left(\frac{\tau}{\bar{\tau}}\right)^{H-1} \log\left(\frac{K}{x}\right) \right] \right\},
\end{aligned} \tag{5.2}
$$

with

$$
d_1 = \sqrt{\frac{\bar{\tau}}{2\tau}} \Big[\frac{\tau}{\bar{\tau}} - \log\Big(\frac{K}{x}\Big)\Big]. \tag{5.3}
$$

Here, the dimensionless random and deterministic correction coefficients are small of order

$$\phi_t^\varepsilon = O\left(\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^{H}\right), \quad a_F = O\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}, \tag{5.4}$$

where we used that ϕ_t^ε as defined in Proposition 4.1 is centered and with standard deviation

$$\mathrm{Var}\big(\phi_t^\varepsilon\big)^{1/2} = \left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^{H}(\bar{\tau}\sigma_\phi) + o(\varepsilon^{1-H}), \tag{5.5}$$

with σ_ϕ defined by Eq. (4.10) (see also Eq. (B.14) in Lemma B.3). We comment in more detail about the statistical structure of ϕ_t^ε in the next section.

It follows from the above that the normalized price correction depends on the two parameters, the moneyness K/x and the relative time to maturity $\tau/\bar{\tau}$, and exhibits a term structure in fractional powers of relative time to maturity.

In Figure 5.1 we show the relative price correction in Eq. (5.2) as function of relative time to maturity $\tau/\bar{\tau}$ for three values of the moneyness K/x. The solid lines plot the mean relative price correction and the dashed lines give the mean plus/minus one standard deviation. We use here $H = 0.6$, $a_F = 0.1$, and $\big((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi\big) = 0.04$. The mean relative price correction is largest for a mid range of times to maturity. For very short times to maturity relative to the effective diffusion time the effect of the volatility fluctuations are small, while for large times the rapid mean reversion "averages" out the effect of the fluctuations. Note, however, that at the money the random component of the price correction decays slowly as

$$\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2},$$

as $\tau \to 0$ while "around the money" with moneyness K/x different from unity the decay is like

$$\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2}\exp\Big(-\frac{\bar{\tau}|\log(K/x)|^2}{4\tau}\Big).$$

This reflects the fact that the vega is diverging in this limit so that the sensitivity to volatility fluctuations becomes large. We remark that this would affect calibration schemes using at the money data. Moreover, results regarding small time asymptotics for the coherent implied volatility becomes questionable in this context as the dominating contribution comes from the random component of the price correction.

Note also that the parameters chosen are not calibrated to market data, this will be considered in another publication.

In Figure 5.2 we show the price correction surface as function of relative time to maturity $\tau/\bar{\tau}$ and moneyness K/x.

We next present the proof of Lemma 5.1.

Proof. For the European call option with payoff $h(x) = (x-K)_+$ we have explicitly

$$C_{\rm BS}(t,x;K,T;\sigma) = x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) + \frac{\sigma\sqrt{T-t}}{2}\right)$$
$$-K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) - \frac{\sigma\sqrt{T-t}}{2}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. We then have in particular the "Greek" relationships for the call price:

$$\partial_\sigma C_{\rm BS} = (T-t)\overline{\sigma}x^2\partial_x^2 C_{\rm BS}, \qquad x\partial_x\partial_\sigma C_{\rm BS} = \left(\frac{1}{2} + \frac{\log\frac{K}{x}}{\overline{\sigma}^2(T-t)}\right)\partial_\sigma C_{\rm BS}.$$

We then get

$$x^2\partial_x^2 Q_t^{(0)}(x) = \frac{1}{\overline{\sigma}(T-t)}\partial_{\bar{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{5.6}$$

$$x\partial_x x^2\partial_x^2 Q_t^{(0)}(x) = \left[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^2}\right]\partial_{\bar{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{5.7}$$

where the "Vega" is given by

$$\partial_\sigma C_{\rm BS}(t,x;K,T;\overline{\sigma}) = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}}, \qquad d_1 = \frac{\frac{1}{2}\sigma^2(T-t) - \log\frac{K}{x}}{\sigma\sqrt{T-t}}. \tag{5.8}$$

Then, with $Q_t^{(1)}(x)$ given in Eq. (4.8) we can identify the form of the price correction as:

$$\begin{aligned}
&\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x)\\
&= \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}D(t)x\partial_x x^2\partial_x^2 Q_t^{(0)}(x)\\
&= \phi_t^\varepsilon\left(\frac{xe^{-d_1^2/2}}{\overline{\sigma}\sqrt{2\pi(T-t)}}\right) + \varepsilon^{1-H}\left(\frac{x\rho\widetilde{\sigma}\overline{D}e^{-d_1^2/2}}{\sqrt{2\pi}}\right)\left[\frac{(T-t)^H}{2\overline{\sigma}} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^{1-H}}\right],
\end{aligned} \tag{5.9}$$

which in turn gives (5.2). ☐

We next consider the implied volatility associated with the price correction. For the stochastic volatility model in Eq. (1.5) we want to identify the implied volatility I_t so that in terms of the corrected price in Lemma 4.1 we have:

$$C_{\rm BS}(t,x;K,T;I_t) = Q_t^{(0)}(x) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x). \tag{5.10}$$

We define the relative implied volatility correction, δI_t, by

$$I_t = \overline{\sigma}(1 + \delta I_t). \tag{5.11}$$

LEMMA 5.2. *The relative implied volatility correction has the form:*

$$\delta I_t = \frac{\phi_t^\varepsilon}{2}\left(\frac{\tau}{\overline{\tau}}\right)^{-1} + a_F\Big[\left(\frac{\tau}{\overline{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\overline{\tau}}\right)^{H-3/2}\log\left(\frac{K}{X_t}\right)\Big] + o(\varepsilon^{1-H}), \tag{5.12}$$

where ϕ_t^ε is defined by (4.7) and a_F by (5.1).

In Figure 5.3 we show the implied volatility correction in Eq. (5.12) as function of relative time to maturity $\tau/\overline{\tau}$ for three values of the moneyness K/x. We again used $H = 0.6, a_F = 0.1$ and $\big((\varepsilon/\overline{\tau})^{(1-H)}\overline{\tau}\sigma_\phi\big) = 0.04$. Note that due to the form of the "vega" , the sensitivity of the price to the volatility, the form of the implied volatility surface is very different from that of the price correction. In Figure 5.4 we show the implied volatility correction surface as function of relative time to maturity $\tau/\overline{\tau}$ and moneyness K/x.

Proof. We find by using Eqs. (5.9) and (5.8) that the implied volatility is given by

$$I_t = \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} + \varepsilon^{1-H}\widetilde{\sigma}\rho D_t\Big[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{X_t}}{\overline{\sigma}^3(T-t)^2}\Big] + o(\varepsilon^{1-H}). \tag{5.13}$$

Since D_t is deterministic and given by (4.9), we can then write

$$\begin{aligned} I_t &= \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} \qquad (5.14)\\ &\quad + \varepsilon^{1-H}\frac{\widetilde{\sigma}\sigma_{\rm ou}\rho\left\langle FF'\right\rangle}{\overline{\sigma}\Gamma(H+\frac{3}{2})}\Big[\frac{1}{2}(T-t)^{H-\frac{1}{2}} + \frac{\log\frac{K}{X_t}}{\overline{\sigma}^2(T-t)^{\frac{3}{2}-H}}\Big] + o(\varepsilon^{1-H}), \end{aligned}$$

and the Lemma follows. ∎

The first two terms in Eq. (5.14) can be combined and rewritten as (up to terms of order $o(\varepsilon^{1-H})$):

$$\overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T(\sigma_s^\varepsilon)^2 ds\big|\mathcal{F}_t\Big]^{1/2} + o(\varepsilon^{1-H}). \tag{5.15}$$

Since D_t is deterministic and given by (4.9), we can then write

$$\begin{aligned} I_t &= \mathbb{E}\Big[\frac{1}{T-t}\int_t^T(\sigma_s^\varepsilon)^2 ds\big|\mathcal{F}_t\Big]^{1/2}\\ &\quad + \overline{\sigma}a_F\left[\left(\frac{\tau}{\overline{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\overline{\tau}}\right)^{H-3/2}\log\left(\frac{K}{X_t}\right)\right] + o(\varepsilon^{1-H}), \qquad (5.16) \end{aligned}$$

so that the implied volatility is the expected effective volatility over the remaining time horizon conditioned on the present and with an added skewness correction.

In view of Eq. (5.5), for small time to maturity the fourth term (in $\tau^{H-\frac{3}{2}}$) dominates in (5.12). We remark here that this is related to the fact that the small parameter in our problem is the mean reversion time so that for any order one time to maturity in this regime the volatility has enough time to fluctuate and mean revert giving a price correction as in Lemma 5.1. Then with the "Vega", $\partial_\sigma C_{\rm BS}$, being mall away from the money, see Eq. (5.8), we get a strong moneyness dependence and the implied volatility blows up for small time to maturity.

Moreover, for large time to maturity the third term (in $\tau^{H-\frac{1}{2}}$) dominates in (5.12). The long-range dependence gives smooth volatility fluctuations which gives an implied volatility that blows up for large time to maturity and with the current value for the underlying being less important in this large time to maturity regime.

6. The t-T Process and the Stochastic Implied Surface. We introduced in Eq. (4.7) the stochastic correction coefficient $\phi_t^\varepsilon \equiv \phi_{t,T}^\varepsilon$ which gives the random component of the price correction and the implied volatility and where we here explicitly display the dependence on maturity T. Note that if the volatility process had been a Markovian process then the correction would have been deterministic, as in Fouque et al. (2011). The presence of long-range memory in the volatility process means that information from the past (volatility path) must be carried forward and this makes the price correction relative to the price at the homogenized volatility a stochastic process, and correspondingly for the implied volatility.

