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CAN MIXING MATERIALS MAKE ELECTROMAGNETIC SIGNALS TRAVEL FASTER?*

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Abstract. Suppose that a composite is constructed from two phases and that we propagate an electromagnetic signal through it. The velocity of the signal in the composite depends on the microstructure. What microstructures are associated with the maximum and minimum speeds of the electromagnetic signal? Here we show that the group velocity of a pulse can be higher in the composite than in either of the two phases. We derive sharp bounds for the relative increase and decrease in the group velocity of the electromagnetic signal in the composite and give the associated optimal microstructures. We also find that a pulse in a composite can have substantially smaller dispersion and, at the same time, substantially larger group velocity than pulses traveling in the pure phases.

Key words. electromagnetic wave propagation, group velocity, homogenization, effective parameters, dispersion

AMS subject classifications. 35Q60, 35L05, 78-02

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1. Introduction.

For electromagnetic wave propagation, a lot is known about the effective properties of materials in the time harmonic case [3, 13, 14, 16]. Expressions for the effective dielectric tensor $\varepsilon^*(\omega)$ often give an accurate description of a material's response to a field that varies at a constant frequency ω. However, surprisingly little is known about the propagation of signals or pulses despite its importance in applications.

A pulse is a superposition of waves with a range of frequencies. If the range of frequencies is narrow (so that the pulse is spatially broad), then the envelope of the pulse travels with a velocity called the group velocity. It is generally regarded as a good approximation to the velocity at which the energy of a pulse travels and to the velocity at which information (at that frequency range) can be propagated. However, there are examples (see [17, 18, 24] and references therein) of superluminal and even negative group velocities, corresponding to the peak of the wave leaving a region before it even enters that region. Here, we consider the group velocity of a two phase *composite* in the quasi-static limit, where the microstructure is much smaller than the characteristic wavelength, and ask the question of whether the group velocity in the composite can be higher than in either of the constituent phases. We show that it can, and by a large factor, depending on the properties of the phases. It is very surprising that such a phenomenon can occur at all, since normally energy is scattered and thus one would expect the energy to travel more slowly in the composite. The key point is that the group velocity depends on both the value of the refractive index and the dispersion. By combining one phase with high refractive index and low dispersion

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with another phase with low refractive index and high dispersion, the composite can be made to exhibit a comparatively low refractive index and low dispersion and hence a relatively large group velocity. In particular, this can be realized when the dispersion relation of both phases is described by a Lorentzian model and when one phase is close to resonance. In the electromagnetic case, the "speed-up" is largest in a laminate microgeometry but can be made large also in isotropic microstructures, described by an assemblage of spheres corresponding to the Maxwell–Garnett approximation formula. The group velocity can also be smaller in the composite than in the phases, and we derive bounds for the possible "slow-down." These bounds are attained by geometries similar to those that realize the optimal bounds for the speed-up, but with a different orientation of the laminate and with the roles of the phases interchanged in the sphere assemblage. A related phenomenon is the reduction of the velocity of sound in bubbly water [6].

Bounds on the group velocity are of interest, for example, when constructing a composite wave guide or in certain device applications. The characteristics of devices like switches, lasers, and integrated optical filters can be tailored using finely layered materials. In [21] a separation in group velocity for different wave modes is exploited for passive pulse shaping; in [5, 8] fine layering is exploited for this purpose. Recently, there has been a strong interest in *single* phase materials with abnormally low group velocity [23]. An unusual quantum-mechanical interference effect gives an index of refraction associated with a very low group velocity. In such a material one obtains an enormous spatial compression of the electromagnetic pulse due to the low group velocity. In principle this could be used for parallelization of information. Page et al. [19] consider the problem of an appropriate *definition* of group velocity in the case of acoustic wave propagation, with strong scatterers and pulse length on the *same* scale as the scatterers. This has been a long-standing open question. In experiments they obtain a large slow-down using a medium similar to a Maxwell–Garnett model. In addition to the group velocity, the dispersion of a propagating pulse is important. In [22] Santosa and Symes derive an effective medium in the case of waves propagating in a periodic medium with a small cell size. For a one dimensional medium they obtain explicit formulas that characterize the dispersive effects and illustrate these with numerical calculations. Here, we show that it is possible to construct a composite with a group velocity that is higher than the group velocities of the constituent phases and, moreover, a dispersion that is smaller than the dispersion in either phase. We illustrate this with a numerical calculation. For numerical examples of pulse propagation in general linear and nonlinear elastic media, see the results of Fogerty and LeVeque [7] and LeVeque [12].

We start our discussion by reviewing the expression for the group velocity, in section 2. In section 3, we derive the *upper* bound for the group velocity in a two phase composite when the volume fraction of the constituents is a free parameter. We derive the associated bounds when the volume fraction is fixed, in section 4. Finally, in section 5, we adapt the analysis to obtain *lower* bounds for the group velocity.

2. Preliminaries. We consider pulse propagation in a dielectric composite of two isotropic phases that is subject to some time dependent field $\mathbf{E}(t)$. We work in the quasi-static regime, assuming that the time variation of the electric field is sufficiently slow such that the curl of the electric field varies on a scale large relative to the microstructure of the material. A main feature of a propagating pulse or wave packet is its group velocity. Following Jackson [9] we first derive an expression for this velocity. We consider one dimensional wave propagation along the spatial dimension

x and let the wave pulse be characterized by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} dk, \tag{1}$$

with k being the wave number and ω the (temporal) frequency. If the distribution $A(k)$ is concentrated in a relatively narrow set about the wave number k_0, then

$$u(x,t) \approx \frac{e^{i(k_0\omega^{'}(k_0) - \omega(k_0))t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ik(x - \omega^{'}(k_0)t)} dk.$$

This shows that, up to a phase factor, the wave pulse travels approximately undistorted in shape, with a velocity characterized by the group velocity v_g:

$$v_g(k_0) \equiv \omega^{'}(k_0) = \frac{d\omega}{dk}(k_0). \tag{2}$$

The phase velocity of a particular mode is

$$v_p(k) \equiv \frac{\omega(k)}{k}. \tag{3}$$

For electromagnetic waves in a medium with relative magnetic permeability $\mu \equiv 1$ the relation between ω and k is given by

$$\omega(k) = \frac{ck}{\sqrt{\varepsilon^*(k)}}, \tag{4}$$

with c being the velocity of light in a vacuum and ε^* the (effective) relative dielectric permittivity of the medium.

In the next section we present bounds for the group velocity of the composite. The phases are characterized in terms of their dielectric permittivities $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$; here these are expressed as functions of the temporal frequency ω. The effective dielectric permittivity ε^* is determined in terms of these. It follows from (2) and (4) that

$$v_g(\omega_0) = \frac{c}{\sqrt{\varepsilon^*(\omega_0)} + \frac{\omega_0}{2\sqrt{\varepsilon^*(\omega_0)}} \frac{d\varepsilon^*(\omega_0)}{d\omega_0}}. \tag{5}$$

Thus, the group velocity depends on both the effective electrical permittivity and its variation with frequency. The idea underpinning the results of the next section is that by combining a material with high permittivity but slow frequency variation with a material with low permittivity but high frequency variation one might obtain a composite which has a comparatively low permittivity and a comparatively slow frequency variation, therefore having a higher group velocity than either phase.

3. An upper bound for the group velocity.

3.1. Summary of results. We present a bound for the group velocity of the dielectric composite for a general anisotropic medium, and also the bound when the medium is constrained to be isotropic.

The composite is constructed from two isotropic dielectric phases. The parameters

$$\varepsilon_i(\omega_0) \quad \text{and} \quad \frac{d\varepsilon_i}{d\omega}(\omega_0) \tag{6}$$

characterize material i, with $i \in \{1,2\}$ and ω_0 being the frequency at which we evaluate the group velocity, which we call the center frequency. The phases are labeled so that $\varepsilon_1(\omega_0) \geq \varepsilon_2(\omega_0)$, and we let p denote the volume fraction of material one. We assume the following:

(i) the quasi-static regime, i.e., that the scale of the microstructure is much smaller than the wavelength (yet large compared to the mean free path of electrons);
(ii) that the relative magnetic permeability is constant: $\mu \equiv 1$;
(iii) that the dielectric permittivity is *real*;
(iv) $d\varepsilon_i/d\omega > 0$.

Thus, we do not consider damping or anomalous dispersion effects, and the relationship between ω and k is given by (4).

Our main interest lies in the speed-up of the composite relative to the constituents. Let $\{v_1, v_2, v^*\}$ be, respectively, the group velocities of the phases and the composite, and define

$$G = \min\left[\frac{v^*}{v_1}, \frac{v^*}{v_2}\right]. \tag{7}$$

We want to maximize this quantity with respect to the *microgeometry* of the composite and the *volume fraction*. We find the following.

THEOREM 3.1. *Under assumptions* (i)–(iv) *stated above,*

$$G \leq \mathcal{G}(h,\beta) = \begin{cases} \frac{(1-h)\beta}{2\sqrt{h(1-\beta)(\beta-h)}} & \text{for } \frac{2h}{1+h} \leq \beta \leq \sqrt{h}, \\ \frac{1-h}{2\sqrt{(1-\beta)(\beta-h)}} & \text{for } \sqrt{h} \leq \beta \leq \frac{1+h}{2}, \\ 1 & \text{otherwise}, \end{cases} \tag{8}$$

with

$$h = \frac{\varepsilon_2(\omega_0)}{\varepsilon_1(\omega_0)} \leq 1, \tag{9}$$

$$\beta = \frac{[\frac{\omega}{2\varepsilon_1}\frac{d\varepsilon_1}{d\omega}]_0 + 1}{[\frac{\omega}{2\varepsilon_2}\frac{d\varepsilon_2}{d\omega}]_0 + 1}.$$

The notation $[\cdot]_0$ indicates a quantity evaluated at ω_0. The result (8) shows that the bound on the speed-up depends only on the dielectric contrast and on a measure of dispersion contrast. The result is derived in section 3.3, and the analysis we present there gives bounds also for v^*/v_1 and v^*/v_2. Note that h measures the contrast in the dielectric permittivity, while β is a measure of the contrast in dispersion.

The bound (8) is realized by waves propagating in a laminated material, with the layer interfaces normal to the direction of propagation and parallel to the electric field. The optimal volume fraction when $2h/(1+h) < \beta < (1+h)/2$ is

$$p = \frac{h(1-2\beta+h)}{(1-h)(\beta-h)}. \tag{10}$$

If $\beta \geq (1+h)/2$, then it is best to use only phase 2 ($p = 0$), and if $\beta \leq 2h/(1+h)$, it is best to use only phase 1. We also find that

$$\max_{\beta} \mathcal{G}(h,\beta) = \mathcal{G}(h,\sqrt{h}) = \frac{1+\sqrt{h}}{2h^{1/4}} \sim \frac{1}{2h^{1/4}} \qquad \text{as } h \downarrow 0. \tag{11}$$

At this maximal speed-up, $p = \sqrt{h}/(1+\sqrt{h})$, $v_1 = v_2$, and the effective dielectric permittivity of the laminate is $\varepsilon^* = \sqrt{h}\varepsilon_1$. Thus, by choosing h small enough, we can construct a composite where pulses travel arbitrarily faster than they do in either of the two phases but still slower than the speed of light in a vacuum. The volume fraction of the geometry realizing the bound is $p \sim \sqrt{h}$. Hence, large relative speed-ups are obtained by doping phase 2 with a small concentration of phase 1, with the ratio of the dielectric permittivities, h, being small. Let $\{v_{p,1}, v_{p,2}\}$ denote the phase velocities of the constituents; then we can write

$$\max_{\beta} \mathcal{G}(h,\beta) = \mathcal{G}(h,\sqrt{h}) \sim \frac{1}{2h^{1/4}} = \frac{\sqrt{v_{p,2}/v_{p,1}}}{2} \qquad \text{as } h \downarrow 0.$$

Similarly to (11) we find

$$\max_{h} \mathcal{G}(h,\beta) = \mathcal{G}(\beta^2,\beta) \sim \frac{1}{2\sqrt{\beta}} \qquad \text{as } \beta \downarrow 0. \tag{12}$$

Hence, a large speed-up can be achieved in the limit of large dispersion contrast.