We here discuss the statistical structure of the random field which describes the implied volatility surface in the scaling regime that we consider. The implied volatility is the central quantity in typical calibration processes and to design efficient estimators for both the coherent and incoherent parts of the implied volatility, moreover, to characterize the resulting estimation precision, it is important to understand the statistical fluctuations of the observed implied surface. We give a precise characterization of these fluctuations below. The fluctuations of the implied volatility for large times to maturity (relative to $\bar{\tau}$) become stronger for larger Hurst exponent because the larger Hurst exponent gives stronger temporal coherence and a larger correction to the anticipated volatility. On the other hand for small times to maturity the fluctuations become larger for small Hurst exponent because this gives a rougher process with large fluctuations even over very small intervals. It is also interesting to note that the correlation structure of the implied volatility surface in fact encodes information about the long-range character of the underlying stochastic volatility. Observing for instance at the money implied volatility fluctuations as function of current time for fixed time to maturity gives information that makes it possible to estimate the Hurst exponent and check for consistency of the modeling framework. In Livieri et al. (2017) observed at the money implied volatility was used to estimate the Hurst exponent. The authors found a coefficient that was slightly larger than the corresponding estimates using historical data and explained this discrepancy in terms of smoothing effect due to the remaining time to maturity. To construct and interpret estimators of this kind a model for the implied surface as a random field relating it to the underlying volatility parameters is clearly essential.

In under to understand the implied volatility random field note first that it follows from Lemma B.3 that as $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi^{\varepsilon}_{t,T}/[\sigma_\phi(T-t)^H]$, $t < T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian stochastic process $\psi_{t,T}$, $t < T$, *the normalized t-T correction process*, with mean zero, variance one, and covariance $\mathbb{E}[\psi_{t,T}\psi_{t',T'}] = \mathcal{C}_\phi(t,t';T,T')$ for any $t \in [0,T)$, $t' \in [0,T')$. The four-parameter function $\mathcal{C}_\phi$ is given by Eq. (B.16). We discuss next in more detail the t-T process $\psi_{t,T}$, a two-parameter process of current time t and maturity T. This process is scaled to have constant unit variance, however, is a non-stationary Gaussian process supported for $0 < t < T$. As we see below, close to maturity $t \approx T$, the process is strongly affected by the presence of the maturity boundary.

Let us first consider the case of a fixed maturity T and introduce the process

$$\psi_0(t;T) = \psi_{t,T}, \qquad t \in [0,T]. \tag{6.1}$$

On short scales relative to the time to maturity, i.e. for $|t-t'| \ll T-t$, it follows from Eq. (B.16) that the process $(\psi_0(t;T))_{t\in[0,T]}$ decorrelates as

$$\mathbb{E}\big[\psi_0(t;T)\psi_0(t';T)\big] \sim 1 - \frac{|t-t'|}{2(T-t)},$$

that is as a Markov process on short scales. More generally, the autocovariance function of $(\psi_0(t;T))_{t\in[0,T]}$ is

$$\mathbb{E}\big[\psi_0(t;T)\psi_0(t';T)\big] = \mathcal{C}(\Delta_0(t,t';T)),$$
$$\mathcal{C}(\Delta) = \frac{\int_0^\infty du \big[\big(u + \frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\big)^{H-\frac12} - u^{H-\frac12}\big]\big[\big(u + \frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\big)^{H-\frac12} - \big(u + \frac{2|\Delta|}{\sqrt{1-\Delta^2}}\big)^{H-\frac12}\big]}{\int_0^\infty du \big[(1+u)^{H-\frac12} - u^{H-\frac12}\big]^2},$$

with

$$\Delta_0(t,t';T) = \frac{t'-t}{|2T-(t+t')|} \tag{6.2}$$

which shows that the correlation function of the process $(\psi_0(t;T))_{t\in[0,T]}$ depends only on this relative separation so that we have a situation with a canonical relative decorrelation that depends only on the times to maturity $\tau = T-t, \tau' = T-t'$. Therefore, we introduce the process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ defined by

$$\psi_1(\tau;T) = \psi_{T-\tau,T}, \qquad \tau \in [0,T]. \tag{6.3}$$

The process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\big[\psi_1(\tau;T)\psi_1(\tau';T)\big] = \mathcal{C}(\Delta_1(\tau,\tau')),$$

with $\mathcal{C}$ as above and

$$\Delta_1(\tau,\tau') = \frac{\tau-\tau'}{|\tau+\tau'|}. \tag{6.4}$$

For $|\tau-\tau'| \ll \tau$ the process decorrelates on the scale τ so that the process fluctuations become more rapid close to maturity. Close to maturity the price fluctuations become smaller, however, when we magnify them we see fluctuations on smaller scales for smaller time to maturity which reflects the self similarity of the driving volatility factor. In Figure 6.1 we show the correlation function $\Delta_1 \mapsto \mathcal{C}(\Delta_1)$ as function of the relative separation time $\Delta_1 \in [-1,1]$ and $H = 0.6$. The process decorrelates as a Markov process on short scales and indeed as one of the times to maturity goes to zero (relative to the other time) the correlation goes rapidly to zero.

Note that it follows from the expression (6.4) for Δ_1 that it is scale invariant in that $\Delta_1(a\tau,a\tau') = \Delta_1(\tau,\tau')$ for $a>0$, giving rapid fluctuations for small times to maturity. The process has indeed a self-similar property. We have in distribution:

$$\big(\psi_1(\tau;1)\big)_{\tau\in[0,1]} \sim \big(\psi_1(\tau T;T)\big)_{\tau\in[0,1]},$$

for any $T>0$. In Figure 6.2 we show two realizations of the process $\psi_1(\tau;1)$ as a function of time to maturity τ.

One can also investigate the structure of the t-T process for a fixed time to maturity τ, as a function of time t. Thus, if we observe the price for a given time to maturity, we would like to know how the price correction, respectively the implied volatility, would fluctuate with respect to the current time, or time translation. Accordingly we consider the process

$$\psi_2(t;\tau) = \psi_{t,\tau+t}, \qquad t \geq 0, \tag{6.5}$$

for fixed $\tau > 0$. The process $(\psi_2(t;\tau))_{t\in[0,\infty)}$ is Gaussian with mean zero and autocovariance function

$$\begin{aligned}
&\mathbb{E}\big[\psi_2(t;\tau)\psi_2(t';\tau)\big] = \mathcal{C}_2(\Delta_2(t,t';\tau)), \qquad (6.6)\\
&\mathcal{C}_2(\Delta) = \frac{\int_0^\infty du\big[(u+1)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]\big[(u+1+|\Delta|)^{H-\frac{1}{2}} - (u+|\Delta|)^{H-\frac{1}{2}}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]^2},
\end{aligned}$$

with

$$\Delta_2(t,t';\tau) = \frac{t'-t}{\tau}. \tag{6.7}$$

The expression of Δ_2 shows that the coherence time of this process scales with time to maturity τ. We see again that the rescaled implied volatility surface fluctuations are more rapid closer to maturity. We also see that on transects parallel to the maturity boundary in the t, T plane these fluctuations are stationary, this is consistent with the fact that we have an underlying consistent model with a stationary volatility driving factor. The fluctuations moreover have a self-similar property. We have in distribution:

$$\big(\psi_2(t;1)\big)_{t\in[0,\infty)} \sim \big(\psi_2(\tau t;\tau)\big)_{t\in[0,\infty)},$$

for any $\tau > 0$. The autocovariance function of $(\psi_2(t;1))_{t\in[0,\infty)}$ is plotted in Figure 6.3. In the figure one can see the rapid decay at the origin followed by a long-range behavior. This shows how the implied surface decorrelates as we move in time. In Figure 6.4 we show the autocorrelation function in a *log-log* plot with the dashed line corresponding to the correlation decay $|t'-t|^{2H-2}$. In Figure 6.5 we show two realizations of the process $\psi_2(t;1)$.

Finally, it is of interest to consider the case where we evaluate the stochastic correction factor as function of time to maturity for fixed current time t:

$$\psi_3(\tau;t) = \psi_{t,t+\tau}, \qquad \tau \geq 0. \tag{6.8}$$

The process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ is Gaussian with mean zero and autocovariance function

$$\begin{aligned}
&\mathbb{E}\big[\psi_3(\tau;t)\psi_3(\tau';t)\big] = \mathcal{C}_3(\Delta_3(\tau,\tau')),\\
&\mathcal{C}_3(\Delta) = \frac{\int_0^\infty du\big[(u+1/\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]\big[(u+\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]^2},
\end{aligned}$$

with

$$\Delta_3(\tau,\tau') = \frac{\tau-\tau'}{\tau\wedge\tau'}. \tag{6.9}$$

This covariance function is plotted in Figure 6.6. Note that it follows from the expression (6.9) for Δ_3 that it is scale invariant in that $\Delta_3(a\tau, a\tau') = \Delta_3(\tau,\tau')$ for $a>0$, so that again the process fluctuates more rapidly for small maturities. The distribution of the process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ does not depend on t and it has a self-similar property. For any $a>0$, we have in distribution:

$$\big(\psi_3(\tau;t)\big)_{\tau\in[0,\infty)} \sim \big(\psi_3(a\tau;t)\big)_{\tau\in[0,\infty)}.$$

In Figure 6.7 we show two realizations of the process $(\psi_3(\tau;t))_{\tau\in[0,1)}$.