If we constrain the composite to be isotropic, we find the following.

LEMMA 3.2. *Under assumptions* (i)–(iv) *stated above,*

$$G \le \mathcal{G}_d(h,\beta) \le \mathcal{G}_d(h,\sqrt{h}) \sim \frac{1}{2h^{1/4}} \qquad \textit{as } h \downarrow 0. \tag{13}$$

The subscript $d \in \{2,3\}$ denotes the spatial dimension, and $\mathcal{G}_d$ is given in sections 3.4 and 3.5. The upper bound in (13) is attained with $p \sim 2\sqrt{h}$ for $d = 2$, and $p \sim (3/2)\sqrt{h}$ for $d = 3$. In both two and three space dimensions, the bounds are attained by composites realizing the Hashin–Shtrikman bounds [10], namely, the Hashin sphere assemblage [11]. That is, the composite consists of densely packed coated spheres which all have the same volume fractions of the two phases and which range in size to the infinitesimal (so they can be packed to fill all space). The spheres have a core made up of phase 1 and are coated with phase 2. In Figure 1 we plot $\mathcal{G}(h,\beta)$ and $\mathcal{G}_3(h,\beta)$ for a range of parameter values. Note that the bounds on the speed-up almost coincide in the isotropic (dashed lines) and anisotropic (solid lines) cases.

For a given ratio of the dielectric permittivities h, the value of the parameter β determines whether we can construct a composite having a larger group velocity than both of the phases, that is, $\mathcal{G} > 1$. In Figure 2 we show, as a function of h, the range of β values for which we can construct such a composite. These are the sets enveloped by the solid, dashed, and dotted lines corresponding to the anisotropic and isotropic ($d = 3$ and $d = 2$) cases, respectively. The more interesting parameter regime, h small, is associated with a relatively large range of β values. A relative speed-up can only be obtained when $0 < \beta < 1$, which requires that

$$\frac{1}{\varepsilon_1(\omega_0)}\frac{d\varepsilon_1(\omega_0)}{d\omega} < \frac{1}{\varepsilon_2(\omega_0)}\frac{d\varepsilon_2(\omega_0)}{d\omega}.$$

3.2. Illustration with Lorentzian phases. Motivated by the above results we consider some specific models for the dielectric permittivities and seek to construct a composite such that it is associated with a large speed-up. We discuss the anisotropic case and thus let the medium be laminated in the direction parallel to the electric field and orthogonal to the propagation direction.

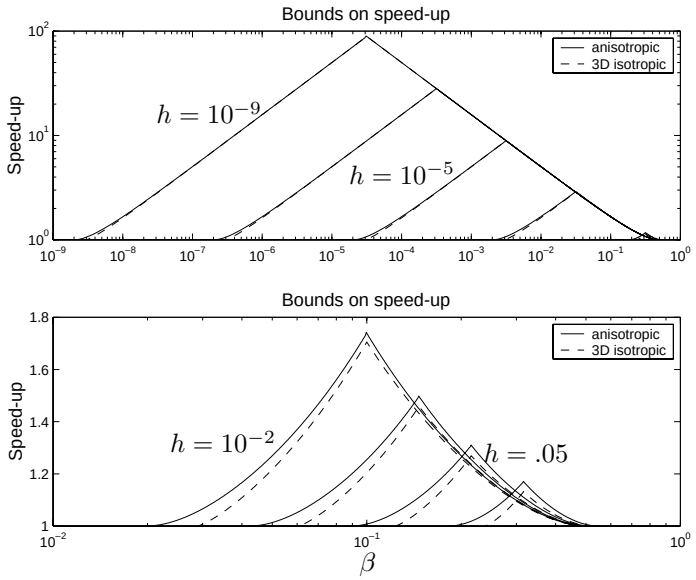

FIG. 1. *The figure shows the optimal speed-up as a function of β for a range of values of the ratio of the dielectric permittivities h. The solid lines correspond to the anisotropic case and the dashed line to the isotropic bound. Note that these almost coincide. In the top plot we use $h \in \{10^{-9}, 10^{-7}, 10^{-5}, 10^{-3}, 10^{-1}\}$ and obtain the largest speed-ups for small h. In the bottom plot we use $h \in \{.01, .02, .05, .1\}$.*

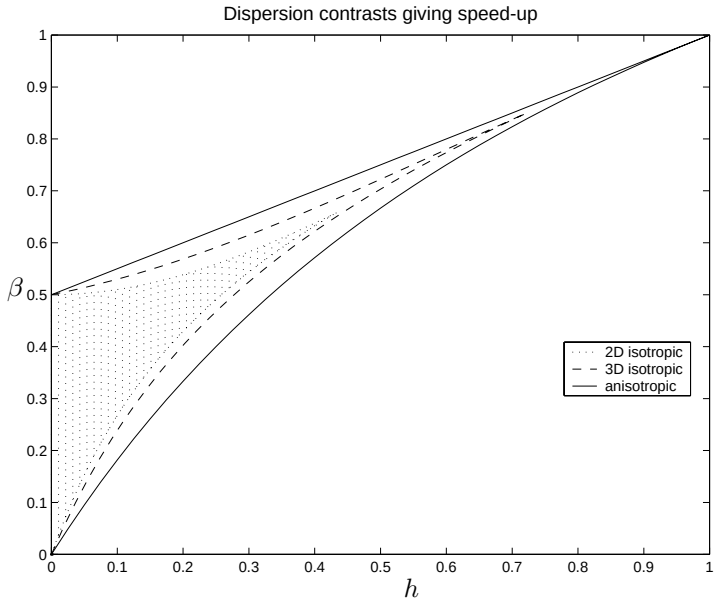

FIG. 2. *The figure shows the set of β values for which a composite can be constructed that has a larger group velocity than* both *of the phases. Note that the set depends only on the value of h. We show the set when we consider, respectively, a general anisotropic geometry for the composite (outlined by the solid lines) and an isotropic geometry in, respectively, 3D (dashed lines) and 2D (dotted area). In the latter two cases the set is empty for large h.*

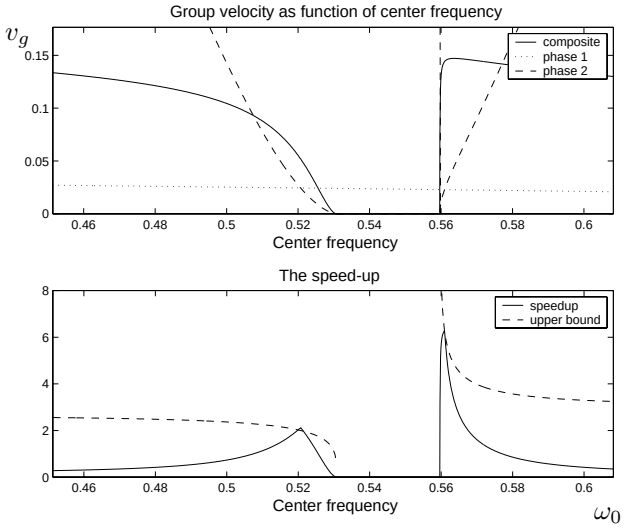

FIG. 3. *The top plot shows the group velocities as a function of the center frequency for a composite of materials defined by the Lorentzian model. The dotted, dashed, and solid lines correspond, respectively, to materials one, two, and the composite. In the bottom plot we show the speed-up of the composite relative to the phases. The dashed line is the asymptotic upper bound on the speed-up for the given (small) ratio of the dielectric permittivities. Note that the bound is attained when the group velocities of the phases coincide.*

Consider the following models for the dielectric permittivities:

$$\varepsilon_1(\omega) = 1 + \frac{\Delta^{-1}}{5/4 - \omega^2}, \tag{14}$$

$$\varepsilon_2(\omega) = 1 + \frac{\sqrt{\Delta}}{1/4 + \sqrt{\Delta} - \omega^2}, \tag{15}$$

with Δ a small parameter. Both of these models have the standard Lorentzian form given in (7.129) of [9]:

$$\varepsilon_1(\omega) = 1 + \frac{\omega_p^2}{\omega_1^2 - \omega^2 - i\omega\gamma},$$

with $\gamma = 0$, that is, no damping. Note that for $1/4 + \sqrt{\Delta} < \omega < \sqrt{1/4 + 2\sqrt{\Delta}}$ the dielectric permittivity of phase 2, ε_2, is negative, and the group velocity as given by (2) becomes purely imaginary. For these frequency modes the wave is being damped, and we *define* the group velocity to be zero. We denote the center frequency of the propagating pulse by ω_0 and choose the volume fraction of the first material to be $p = p(\omega_0) = \sqrt{\varepsilon_2(\omega_0)/\varepsilon_1(\omega_0)} = \sqrt{h}$. The effective dielectric permittivity of the composite in the direction of the electric field (parallel to the layers) is $\varepsilon^* = p\varepsilon_1 + (1-p)\varepsilon_2$.

In the top plot of Figure 3 we show the group velocities as a function of ω_0. The group velocity is normalized by the speed of light c. The displayed frequency range is close to the resonance frequency of the second material. The "nonpropagating" regime corresponds to the frequencies at which the group velocity of phase 2, shown with the dashed line, is zero. We let $\Delta = 10^{-3}$ and thus are doping phase 2 with a material which has a high dielectric permittivity and low dispersion. The group velocity of

phase 1 is shown with a dotted line. The bottom plot shows the actual speed-up as a function of ω_0. The dashed line is the bound for the speed-up in the small h limit, that is, $1/(2h^{1/4})$. Recall that h is the ratio of the dielectric permittivity of the phases. For the models we use $h = \mathcal{O}(\sqrt{\Delta})$. The figure shows that when the group velocities of the phases coincide, the realized speed-up approximately equals the bound. If the group velocity depends sensitively on ω_0, the material is dispersive. From the figure we see that a large speed-up is obtained by combining one phase with a large dielectric permittivity and low dispersion with another phase that has a relatively low dielectric permittivity and relatively high dispersion. The composite can then be made to exhibit a comparatively low dielectric permittivity and low dispersion and hence a large group velocity.