7. Conclusion. We have considered a continuous time stochastic volatility model with long-range correlation properties. We consider the regime of fast mean reversion. This makes it possible to derive an explicit expression for the European call option price and the implied volatility. Specifically the volatility is a smooth function of a fractional Ornstein-Uhlenbeck process. The analysis of such a non-Markovian situation is challenging. To the best of our knowledge we present the first analytical expression for the price for general maturities when the volatility fluctuations are order one. So far the price computations for such situations have been based on numerical approximations. The main result from the applied view point is then the form of the fractional term structure we get for the implied volatility surface. Indeed we get an implied volatility that grows large with time to maturity while generating a strong skew for short times to maturity consistently with common observations. We stress that in our formulation the mean reversion time is small compared to any fixed maturity as we consider a fast mean reverting process. Note finally that we have considered the case of processes with long-range correlation properties with the Hurst exponent $H>1/2$ explaining the growth of implied volatility for large maturity.

A. Hermite Decomposition of the Stochastic Volatility Model. We denote

$$\widetilde{F}(z) = F(\sigma_{\rm ou} z)^2. \tag{A.1}$$

Because $\mathbb{E}[\widetilde{F}(Z)^2] < \infty$ is finite when Z is a standard normal variable, the function $\widetilde{F}$ can be expanded in terms of the Hermite polynomials

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2} \tag{A.2}$$

and the series

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(z), \tag{A.3}$$

with

$$C_k = \mathbb{E}\big[H_k(Z)\widetilde{F}(Z)\big] = \int_{\mathbb{R}} H_k(z)\widetilde{F}(z)p(z)dz, \tag{A.4}$$

converges in $L^2(\mathbb{R}, p(z)dz)$ to $\widetilde{F}(z)$. The Hermite polynomials satisfy

$$\mathbb{E}[H_k(Z)H_j(Z)] = \int_{\mathbb{R}} H_k(z)H_j(z)p(z)dz = \delta_{kj}k!,$$

and we have $\sum_{k=0}^{\infty} \frac{C_k^2}{k!} = \mathbb{E}[\widetilde{F}(Z)^2] < \infty$. Note that $C_0 = \langle F^2 \rangle$.

Lemma A.1. *If there exists $\alpha > 2$ such that the function $\widetilde{F}$ defined by (A.1) satisfies*

$$\sum_{k=0}^{\infty} \frac{\alpha^k C_k^2}{k!} < \infty, \tag{A.5}$$

then the random process

$$I_t^\varepsilon = \int_0^t F^2(Z_s^\varepsilon) - \langle F^2 \rangle \, ds \tag{A.6}$$

satisfies

$$\sup_{t\in[0,T]} \mathbb{E}[(I_t^\varepsilon)^4] \le K\varepsilon^{4-4H}, \tag{A.7}$$

for some constant K.

Proof. Denoting $\widetilde{Z}_t^\varepsilon = \sigma_{\rm ou}^{-1} Z_t^\varepsilon$, which is a zero-mean Gaussian process with covariance function $\mathbb{E}[\widetilde{Z}_t^\varepsilon \widetilde{Z}_{t+s}^\varepsilon] = \mathcal{C}_Z(s/\varepsilon)$, we have

$$I_t^\varepsilon = \int_0^t \widetilde{F}(\widetilde{Z}_s^\varepsilon) - \langle F^2 \rangle \, ds = \sum_{m=1}^{\infty} C_m I_{t,m}^\varepsilon,$$

where

$$I_{t,m}^\varepsilon = \frac{1}{m!} \int_0^t H_m(\widetilde{Z}_s^\varepsilon) ds, \qquad m \ge 1.$$

From (Taqqu , 1978, Lemma 2.2) the fourth-order moment of $I^{\varepsilon}_{t,m}$ can be expanded as

$$\mathbb{E}[(I^{\varepsilon}_{t,m})^4] = \frac{1}{2^m(2m)!}\sum \int_0^t \cdots \int_0^t dt_1 dt_2 dt_3 dt_4 \prod_{\ell=1}^{m} \mathcal{C}_Z\Big(\frac{t_{i_\ell}-t_{j_\ell}}{\varepsilon}\Big),$$

where the sum is over all indices $i_1, j_1, \dots, i_{2m}, j_{2m}$ such that:

i) $i_1, j_1, \dots, i_{2m}, j_{2m} \in \{1,2,3,4\}$,

ii) $i_1 \neq j_1$, ..., $i_{2m} \neq j_{2m}$,

iii) each number $1,2,3,4$ appears exactly m times in $(i_1, j_1, \dots, i_{2m}, j_{2m})$.

The number N_{2m} of terms in this sum is therefore smaller than $(4m)!/m!^4$ (it would be exactly this cardinal without the second condition, therefore it is smaller than this number).

Since $\mathcal{C}_Z(s) \leq 1 \wedge K|s|^{2H-2}$ for some constant K, we have, for any $t \in [0,T]$,

$$\mathbb{E}[(I^{\varepsilon}_{t,m})^4] \leq \frac{1}{2^{2m}(2m)!}\sum \int_0^T \cdots \int_0^T dt_1 dt_2 dt_3 dt_4 \prod_{\ell=1}^{2m} 1 \wedge K\Big(\frac{|t_{i_\ell}-t_{j_\ell}|}{\varepsilon}\Big)^{2H-2}.$$

For each term of the sum, we apply the change of variables $s_1 = t_{i_1}$, $s_2 = t_{j_1}$, $s_3 = t_{\min(\{1,2,3,4\}\setminus\{i_1,j_1\})}$, $s_4 = t_{\max(\{1,2,3,4\}\setminus\{i_1,j_1\})}$. In the product we keep the first term: $K(|s_1-s_2|/\varepsilon)^{2H-2}$, and the first term that has s_3 in it: $K(|s_3-s_j|/\varepsilon)^{2H-2}$, so that we can write, for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}[(I^{\varepsilon}_{t,m})^4] &\leq \frac{N_{2m}K^2}{2^{2m}(2m)!}\int_0^T \cdots \int_0^T ds_1 ds_2 ds_3 ds_4 \Big(\frac{|s_1-s_2|}{\varepsilon}\Big)^{2H-2}\Big[\Big(\frac{|s_3-s_1|}{\varepsilon}\Big)^{2H-2} \\
&\qquad + \Big(\frac{|s_3-s_2|}{\varepsilon}\Big)^{2H-2} + \Big(\frac{|s_3-s_4|}{\varepsilon}\Big)^{2H-2}\Big] \\
&\leq K' \frac{(4m)!}{2^{2m}(2m)!m!^4}\varepsilon^{4-4H},
\end{aligned}$$

for some constant K' (that depends on H and T), because s^{2H-2} is integrable over $[0,T]$. By Stirling's formula,

$$\frac{(4m)!}{2^{2m}(2m)!m!^4} \simeq \frac{2^{2m}}{m!^2}\frac{1}{\sqrt{2\pi m}}.$$

Therefore, by Minkowski's inequality, for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}[(I^{\varepsilon}_t)^4)]^{1/4} &\leq \sum_{m=1}^{\infty}|C_m|\mathbb{E}[(I^{\varepsilon}_m)^4)]^{1/4} \leq K''\varepsilon^{1-H}\sum_{m=1}^{\infty}|C_m|\Big(\frac{2^m}{m!}\Big)^{1/2} \\
&\leq K''\varepsilon^{1-H}\Big(\sum_{m=1}^{\infty}\frac{\alpha^m C_m^2}{m!}\Big)^{1/2}\Big(\sum_{m=1}^{\infty}\frac{2^m}{\alpha^m}\Big)^{1/2},
\end{aligned}$$

for some constant K'', which gives the desired result. □

The hypothesis (A.5) in Lemma A.1 requires some smoothness for the function $\widetilde{F}$. The following lemma gives a sufficient condition.

LEMMA A.2. *If the function $\widetilde{F}$ defined by (A.1) is of the form*

$$\widetilde{F}(x) = \int_{-\infty}^{x} f(y)dy, \tag{A.8}$$

where the Fourier transform of the function f satisfies $|\hat{f}(\nu)| \leq C \exp(-\nu^2)$ for some $C > 0$, then there exists $K > 0$ such that, for any $k \geq 0$,

$$\frac{C_k^2}{k!} \leq K 3^{-k}. \tag{A.9}$$

The inequality (A.9) is sufficient to ensure that the hypothesis (A.5) is fulfilled. We may for instance consider :

$$\widetilde{F}(x) = \int_{-\infty}^{x} e^{-y^2/4}dy \text{ or } \widetilde{F}(x) = \int_{-\infty}^{x} \mathrm{sinc}^2(y)dy. \tag{A.10}$$

Proof. The function $\widetilde{F}$ is of class $\mathcal{C}^\infty$ and we have, for any $k \geq 1$, using integration by parts,

$$C_k = \int_{\mathbb{R}} \widetilde{F}(z)H_k(z)p(z)dz = \int_{\mathbb{R}} \widetilde{F}^{(k)}(z)p(z)dz = \int_{\mathbb{R}} f^{(k-1)}(z)p(z)dz$$

By Parseval formula,

$$C_k = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\nu^2/2}(i\nu)^{k-1}\hat{f}(\nu)d\nu.$$

Since $|\hat{f}(\nu)| \leq C\exp(-\nu^2)$,

$$|C_k| \leq C \int_{\mathbb{R}} e^{-3\nu^2/2}|\nu|^{k-1}d\nu = C\Big(\frac{2}{3}\Big)^{\frac{k}{2}} \int_0^\infty e^{-s}s^{\frac{k}{2}-1}ds = C\Big(\frac{2}{3}\Big)^{\frac{k}{2}}\Gamma\Big(\frac{k}{2}\Big),$$

which gives the desired result using Stirling's formula $\Gamma(z) \sim z^{z-1/2}e^{-z}\sqrt{2\pi}$. □

B. Technical Lemmas. We denote

$$G(z) = \frac{1}{2}\big(F(z)^2 - \overline{\sigma}^2\big). \tag{B.1}$$

The martingale ψ_t^ε defined by (4.12) has the form

$$\psi_t^\varepsilon = \mathbb{E}\Big[\int_0^T G(Z_s^\varepsilon)ds \big| \mathcal{F}_t\Big]. \tag{B.2}$$

LEMMA B.1. *$(\psi_t^\varepsilon)_{t\in[0,T]}$ is a square-integrable martingale and*

$$d\left<\psi^\varepsilon, W\right>_t = \vartheta_t^\varepsilon dt, \qquad \vartheta_t^\varepsilon = \sigma_{\mathrm{ou}} \int_t^T \mathbb{E}\big[G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\mathcal{K}^\varepsilon(s-t)ds. \tag{B.3}$$

An alternative expression of the bracket $\langle \psi^\varepsilon, W\rangle_t$ is given in (B.5-B.6).

Proof. For $t \le s$, the conditional distribution of Z_s^ε given $\mathcal{F}_t$ is Gaussian with mean

$$\mathbb{E}\big[Z_s^\varepsilon|\mathcal{F}_t\big] = \sigma_{\mathrm{ou}} \int_{-\infty}^{t} \mathcal{K}^\varepsilon(s-u) dW_u$$

and deterministic variance given by

$$\mathrm{Var}\big(Z_s^\varepsilon|\mathcal{F}_t\big) = (\sigma_{0,s-t}^\varepsilon)^2,$$

where we have defined for any $0 \le t \le s \le \infty$:

$$(\sigma_{t,s}^\varepsilon)^2 = \sigma_{\mathrm{ou}}^2 \int_t^s \mathcal{K}^\varepsilon(u)^2 du. \tag{B.4}$$

We thus have that the distribution of

$$\frac{1}{\sigma_{0,s-t}^\varepsilon}\Big(\big(Z_s^\varepsilon - \int_{-\infty}^{t} \mathcal{K}^\varepsilon(s-u) dW_u\big)\big|\mathcal{F}_t\Big)$$

is standard normal. Therefore we have

$$\mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_t\big] = \int_{\mathbb{R}} G\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^{t} \mathcal{K}^\varepsilon(s-u) dW_u + \sigma_{0,s-t}^\varepsilon z\Big) p(z) dz,$$

where $p(z)$ is the pdf of the standard normal distribution. As a random process in t it is a continuous martingale. By Itô's formula, for any $t \le s$:

$$\begin{aligned}
\mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_t\big] = & \int_{\mathbb{R}} G\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^{0} \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s}^\varepsilon z\Big) p(z) dz \\
& + \int_0^t \int_{\mathbb{R}} G'\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^{u} \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon z\Big) z p(z) dz \partial_u \sigma_{0,s-u}^\varepsilon du \\
& + \sigma_{\mathrm{ou}} \int_0^t \int_{\mathbb{R}} G'\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^{u} \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon z\Big) p(z) dz \mathcal{K}^\varepsilon(s-u) dW_u \\
& + \frac{\sigma_{\mathrm{ou}}^2}{2} \int_0^t \int_{\mathbb{R}} G''\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^{u} \mathcal{K}^\varepsilon(s-v) dW_v + \sigma_{0,s-u}^\varepsilon z\Big) p(z) dz \mathcal{K}^\varepsilon(s-u)^2 du
\end{aligned}$$

and

$$
\begin{aligned}
G(Z_s^\varepsilon) &= G\Big(\sigma_{\rm ou}\int_{-\infty}^s \mathcal{K}^\varepsilon(s-v)dW_v\Big)\\
&= \int_{\mathbb{R}} G\Big(\sigma_{\rm ou}\int_{-\infty}^s \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,0}^\varepsilon z\Big)p(z)dz\\
&= \int_{\mathbb{R}} G\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\Big)p(z)dz\\
&\quad + \int_0^s \int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon du\\
&\quad + \sigma_{\rm ou}\int_0^s \int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)dW_u\\
&\quad + \frac{\sigma_{\rm ou}^2}{2}\int_0^s \int_{\mathbb{R}} G''\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)^2du.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\psi_t^\varepsilon &= \int_0^t G(Z_s^\varepsilon)ds + \int_t^T \mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_t\big]ds\\
&= \Big[\int_{\mathbb{R}}\int_0^T G\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\Big)dsp(z)dz\Big]\\
&\quad + \int_0^t \Big[\int_u^T \int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon ds\Big]du\\
&\quad + \sigma_{\rm ou}\int_0^t \Big[\int_u^T \int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)ds\Big]dW_u\\
&\quad + \frac{\sigma_{\rm ou}^2}{2}\int_0^t \Big[\int_u^T \int_{\mathbb{R}} G''\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)^2ds\Big]du.
\end{aligned}
$$

This gives

$$
d\left\langle \psi^\varepsilon, W\right\rangle_t = \vartheta_t^\varepsilon dt, \tag{B.5}
$$

with

$$
\vartheta_t^\varepsilon = \sigma_{\rm ou}\int_t^T \int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-t}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-t)ds, \tag{B.6}
$$

which can also be written as stated in the Lemma. □

The important properties of the random process ϑ_t^ε are stated in the following lemma.

LEMMA B.2. *For any $t \in [0,T]$, we have*

$$
\vartheta_t^\varepsilon = \varepsilon^{1-H}\theta_t + \widetilde{\theta}_t^\varepsilon, \tag{B.7}
$$

where θ_t is deterministic

$$
\theta_t = \overline{\theta}(T-t)^{H-\frac{1}{2}}, \qquad \overline{\theta} = \frac{\sigma_{\rm ou}\left\langle G'\right\rangle}{\Gamma(H+\frac{1}{2})}, \tag{B.8}
$$

and $\widetilde{\theta}^\varepsilon_t$ is random but smaller than ε^{1-H}:

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\widetilde{\theta}^\varepsilon_t)^2\big]^{1/2} = 0. \tag{B.9}$$

Proof. Recall first from Eq. (2.8)

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}\Big(\frac{t}{\varepsilon}\Big), \qquad \mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}} e^{-s} ds\Big].$$

The expectation of ϑ^ε_t is then equal to

$$\mathbb{E}\big[\vartheta^\varepsilon_t\big] = \sigma_{\rm ou}\left\langle G'\right\rangle \int_0^{T-t} \mathcal{K}^\varepsilon(s)ds = \sigma_{\rm ou}\left\langle G'\right\rangle \sqrt{\varepsilon}\int_0^{(T-t)/\varepsilon} \mathcal{K}(s)ds.$$

Therefore the difference

$$\mathbb{E}\big[\vartheta^\varepsilon_t\big] - \varepsilon^{1-H}\theta_t = \sigma_{\rm ou}\left\langle G'\right\rangle \varepsilon^{1/2}\int_0^{(T-t)/\varepsilon} \mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})} ds$$

can be bounded by

$$\big|\mathbb{E}\big[\vartheta^\varepsilon_t\big] - \varepsilon^{1-H}\theta_t\big| \leq C\varepsilon^{1/2}, \tag{B.10}$$

uniformly in $t\in[0,T]$, for some constant C, because $\mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})}$ is in L^1.

We have

$$\begin{aligned}
{\rm Var}(\vartheta^\varepsilon_t) &= \sigma_{\rm ou}^2 \int_t^T ds \int_t^T ds' \mathcal{K}^\varepsilon(s-t)\mathcal{K}^\varepsilon(s'-t){\rm Cov}\big(\mathbb{E}\big[G'(Z^\varepsilon_s)|\mathcal{F}_t\big], \mathbb{E}\big[G'(Z^\varepsilon_{s'})|\mathcal{F}_t\big]\big) \\
&\leq \sigma_{\rm ou}^2 \Big(\int_t^T ds \mathcal{K}^\varepsilon(s-t){\rm Var}\big(\mathbb{E}\big[G'(Z^\varepsilon_s)|\mathcal{F}_t\big]\big)^{1/2}\Big)^2 \\
&= \sigma_{\rm ou}^2 \Big(\int_0^{T-t} ds \mathcal{K}^\varepsilon(s){\rm Var}\big(\mathbb{E}\big[G'(Z^\varepsilon_s)|\mathcal{F}_0\big]\big)^{1/2}\Big)^2.
\end{aligned}$$

The conditional distribution of Z^ε_t given $\mathcal{F}_0$ is Gaussian with mean

$$\mathbb{E}\big[Z^\varepsilon_t|\mathcal{F}_0\big] = \sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(t-u)dW_u$$

and variance

$${\rm Var}\big(Z^\varepsilon_t|\mathcal{F}_0\big) = (\sigma^\varepsilon_{0,t})^2 = \sigma_{\rm ou}^2 \int_0^t \mathcal{K}^\varepsilon(u)^2 du.$$

Therefore

$${\rm Var}\big(\mathbb{E}\big[G'(Z^\varepsilon_t)|\mathcal{F}_0\big]\big) = {\rm Var}\Big(\int_{\mathbb{R}} G'\big(\mathbb{E}\big[Z^\varepsilon_t|\mathcal{F}_0\big] + \sigma^\varepsilon_{0,t}z\big)p(z)dz\Big).$$

The random variable $\mathbb{E}\big[Z_t^\varepsilon|\mathcal{F}_0\big]$ is Gaussian with mean zero and variance

$$(\sigma_{t,\infty}^\varepsilon)^2 = \sigma_{\rm ou}^2 \int_t^\infty \mathcal{K}^\varepsilon(u)^2 du,$$

so that

$$\begin{aligned}
{\rm Var}\big(\mathbb{E}\big[G'(Z_t^\varepsilon)|\mathcal{F}_0\big]\big) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dzdz'p(z)p(z')\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u') \\
&\quad \times\Big[G'\big(\sigma_{t,\infty}^\varepsilon u+\sigma_{0,t}^\varepsilon z\big)-G'\big(\sigma_{t,\infty}^\varepsilon u'+\sigma_{0,t}^\varepsilon z\big)\Big] \\
&\quad \times\Big[G'\big(\sigma_{t,\infty}^\varepsilon u+\sigma_{0,t}^\varepsilon z'\big)-G'\big(\sigma_{t,\infty}^\varepsilon u'+\sigma_{0,t}^\varepsilon z'\big)\Big] \\
&\leq \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u')(u-u')^2 \\
&= \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2. \qquad \text{(B.11)}
\end{aligned}$$