Next, we examine the dispersion and actual shape of a pulse transmitted, respectively, through the phases and the composite. We let the halfspace $x > 0$ be occupied by the dielectric medium. A wave pulse $v(x,t)$ is impinging upon the medium at $x = 0$, arriving from the vacuous halfspace $x < 0$. We let the impinging pulse or wave packet traveling in the positive x-direction be defined by

$$v(0,t) = e^{-t^2/(2T^2)} \cos(\overline{\omega}_0 t), \tag{16}$$

with T the temporal support of the initial pulse. Thus, the pulse is a (Gaussian) amplitude modulated cosine wave. Its speed of propagation is approximately the group velocity evaluated at the "center frequency" $\overline{\omega}_0$. Our objective is to characterize the way in which the pulse is being degraded by dispersion as it propagates into the medium. We normalize by the speed of light in a vacuum through the change of variables:

$$\begin{aligned} \omega_0 &= \frac{\overline{\omega}_0}{c}, \\ z &= tc, \\ u(x,z) &= v\left(x, \frac{z}{c}\right). \end{aligned} \tag{17}$$

The Fourier transformed pulse

$$\hat{u}(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,z) e^{i\omega z} dz$$

solves

$$\frac{\partial^2 \hat{u}}{\partial x^2} + \varepsilon \omega^2 \hat{u} = 0. \tag{18}$$

Using (16) and (18), we find that the transmitted pulse is

$$u(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) e^{ik(\omega)x - i\omega z} d\omega, \tag{19}$$

with $k(\omega) = \omega\sqrt{\varepsilon(\omega)}$, ε the dielectric permittivity of the medium, and

$$\begin{aligned} A(\omega) &= \frac{L}{2}\left[e^{-L^2(\omega-\omega_0)^2/2} + e^{-L^2(\omega+\omega_0)^2/2}\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(0,z) e^{i\omega z} dz, \\ L &= cT. \end{aligned}$$

We center in the normalized time coordinate according to the "travel time" as defined by the group velocity v_g, introducing new coordinates

$$d = \frac{x}{L}, \qquad \tau = \frac{z - \frac{x}{v_g}}{L},$$

and we seek an expression for the transmitted pulse in these coordinates:

$$U_L(d,\tau) \equiv u\left(Ld, L\left(\tau + \frac{d}{v_g}\right)\right).$$

In this section we normalize the group velocity with respect to the speed of light in a vacuum: $v_g \equiv [d\omega/dk]_0 = 1/k^{'}(\omega_0)$. From (19) it then follows that

$$U_L(d,\tau) = \Re[e^{iLd(k(\omega_0)-\omega_0 k^{'}(\omega_0))} e^{-i\omega_0 L\tau} w(d,\tau)], \tag{20}$$

with

$$w(d,\tau) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2} e^{iL\kappa(s/L)d} e^{-is\tau} ds, \tag{21}$$
$$\kappa(\omega) \equiv k(\omega+\omega_0) - k(\omega_0) - k^{'}(\omega_0)\omega.$$

Here we expand the wave number k around the center frequency ω_0. If the medium is nondispersive, then k is linear in ω and $\kappa \equiv 0$; thus

$$w(d,\tau) = e^{-\tau^2/2},$$

which is the initial pulse envelope. However, in general the material is dispersive, $\kappa \neq 0$, and this implies a gradual smearing of the envelope. We discuss next a measure for dispersion. Note that

$$e^{iL\kappa(s/L)d} = e^{id([k^{''}]_0 s^2/(2L) + [k^{'''}]_0 s^3/(6L^2) + \cdots)}; \tag{22}$$

hence, we let the quantity

$$\mathcal{D} \equiv |k^{''}|_0 = \left|\frac{d^2 k(\omega_0)}{d\omega^2}\right| = v_g^{-3} |\omega^{''}|_0 \tag{23}$$

define dispersion. Observe that, if

$$T \gg T_{\min} \equiv \frac{1}{c}\left[\frac{k^{'''}}{k^{''}}\right]_0, \tag{24}$$

then the third order term in the Taylor series in (22) will be dominated by the dispersion term. Note also that since the slowness is

$$s \equiv \frac{1}{v_g} = [k^{'}]_0 = \left[\frac{d(\omega n)}{d\omega}\right]_0,$$

with $n = \sqrt{\varepsilon}$ being the index of refraction, we can write

$$\mathcal{D} = |k^{''}|_0 = \left|\frac{ds}{d\omega}\right|_0 = \left|2\frac{dn(\omega_0)}{d\omega} + \omega_0 \frac{d^2 n(\omega_0)}{d\omega^2}\right|.$$

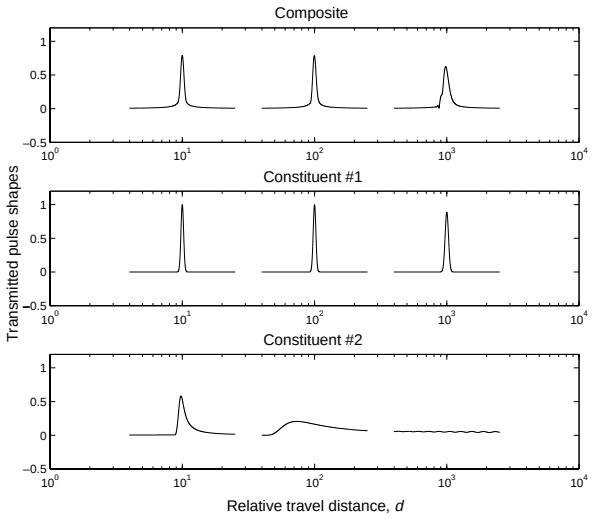

FIG. 4. *The top, middle, and bottom plots show the transmitted pulse shapes for three propagation distances. The top plot corresponds to the composite, which exhibits a smaller dispersion than phase* 2 *(bottom plot) but somewhat larger than phase* 1 *(middle plot).*

Let $\mathcal{L}$ denote the propagation length at which the initial support of the pulse has approximately doubled due to dispersive effects. This length depends on the dispersion, and we seek an expression for it. The initial temporal width of the pulse is approximately T, and therefore the support of $A(\omega)$ in (19) is about $\Delta\omega = 1/(cT)$. Thus, the variation in the slowness over the relevant frequencies is approximately

$$\Delta s = |k^{''}|_0 \Delta\omega = \frac{\mathcal{D}}{cT}.$$

The pulse has approximately doubled its support when $x \approx \mathcal{L}$ with $\mathcal{L}\Delta s = cT$, or

$$\mathcal{L} = \frac{c^2T^2}{\mathcal{D}} = \frac{L^2}{\mathcal{D}}. \tag{25}$$

Note that in untransformed coordinates

$$\frac{d^2k(\overline{\omega}_0)}{d\overline{\omega}^2} = \frac{\mathcal{D}}{c^2} \equiv \overline{\mathcal{D}}$$

and $\mathcal{L} = T^2/\overline{\mathcal{D}}$. If the initial spatial support of the pulse $\Delta x = v_g cT$ is fixed, then $v_g^2\mathcal{D}$ becomes the measure that determines the penetration depth.