Therefore

$$\begin{aligned}
{\rm Var}(\vartheta_t^\varepsilon)^{1/2} &\leq \|G''\|_\infty \sigma_{\rm ou}^2 \int_0^{T-t} ds\mathcal{K}^\varepsilon(s)\Big(\int_s^\infty du\mathcal{K}^\varepsilon(u)^2\Big)^{1/2} \\
&\leq \|G''\|_\infty \sigma_{\rm ou}^2 \varepsilon^{1/2} \int_0^{(T-t)/\varepsilon} ds\mathcal{K}(s)\Big(\int_s^\infty du\mathcal{K}(u)^2\Big)^{1/2}.
\end{aligned}$$

Since $\mathcal{K}(s) \leq 1\wedge Ks^{H-\frac{3}{2}}$, this gives

$$ {\rm Var}(\vartheta_t^\varepsilon)^{1/2} \leq C \begin{cases} \varepsilon^{1/2} & \text{if } H<3/4, \\ \varepsilon^{1/2}\ln(\varepsilon) & \text{if } H=3/4, \\ \varepsilon^{2-2H} & \text{if } H>3/4, \end{cases} \qquad \text{(B.12)}$$

uniformly in $t\in[0,T]$, for some constant C. This completes the proof of the lemma. □

The random term ϕ_t^ε defined by (4.7) has the form

$$\phi_{t,T}^\varepsilon = \mathbb{E}\Big[\int_t^T G(Z_s^\varepsilon)ds\big|\mathcal{F}_t\Big]. \qquad \text{(B.13)}$$

Here we write explicitly the argument T (maturity) as we compute the correlations of these random terms for different maturities.

LEMMA B.3.

1. *For any $t\leq T$, $\phi_{t,T}^\varepsilon$ is a zero-mean random variable with standard deviation of order ε^{1-H}:*

$$\varepsilon^{2H-2}\mathbb{E}[(\phi_{t,T}^\varepsilon)^2] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2(T-t)^{2H}, \qquad \text{(B.14)}$$

where σ_ϕ is defined by (4.10).

2. *The covariance function of $\phi^\varepsilon_{t,T}$ has the following limit for any $t \le T$, $t' \le T'$, with $t \le t'$:*

$$\varepsilon^{2H-2}\mathbb{E}[\phi^\varepsilon_{t,T}\phi^\varepsilon_{t',T'}] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2 (T-t)^H (T'-t')^H \mathcal{C}_\phi(t,t';T,T'), \tag{B.15}$$

where the limit correlation is

$$\mathcal{C}_\phi(t,t';T,T') = \frac{\int_0^\infty du\big[(u+r)^{H-\frac12} - u^{H-\frac12}\big]\big[(u+s)^{H-\frac12} - (u+q)^{H-\frac12}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac12} - u^{H-\frac12}\big]^2}, \tag{B.16}$$

with

$$q = \frac{t'-t}{\sqrt{(T-t)(T'-t')}}, \qquad r = \frac{\sqrt{T-t}}{\sqrt{T'-t'}}, \qquad s = \frac{T'-t}{\sqrt{(T-t)(T'-t')}}.$$

3. *As $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$, $t \le T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian random process $\phi_{t,T}$, $t \le T$, with mean zero and covariance $\varepsilon^{2(H-1)}\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T')$ for any $t \in [0,T]$, $t' \in [0,T']$, with $t \le t'$.*
4. *The fourth-order moments of $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$ are uniformly bounded: there exists a constant K_T independent of ε such that*

$$\sup_{t\in[0,T]} \mathbb{E}[(\phi^\varepsilon_{t,T})^4]^{1/4} \le K_T \varepsilon^{1-H}. \tag{B.17}$$

Note that the mean square increment of the limit process $\phi_{t,T}$ satisfies for $t, t+h \in [0,T]$:

$$\begin{aligned}\mathbb{E}\big[(\phi_{t,T} - \phi_{t+h,T})^2\big] &= \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac12)^2}\int_0^\infty du\big[(T-t-h+u)^{H-\frac12} - u^{H-\frac12}\big]^2 \\ &\quad -\big[(T-t+u)^{H-\frac12} - (u+h)^{H-\frac12}\big]^2 + \big[(u+h)^{H-\frac12} - u^{H-\frac12}\big]^2 \\ &= \frac{\sigma_{\rm ou}^2 (T-t)^{2H-1}}{\Gamma(H+\frac12)^2} h + o(h), \quad h \to 0. \end{aligned} \tag{B.18}$$

This shows that the limit Gaussian process $\phi_{t,T}$ has the same local regularity (as a function of t) as a standard Brownian motion. We also have for any $t < T \le T+h$:

$$\mathbb{E}\big[(\phi_{t,T+h} - \phi_{t,T})^2\big] = \frac{\sigma_{\rm ou}^2 (T-t)^{2H-2}}{(2-2H)\Gamma(H-\frac12)^2} h^2 + o(h^2), \quad h \to 0. \tag{B.19}$$

This shows that the limit Gaussian process $\phi_{t,T}$ is smooth (mean square differentiable) as a function of the maturity T.

Proof. Let us fix $T_0 > 0$. For $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \le T_0$, and $t \le t'$, the covariance of $\phi^\varepsilon_{t,T}$ is

$$\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) &= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\mathbb{E}\Big[\int_{t'}^{T'} G(Z^\varepsilon_s)ds\big|\mathcal{F}_{t'}\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\mathbb{E}\Big[\int_{t'}^{T'} G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\Big] \\
&= \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \mathrm{Cov}\big(\mathbb{E}\big[G(Z^\varepsilon_s)|\mathcal{F}_0\big], \mathbb{E}\big[G(Z^\varepsilon_{s'})|\mathcal{F}_0\big]\big).
\end{aligned}$$

Then, proceeding as in the proof of the previous lemma,

$$\mathrm{Var}(\phi^\varepsilon_{t,T}) \le \Big(\int_0^{T-t} ds \mathrm{Var}\big(\mathbb{E}\big[G(Z^\varepsilon_s)|\mathcal{F}_0\big]\big)^{1/2}\Big)^2 \le \|G'\|^2_\infty \Big(\int_0^{T-t} ds \sigma^\varepsilon_{s,\infty}\Big)^2.$$

Since $\mathcal{K}(s) \le 1 \wedge K s^{H-\frac{3}{2}}$, this gives

$$\mathrm{Var}(\phi^\varepsilon_{t,T}) \le C_{T_0} \varepsilon^{2-2H},$$

uniformly in $t \le T \le T_0$, for some constant C_{T_0}. More precisely, for $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \le T_0$, and $t \le t'$, we have

$$\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) &= \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') \\
\times \mathbb{E}\Big[G\Big(\sigma_{\mathrm{ou}} &\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + \sigma^\varepsilon_{0,s} z\Big) G\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + \sigma^\varepsilon_{0,s'} z'\Big)\Big].
\end{aligned}$$

Using the fact that $\langle G \rangle = 0$, we can write

$$\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = &\int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') \\
&\times \mathbb{E}\Big[\Big(G\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + \sigma^\varepsilon_{0,s} z\Big) - G(\sigma_{\mathrm{ou}} z)\Big) \\
&\quad \times \Big(G\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + \sigma^\varepsilon_{0,s'} z'\Big) - G(\sigma_{\mathrm{ou}} z')\Big)\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = &\int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') G'(\sigma_{\mathrm{ou}} z) G'(\sigma_{\mathrm{ou}} z') \\
&\times \mathbb{E}\Big[\Big(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + (\sigma^\varepsilon_{0,s} - \sigma_{\mathrm{ou}}) z\Big) \\
&\quad \times \Big(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + (\sigma^\varepsilon_{0,s'} - \sigma_{\mathrm{ou}}) z'\Big)\Big] + V^\varepsilon_3,
\end{aligned}$$

up to a term V_3^ε which is of order ε^{3-3H}:

$$
\begin{aligned}
V_3^\varepsilon \leq\ & 2\|G'\|_\infty \|G''\|_\infty \int_0^{T-t} ds \int_0^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dzdz' p(z)p(z') \\
&\times \mathbb{E}\Big[\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u + (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})z\Big)^2 \\
&\quad \times \Big|\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')dW_{u'} + (\sigma_{0,s'}^\varepsilon - \sigma_{\rm ou})z'\Big|\Big] \\
\leq\ & C\|G'\|_\infty \|G''\|_\infty \int_0^{T_0-t} ds \int_0^{T_0-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dzdz' p(z)p(z') \\
&\times \Big(\sigma_{\rm ou}^2\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)^2 du + (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})^2 z^2\Big) \\
&\quad \times \Big(\sigma_{\rm ou}^2\int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')^2 du' + (\sigma_{0,s'}^\varepsilon - \sigma_{\rm ou})^2 z'^2\Big)^{1/2} \\
\leq\ & C'\|G'\|_\infty \|G''\|_\infty \Big[\int_0^{T_0-t} ds \int_{\mathbb{R}} dz p(z) \Big((\sigma_{s,\infty}^\varepsilon)^2 + (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})^2 z^2\Big)\Big]^{3/2} \\
\leq\ & C'\|G'\|_\infty \|G''\|_\infty \Big[\int_0^{T_0-t} ds (\sigma_{s,\infty}^\varepsilon)^2 + (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})^2\Big]^{3/2}.
\end{aligned}
$$

Using $(\sigma_{s,\infty}^\varepsilon)^2 + (\sigma_{0,s}^\varepsilon)^2 = \sigma_{\rm ou}^2$ and

$$
\begin{aligned}
|\sigma_{\rm ou} - \sigma_{0,s}^\varepsilon| &= \sigma_{\rm ou}\Big(1 - \Big(\int_0^{s/\varepsilon} \mathcal{K}(u)^2 du\Big)^{1/2}\Big) = \sigma_{\rm ou}\Big(1 - \Big(1 - \int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du\Big)^{1/2}\Big) \\
&\leq \sigma_{\rm ou}\int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du \leq \sigma_{\rm ou}\Big(1 \wedge K\big(\frac{s}{\varepsilon}\big)^{2H-2}\Big),
\end{aligned}
\tag{B.20}
$$

where the fist inequality follows from $\sqrt{1-x} > 1-x$ for $0 \leq x \leq 1$, we get

$$
\begin{aligned}
V_3^\varepsilon &\leq C'\|G'\|_\infty \|G''\|_\infty \Big[\int_0^{T_0-t} ds 2\sigma_{\rm ou}(\sigma_{\rm ou} - \sigma_{0,s}^\varepsilon)\Big]^{3/2} \\
&\leq C''\|G'\|_\infty \|G''\|_\infty \varepsilon^{3-3H}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
{\rm Cov}(\phi_{t,T}^\varepsilon, \phi_{t',T'}^\varepsilon) =\ & \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dzdz' p(z)p(z') G'(\sigma_{\rm ou} z) G'(\sigma_{\rm ou} z') \\
&\times \Big(\sigma_{\rm ou}^2 \int_0^\infty \mathcal{K}^\varepsilon(s+u)\mathcal{K}^\varepsilon(s'+u) du + (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})(\sigma_{0,s'}^\varepsilon - \sigma_{\rm ou}) zz'\Big) + V_3^\varepsilon \\
=\ & V_1^\varepsilon \left\langle G' \right\rangle^2 + V_2^\varepsilon \sigma_{\rm ou}^2 \left\langle G'' \right\rangle^2 + V_3^\varepsilon,
\end{aligned}
$$

with

$$
\begin{aligned}
V_1^\varepsilon &= \sigma_{\rm ou}^2 \int_0^\infty du \Big(\int_0^{T-t} ds \mathcal{K}^\varepsilon(s+u)\Big)\Big(\int_{t'-t}^{T'-t} ds' \mathcal{K}^\varepsilon(s'+u)\Big), \\
V_2^\varepsilon &= \Big(\int_0^{T-t} ds (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou})\Big)\Big(\int_{t'-t}^{T'-t} ds' (\sigma_{0,s'}^\varepsilon - \sigma_{\rm ou})\Big).
\end{aligned}
$$

Using again (B.20) we find that

$$V_2^\varepsilon \le C\varepsilon^{4-4H},$$

while

$$V_1^\varepsilon = \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((T-t+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big) \times\big((T'-t+u)^{H-\frac{1}{2}} - (u+t'-t)^{H-\frac{1}{2}}\big)du\,\varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

Applying the change of variable

$$u \to (T-t)^{\frac{1}{2}}(T'-t')^{\frac{1}{2}}u$$

gives the first and second items of the lemma with

$$\sigma_\phi^2 = \frac{\sigma_{\rm ou}^2\,\langle G'\rangle^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du,$$

which is equivalent to (4.10).

In order to prove the third item we introduce

$$\check{\phi}_{t,T}^\varepsilon = \mathbb{E}\Big[\int_t^T Z_s^\varepsilon ds\big|\mathcal{F}_t\Big], \tag{B.21}$$

which is a Gaussian random process with mean zero and covariance, for $t\in[0,T]$, $t'\in[0,T']$, with $t\le t'$:

$$\begin{aligned}
{\rm Cov}\big(\check{\phi}_{t,T}^\varepsilon,\check{\phi}_{t',T'}^\varepsilon\big) &= \int_t^T ds\int_{t'}^{T'} ds'\mathbb{E}\Big[\mathbb{E}[Z_s^\varepsilon|\mathcal{F}_t]\mathbb{E}[Z_s^\varepsilon|\mathcal{F}_{t'}]\Big]\\
&= \int_t^T ds\int_{t'}^{T'} ds'\mathbb{E}\Big[\mathbb{E}[Z_s^\varepsilon|\mathcal{F}_t]\mathbb{E}[Z_s^\varepsilon|\mathcal{F}_t]\Big]\\
&= \sigma_{\rm ou}^2\int_0^{T-t} ds\int_{t'-t}^{T'-t} ds'\mathbb{E}\Big[\Big(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u\Big)\Big(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u)dW_u\Big)\Big]\\
&= \sigma_{\rm ou}^2\int_0^\infty du\Big(\int_0^{T-t} ds\mathcal{K}^\varepsilon(s+u)\Big)\Big(\int_{t'-t}^{T'-t} ds'\mathcal{K}^\varepsilon(s'+u)\Big).
\end{aligned}$$

Therefore, for $t_j \in [0,T_j]$, with $t_1\le\cdots\le t_n$, the random vector $(\varepsilon^{H-1}\langle G'\rangle\,\check{\phi}_{t_1,T_1}^\varepsilon,\dots,\varepsilon^{H-1}\langle G'\rangle\,\check{\phi}_{t_n,T_n}^\varepsilon)$ converges to a Gaussian random vector with mean 0 and covariance matrix $(\sigma_\phi^2(T_j-t_j)^H(T_l-t_l)^H\mathcal{C}_\phi(t_j,t_l;T_j,T_l))_{j,l=1}^n$. In other words, the random process $\varepsilon^{H-1}\langle G'\rangle\,\check{\phi}_{t,T}^\varepsilon$, $t\le T$, converges in the sense of finite-dimensional distributions to a Gaussian process $\phi_{t,T}$, $t\le T$, with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T')$, for $t\in[0,T]$, $t'\in[0,T']$, with $t\le t'$.

Moreover, we have

$$\mathrm{Var}\big(\check{\phi}^{\varepsilon}_{t,T}\big) = \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du\,(T-t)^{2H}\varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

Similarly,

$$\mathbb{E}\big[\check{\phi}^{\varepsilon}_{t,T}\phi^{\varepsilon}_{t,T}\big] = \frac{\sigma_{\rm ou}^2\,\langle G'\rangle}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du\,(T-t)^{2H}\varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

As a result,

$$\varepsilon^{2H-2}\mathbb{E}\big[(\phi^{\varepsilon}_{t,T} - \langle G'\rangle\,\check{\phi}^{\varepsilon}_{t,T})^2\big] \stackrel{\varepsilon\to 0}{\longrightarrow} 0,$$

and the random process $\varepsilon^{H-1}\langle G'\rangle\,\check{\phi}^{\varepsilon}_{t,T}$, $t\leq T$, converges in the sense of finite-dimensional distributions to a Gaussian process $\phi_{t,T}$, $t\leq T$, with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T')$ for $t\in[0,T]$, $t'\in[0,T']$, with $t\leq t'$. This gives the third item of the lemma.

To prove the fourth item of the lemma, we note that

$$\phi^{\varepsilon}_{t,T} = \frac{1}{2}\mathbb{E}\big[I^{\varepsilon}_T|\mathcal{F}_t\big] - \frac{1}{2}I^{\varepsilon}_t,$$

where I^{ε}_t is defined by (A.6). Therefore

$$\sup_{t\in[0,T]}\mathbb{E}\big[(\phi^{\varepsilon}_{t,T})^4\big] \leq \sup_{t\in[0,T]}\mathbb{E}\big[(I^{\varepsilon}_t)^4\big],$$

and the result follows from Lemma A.1, Eq. (A.7). ◻

Lemma B.4. *Let us define for any $t\in[0,T]$:*

$$\gamma^{\varepsilon}_t = \frac{1}{2}\int_0^t \big((\sigma^{\varepsilon}_s)^2 - \overline{\sigma}^2\big)\phi^{\varepsilon}_s ds, \tag{B.22}$$

as in (4.20). We have

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(\gamma^{\varepsilon}_t)^2\big]^{1/2} = 0. \tag{B.23}$$

Proof. Let us define for any $t\in[0,T]$:

$$\Gamma^{\varepsilon}_t = \int_t^T \big((\sigma^{\varepsilon}_s)^2 - \overline{\sigma}^2\big)\phi^{\varepsilon}_s ds. \tag{B.24}$$

By the definition (4.12) of ϕ^{ε}_s, we have

$$\Gamma^{\varepsilon}_t = 2\int_t^T ds\int_s^T du\mathbb{E}\big[G(Z^{\varepsilon}_s)G(Z^{\varepsilon}_u)|\mathcal{F}_s\big].$$