For the model (14) with $\omega_0 = 1/2$ we find that for phase 1, 2, and the composite this measure $\mathcal{L}$ of "penetration depth" is, respectively, $\mathcal{O}(\Delta^{1/2})$, $\mathcal{O}(\Delta)$, and $\mathcal{O}(\Delta^{3/4})$. This implies that this dispersion measure for the composite is in between those of the phases.

In order to examine more closely the transformation in pulse shape, we next illustrate it with a numerical wave propagation example. The pulse $U_L(d,\cdot)$ in (20) consists of rapid oscillations on the scale $1/(\omega_0 L)$ with amplitude or envelope defined by $|w(d,\cdot)|$. In Figure 4 we show the transmitted temporal pulse shape as defined by the envelope $|w(d,\cdot)|$. We use $\Delta = 10^{-3}$, $L = 10^3$m, $d \in \{10, 100, 1000\}$, and a center frequency $\omega_0 = .5605\text{m}^{-1}$. This center frequency corresponds to the maximum

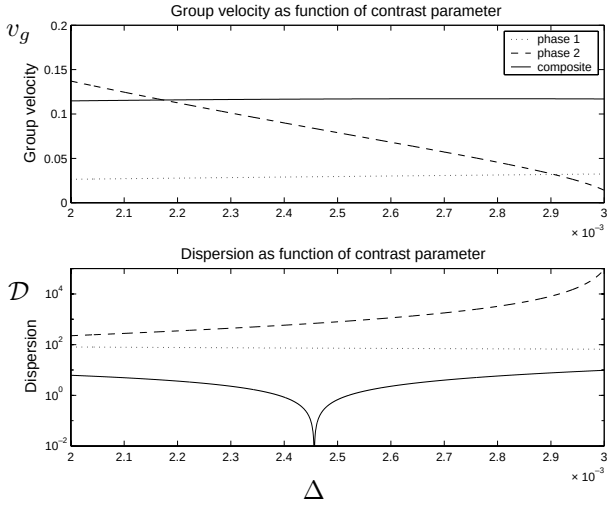

FIG. 5. *The top plot in the figure shows the group velocities and the bottom plot the dispersion for, respectively, the composite (solid line), phase* 1 *(dotted line), and phase* 2 *(dashed line). Note that the group velocity of the composite is larger and the dispersion smaller than the corresponding values for the individual phases.*

speed-up seen in Figure 3, which is ≈ 7!. The calculations were carried out in Matlab on a Sun work-station and involve using the fast Fourier transform (FFT) to compute $|w(d,\cdot)|$. The envelopes are plotted as functions of τ (on a linear scale) and are in the plot centered according to d. Note that the speed of propagation is well described by the group velocity; moreover, note that the dispersion in the composite is in between those in the phases.

Next we show that we can, in fact, construct a composite having a group velocity that is *larger* than that of either phase and *in addition* having a *smaller* dispersion than both phases! We accomplish this task by a slight modification of the parameters in the above example. First, we move the center frequency further away from the resonance frequency of phase 2: $\omega_0 = 0.6\text{m}^{-1}$. Second, we shift the resonance frequency of phase 1 somewhat closer to ω_0:

$$\varepsilon_1(\omega) = 1 + \frac{\Delta^{-1}}{3/4 - \omega^2}. \tag{26}$$

We use the same phase 2 and the same layered geometry for the composite as above, and for the volume fraction of phase 1 we use $p = p(\Delta) = \sqrt{\Delta}$. The solid line in the top plot of Figure 5 shows the resulting group velocity for the composite as a function of the (small) parameter Δ. The dotted and dashed lines give the group velocities for phases 1 and 2, respectively. Recall that we define the group velocity relative to the speed of light. In the bottom plot we graph the corresponding dispersion measure $\mathcal{D}$. The figure shows that for a certain regime of Δ values we have indeed constructed a composite having increased group velocity and a substantially reduced dispersion relative to the constituents.

Next, we confirm this by numerical wave propagation. We use $L = 10^3\text{m}$ and $d \in \{0.1, 1, 100\}$. We choose the small parameter $\Delta = 2.8 \times 10^{-3}$ and let $\omega_0 = 0.6\text{m}^{-1}$. Figure 6 shows the transmitted pulses. With the chosen parameters, the group velocity of the composite is approximately twice the group velocity of phase 2, and four times

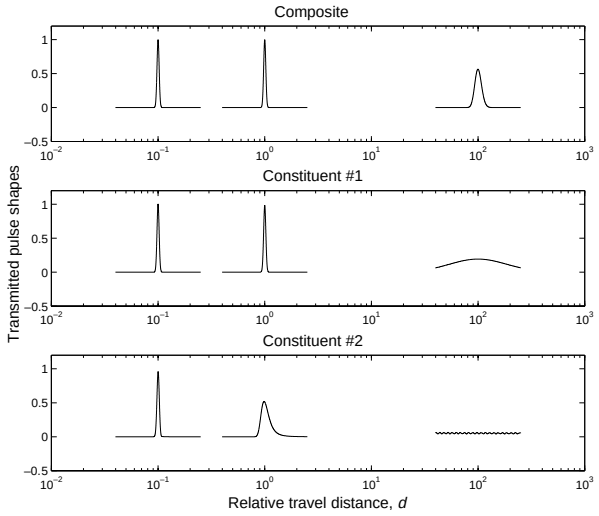

FIG. 6. *The figure plots the transmitted pulses in the composite (top plot) and phase* 1 *(middle plot) and* 2 *(bottom plot) for three travel-lengths. The group velocity is largest in the composite; still, the pulse is subject to the smallest dispersion here.*