Therefore

$$\begin{aligned}
\mathbb{E}\Big[(\Gamma_t^\varepsilon)^2\Big] &= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\Big[\mathbb{E}\big[G(Z_s^\varepsilon)G(Z_u^\varepsilon)|\mathcal{F}_s\big]\mathbb{E}\big[G(Z_{s'}^\varepsilon)G(Z_{u'}^\varepsilon)|\mathcal{F}_{s'}\big]\Big] \\
&= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\Big[G(Z_s^\varepsilon)G(Z_u^\varepsilon)\mathbb{E}\big[G(Z_{s'}^\varepsilon)G(Z_{u'}^\varepsilon)|\mathcal{F}_s\big]\Big] \\
&= \int_t^T ds \int_s^T du \mathbb{E}\Big[G(Z_s^\varepsilon)G(Z_u^\varepsilon)\mathbb{E}\big[\big(\int_s^T G(Z_{s'}^\varepsilon)ds'\big)^2\big|\mathcal{F}_s\big]\Big] \\
&= \int_t^T ds \mathbb{E}\Big[G(Z_s^\varepsilon)\mathbb{E}\big[\int_s^T G(Z_u^\varepsilon)du\big|\mathcal{F}_s\big]\mathbb{E}\big[\big(\int_s^T G(Z_{s'}^\varepsilon)ds'\big)^2\big|\mathcal{F}_s\big]\Big] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big|\mathbb{E}\big[\big(\int_s^T G(Z_{s'}^\varepsilon)ds'\big)^2\big|\mathcal{F}_s\big]\Big|^{3/2}\Big] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big|\int_s^T G(Z_{s'}^\varepsilon)ds'\Big|^3\Big] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big(\int_s^T G(Z_{s'}^\varepsilon)ds'\Big)^4\Big]^{3/4},
\end{aligned}$$

where we in the first inequality used that

$$\Big|\mathbb{E}\big[\int_s^T G(Z_u^\varepsilon)du\big|\mathcal{F}_s\big]\Big| \leq \Big|\mathbb{E}\big[\big(\int_s^T G(Z_u^\varepsilon)du\big)^2\big|\mathcal{F}_s\big]\Big|^{1/2},$$

which follows from the conditional version of Jensen's inequality. It follows by Lemma A.1 that $\mathbb{E}\big[(\Gamma_t^\varepsilon)^2\big]$ is smaller than $K'\varepsilon^{3-3H}$ for some constant K'. This proves

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\Gamma_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.25}$$

Note that γ_t^ε defined by (4.20) is related to Γ_t^ε through

$$\gamma_t^\varepsilon = 2\left(\Gamma_0^\varepsilon - \Gamma_t^\varepsilon\right),$$

therefore Eq. (B.25) also implies

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\gamma_t^\varepsilon)^2\big]^{1/2} = 0,$$

which is the desired result. □

Lemma B.5. *Let us define for any $t \in [0,T]$:*

$$\eta_t^\varepsilon = \varepsilon^{1-H}\int_0^t \big(\sigma_s^\varepsilon - \widetilde{\sigma}\big)ds, \tag{B.26}$$

as in (4.22). We have

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\eta_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.27}$$

Proof. By Lemma 3.1,

$$\begin{aligned}
\mathbb{E}\big[(\eta_t^\varepsilon)^2\big] &= \varepsilon^{2-2H}\mathbb{E}\Big[\Big(\int_0^t \big(\sigma_s^\varepsilon - \widetilde{\sigma}\big)ds\Big)^2\Big] \\
&= \varepsilon^{2-2H}\int_0^t\int_0^t \mathrm{Cov}\big(F(Z_s^\varepsilon), F(Z_{s'}^\varepsilon)\big)dsds' \\
&= \varepsilon^{2-2H}\big(\langle F^2\rangle - \langle F\rangle^2\big)\int_0^t\int_0^t \mathcal{C}_\sigma\big(\frac{s-s'}{\varepsilon}\big)dsds' \\
&\leq K\varepsilon^{2-2H}\int_0^T\int_0^T \big(\frac{|s-s'|}{\varepsilon}\big)^{2H-2}dsds' \\
&\leq K'\varepsilon^{4-4H},
\end{aligned}$$

for some constants K, K', because s^{2H-2} in integrable over $(0, T)$, which gives the desired result. □

Lemma B.6. *Let us define for any $t \in [0, T]$:*

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1-H}}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds, \tag{B.28}$$

as in (4.21). We have

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(\kappa_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.29}$$

Proof. The proof is similar to the one of Lemma B.5. □

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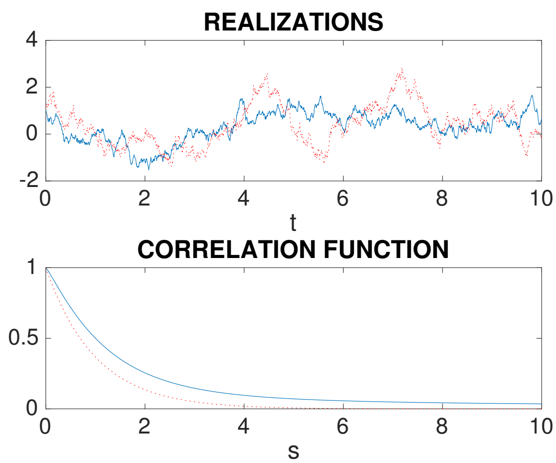

FIG. 2.1. *The top plot shows a realization, $Z_t^\varepsilon, \quad t \in (0,10)$, of the fractional OU process with Hurst index $H = 0.6$ and correlation time $\varepsilon = 1$ (blue solid line) and a realization of the standard OU process with $H = 1/2$ and $\varepsilon = 1$ (red dashed line). The trajectories are more regular when H is larger. The bottom plot shows the corresponding correlation functions, $\mathcal{C}_Z(s)$, and the "heavy" tail of the blue solid line of the case $H = 0.6$ gives the long-range property.*

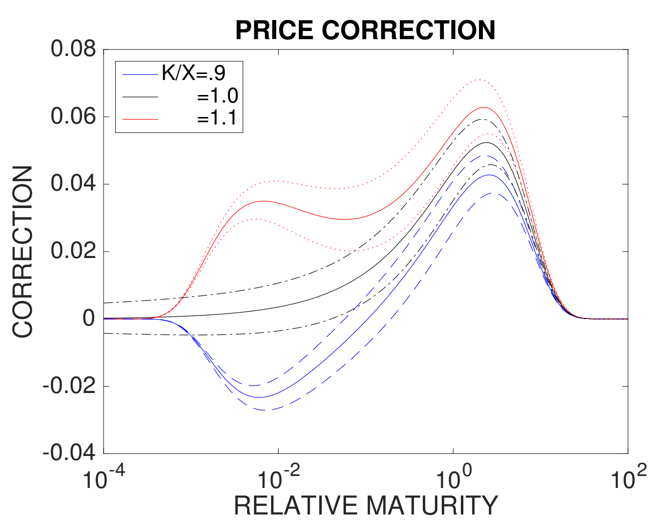

FIG. 5.1. *Price correction as function of relative time to maturity $\tau/\bar{\tau}$. The three solid lines correspond (from bottom to top) to the mean price correction for $K/X = 0.9$, 1.0, and 1.1 respectively. The dashed/dotted lines correspond to the mean $\pm$ one standard deviation. Here $H = 0.6, a_F = 0.1$, and $\left((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi\right) = 0.04$.*

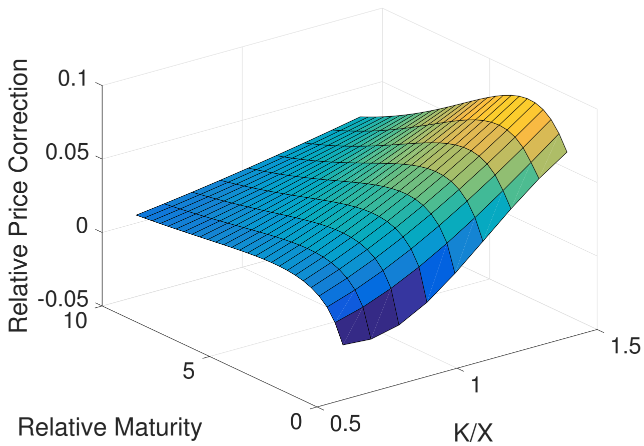

FIG. 5.2. *The price correction surface as function of relative time to maturity $\tau/\bar{\tau}$ and moneyness K/X. The parameters are as in Figure 5.1.*

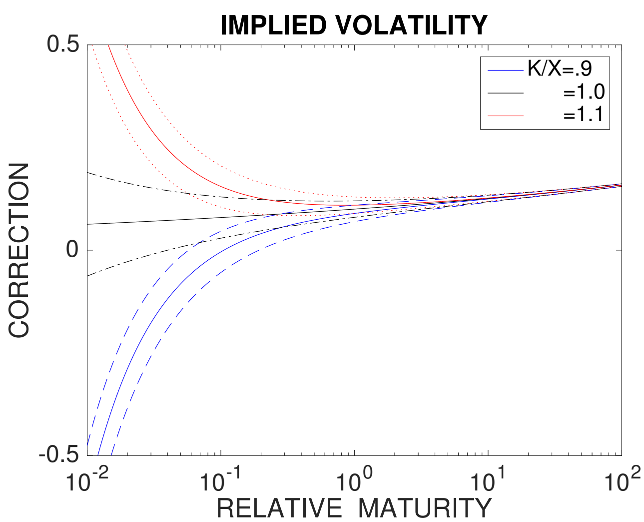

FIG. 5.3. *The implied volatility correction as function of relative time to maturity $\tau/\bar{\tau}$. The three solid lines correspond (from bottom to top) to the mean implied volatility correction for $K/X =$ 0.9, 1.0, and 1.1 respectively. The dashed/dotted lines correspond to the mean $\pm$ one standard deviation.*

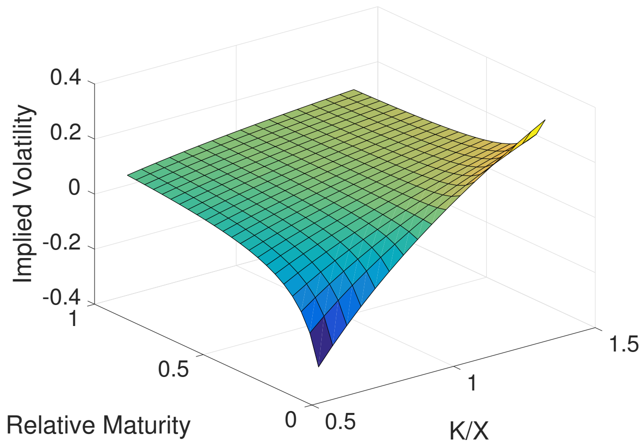

FIG. 5.4. *The mean implied volatility correction surface as function of relative time to maturity $\tau/\bar{\tau}$ and moneyness K/X. The parameters are as in Figure 5.3.*

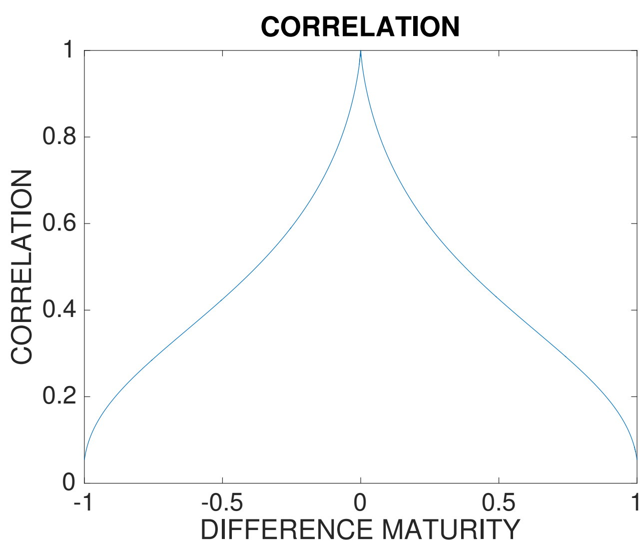

FIG. 6.1. *Autocovariance function of the t-T process $\psi_1(\tau; 1)$ as function of relative time to maturity separation $\Delta_1 = (\tau - \tau')/|\tau + \tau'|$ with $H = 0.6$. The correlation decays approximately linearly at the origin and rapidly as one of the times to maturity goes to zero.*

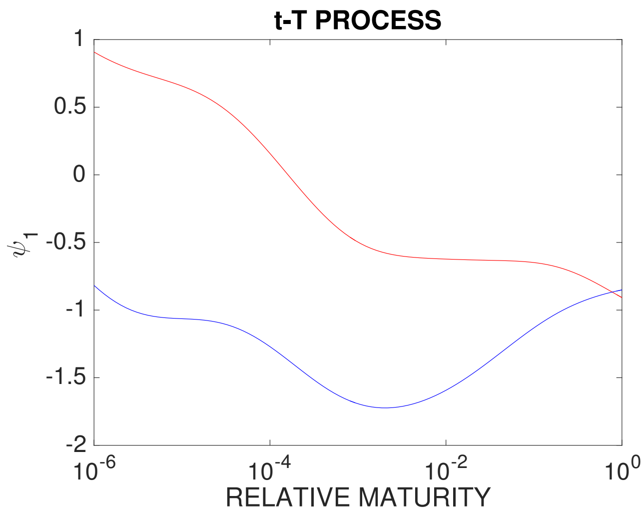

FIG. 6.2. *Realizations of the process $\psi_1(\tau; 1)$ as function of time to maturity τ for fixed maturity $T = 1$ with $H = 0.6$.*

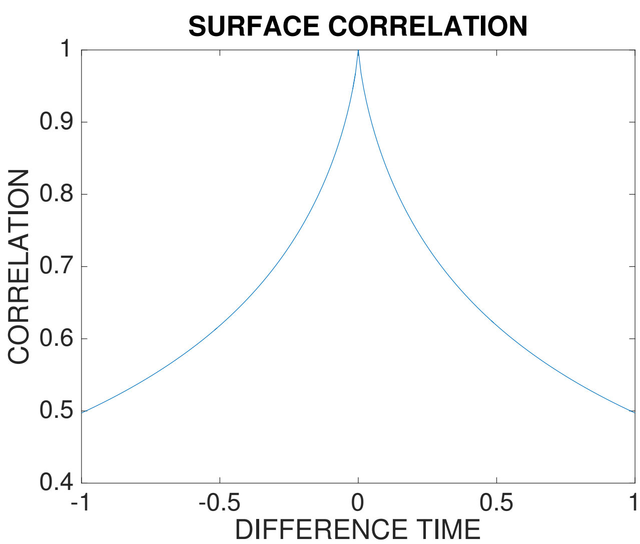

FIG. 6.3. *Autocovariance function of the t-T process $\psi_2(t; 1)$ as function of time $t' - t$ for fixed time to maturity $\tau = 1$ with $H = 0.6$. On the short scales the process decorrelates as a Markov process and on the long scales it exhibits long-range correlations.*

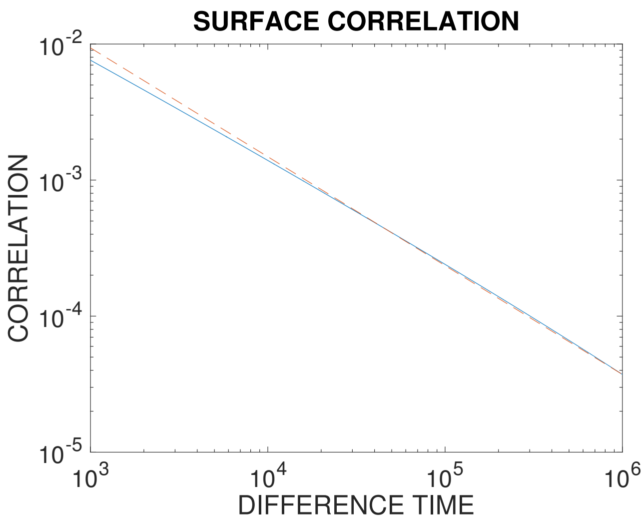

Fig. 6.4. *Autocovariance function of the t-T process $\psi_2(t;1)$ as in Figure 6.3, but on a log-log scale with the dashed line showing the decay $|t'-t|^{2H-2}$.*

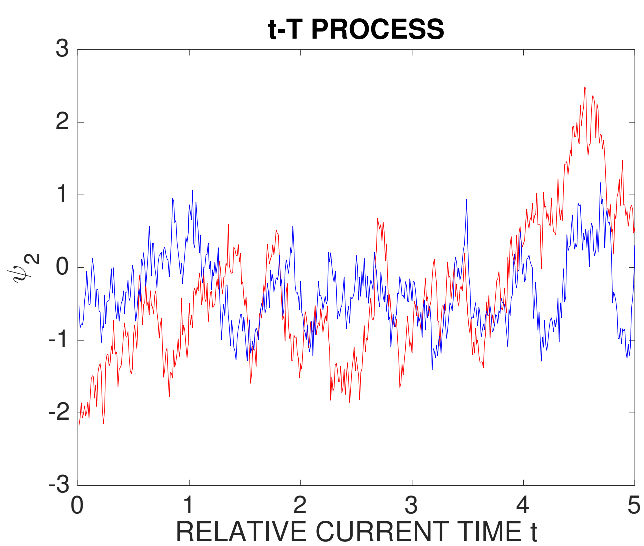

Fig. 6.5. *Realizations of the process $\psi_2(t;1)$ with $H = 0.6$.*

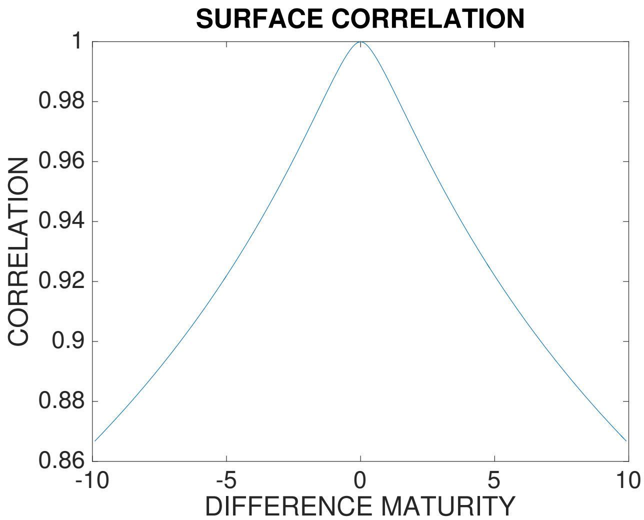

FIG. 6.6. *Autocovariance function of the t-T process $\psi_3(\tau;1)$ as function of the relative time to maturity separation $\Delta_3 = (\tau - \tau')/(\tau \wedge \tau')$ with $H = 0.6$. Note that the correlation function exhibits slow decay.*

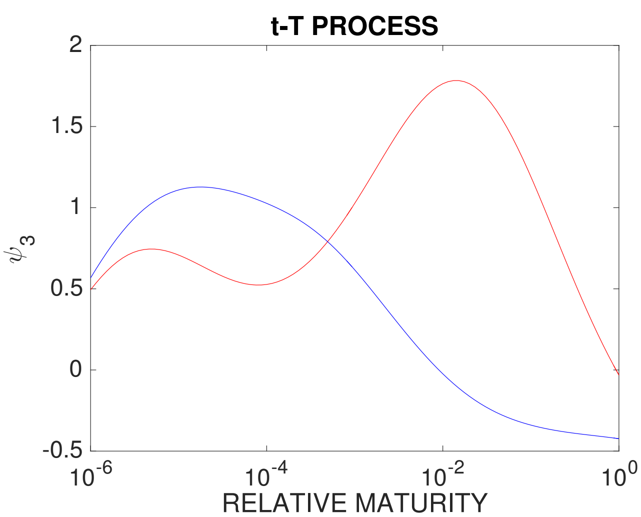

FIG. 6.7. *Realizations of the process $\psi_3(\tau;1)$ for fixed current time $t = 1$ and $H = 0.6$, with the smooth and slow decay of the correlations giving a smooth time to maturity dependence.*