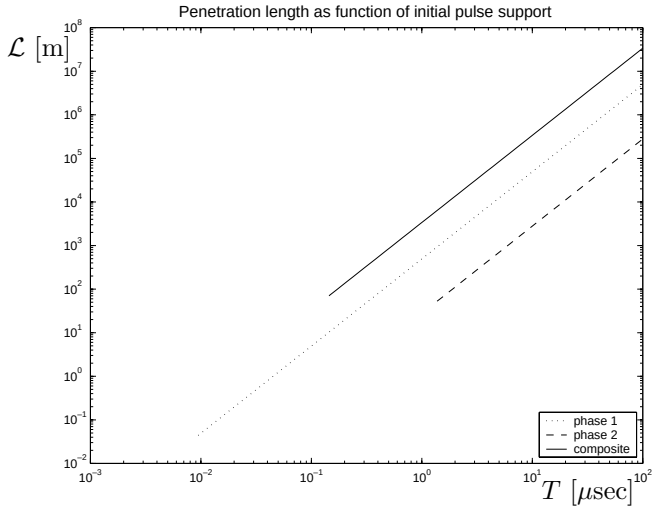

FIG. 7. *The solid line is the penetration depth of a pulse traveling in the composite, as a function of the initial pulse support. The dotted and dashed lines are the penetration depths for the two phases; these are significantly smaller than for the composite.*

that of phase 1; see Figure 5. Despite this, we see that the degradation of the traveling pulse is much greater in the individual phases. This is particularily the case for the pulse traveling in phase 2, the relatively faster constituent.

In (25) we define the "penetration depth," $\mathcal{L}$, of a pulse. This quantity depends on the dispersion of the medium and the temporal support of the initial pulse. In Figure 7 we plot $\mathcal{L}$ as a function of the initial pulse support T, both for the composite at hand and for the pure phases. We do this for for the initial support $T > T_{\min}$, with $T_{\min}$ being defined in (24). The figure shows that the penetration depth is strongly enhanced by combining the two phases in the way discussed above, a way that also

enhances the group velocity.

3.3. Analysis for anisotropic media. In this section we derive the bound (8) for the group velocity. Thus, we consider a dielectric composite with two isotropic phases when there are no constraints on the volume fraction nor on the geometry of the composite.

From (2) and (4) it follows that the group velocity of the composite can be expressed as

$$v_g(\omega_0) = \frac{c}{\sqrt{\varepsilon^*(\omega_0)} + \frac{\omega_0}{2\sqrt{\varepsilon^*(\omega_0)}} \frac{d\varepsilon^*(\omega_0)}{d\omega_0}}, \tag{27}$$

with c being the velocity of light, ε^* the effective dielectric permittivity of the composite, and ω_0 the center frequency of the propagating pulse. The main idea behind the derivation is to first bound $d\varepsilon^*/d\omega$ in terms of ε^*. This, together with the fact that $\varepsilon_2 \le \varepsilon^* \le \varepsilon_1$, can be used to bound v_g in (27). We define

$$\begin{aligned} \alpha_i &= \frac{\omega_0}{2\varepsilon_i(\omega_0)} \frac{d\varepsilon_i(\omega_0)}{d\omega_0}, \\ \gamma &= \alpha_1 - \alpha_2, \\ \beta &= \frac{1+\alpha_1}{1+\alpha_2}. \end{aligned}$$

These quantities are determined by the parameters (6). Define, moreover, the parameters

$$\begin{aligned} \alpha^* &= \frac{\omega_0}{2\varepsilon^*(\omega_0)} \frac{d\varepsilon^*(\omega_0)}{d\omega_0}, \\ e^* &= \frac{\varepsilon^*(\omega_0)}{\varepsilon_1(\omega_0)}, \end{aligned}$$

which also depend on the volume fraction and the geometry of the composite. Below we suppress the dependence on ω_0. It is convenient to derive the bounds on the group velocity in terms of the scaled "slowness":

$$s = \frac{c}{v_g\sqrt{\varepsilon_1}} = \sqrt{e^*} + \frac{\alpha^*}{\sqrt{e^*}}, \tag{28}$$

with $h < e^* \le 1$. First, we seek a bound for α^* in terms of ε^*. Since ε^* is a homogeneous function in ε_1 and ε_2, we find, using Euler's theorem,

$$\begin{aligned} \varepsilon^* &= \varepsilon^*_{\varepsilon_1}\varepsilon_1 + \varepsilon^*_{\varepsilon_2}\varepsilon_2, \\ \frac{d\varepsilon^*}{d\omega} &= \varepsilon^*_{\varepsilon_1}\frac{d\varepsilon_1}{d\omega} + \varepsilon^*_{\varepsilon_2}\frac{d\varepsilon_2}{d\omega}, \end{aligned}$$

where $\varepsilon^*_{\varepsilon_1} = \partial\varepsilon^*/\partial\varepsilon_1$ and $\varepsilon^*_{\varepsilon_2} = \partial\varepsilon^*/\partial\varepsilon_2$. From these relations it follows that

$$\frac{d\varepsilon^*}{d\omega} = \varepsilon^*_{\varepsilon_1}\varepsilon_1\left(\frac{1}{\varepsilon_1}\frac{d\varepsilon_1}{d\omega} - \frac{1}{\varepsilon_2}\frac{d\varepsilon_2}{d\omega}\right) + \varepsilon^*\frac{1}{\varepsilon_2}\frac{d\varepsilon_2}{d\omega},$$

and hence also that

$$\alpha^* = \varepsilon^*_{\varepsilon_1}\gamma + e^*\alpha_2.$$

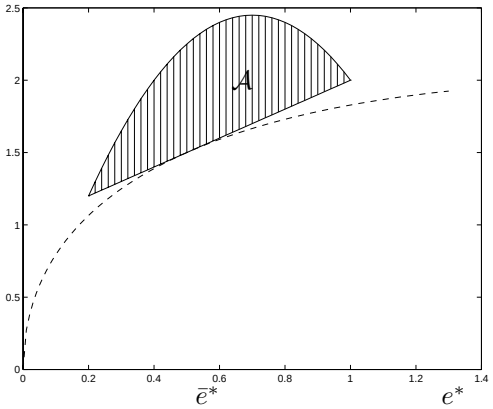

FIG. 8. *The area marked $\mathcal{A}$ in the figure gives an example of the set of possible values for α^* as a function of e^*. The dashed line represents the case with $\bar{s}$ as the minimal value for the slowness for the parameters chosen in this example. The line is tangential to the boundary of the marked set at $e^* = \bar{e}^*$.*

Bounds for $\varepsilon^*_{\varepsilon_1}$ when the volume fraction is *fixed* are derived in [20] and listed in (46). From these we find, by an argument similar to that presented in Appendix A concerning the isotropic case, that the upper and lower bounds on $\varepsilon^*_{\varepsilon_1}$ for each possible value of e^* are

$$\varepsilon^*_{\varepsilon_1} \le l_I(e^*) = \frac{e^* - h}{1 - h}, \tag{29}$$
$$\varepsilon^*_{\varepsilon_1} \ge l_{II}(e^*) = \frac{e^*(e^* - h)}{1 - h}.$$

Thus, the set of possible α^* values are enveloped by

$$\alpha_I(e^*) = l_I(e^*)\gamma + e^*\alpha_2 = \frac{e^* - h}{1 - h}\gamma + e^*\alpha_2, \tag{30}$$
$$\alpha_{II}(e^*) = l_{II}(e^*)\gamma + e^*\alpha_2 = \frac{e^*(e^* - h)}{1 - h}\gamma + e^*\alpha_2.$$

The geometries giving the extremal values for $\varepsilon^*_{\varepsilon_1}$ correspond to wave propagation in a laminated medium with the layers orthogonal or, respectively, parallel to the direction of propagation. We let ε denote the relevant component of the dielectric tensor. To get the upper bound for the group velocity, we minimize $s = s(e^*, \alpha^*)$ subject to $h \le e^* \le 1$ and $\alpha_I(e^*) \le \alpha^* \le \alpha_{II}(e^*)$.

The bound for s is found via geometrical considerations; we next show how. Note that the area marked $\mathcal{A}$ in Figure 8 gives an example of the set of possible values for α^* as a function of e^*. The plot corresponds to $\gamma < 0$. When we solve (28) with respect to α^*, we find

$$\alpha^* = \sqrt{e^*}s - e^*.$$

Denote the minimum possible value of the slowness by $\bar{s}$, and the associated value of e^* by $\bar{e}^*$. Then the curve $\sqrt{e^*}\bar{s} - e^*$ will either be tangential to the boundary of $\mathcal{A}$ at $e^* = \bar{e}^*$ or touch $\mathcal{A}$ at one of the two corners $e^* = h$ or $e^* = 1$. In the latter case the minimal slowness is achieved when the composite is one of the phases. Figure 8 shows an example for which a composite is optimal. The dashed curve in the figure is

$\sqrt{e^*}\bar{s} - e^*$, with $\bar{s}$ being the minimum slowness for the given parameters; the curve is tangential to the boundary of the marked region for $e^* = \bar{e}^*$. Note that if $s > \bar{s}$, the curve $\sqrt{e^*}s - e^*$ intersects the boundary for $e^* \neq \bar{e}^*$. Thus, if we define $S = S(e^*)$ by

$$l_I(e^*)\gamma + e^*\alpha_2 = \sqrt{e^*}S(e^*) - e^*, \tag{31}$$

then $S(e^*)$ is minimum for $e^* = \bar{e}^*$ and $S(\bar{e}^*) = \bar{s}$. From (31) we therefore find

$$\frac{S^{'}(\bar{e}^*)}{1+\alpha_2} = \frac{d}{de}\left[\frac{(\beta-1)l_I(e)+e}{\sqrt{e}}\right]\Bigg|_{e=\bar{e}^*} = 0. \tag{32}$$

When we carry out the differentiation in (32) and solve for $\bar{e}^*$, we find that $\bar{e}^*$ solves

$$\beta = \frac{1+\alpha_1}{1+\alpha_2} = 1 - \frac{\bar{e}^*}{2l_I'(\bar{e}^*)\bar{e}^* - l_I(\bar{e}^*)} = \frac{h(\bar{e}^*+1)}{\bar{e}^*+h}. \tag{33}$$

The resulting value of $\bar{e}^*$ lies in the interval $h < e^* < 1$ only when

$$\frac{2h}{1+h} < \beta < \frac{1+h}{2}, \tag{34}$$

which is the condition for the bound on the speed-up $\mathcal{G}$, as defined in (8) to be larger than unity. This is the area outlined by the solid lines in Figure 2. Note that if $\gamma > 0$, l_I in (33) is replaced by l_{II}. However, in this case (33) has no solution with $\beta > 1$ and $h < \bar{e}^* < 1$.

Let $s(h)$, $s(1)$, and $s(\bar{e}^*)$ denote the slownesses of, respectively, the pure phases and the composite that minimizes the slowness; from (31) we then find

$$\begin{aligned}
\frac{s(h)}{1+\alpha_2} &= \sqrt{h}, \\
\frac{s(1)}{1+\alpha_2} &= \beta, \\
\frac{s(\bar{e}^*)}{1+\alpha_2} &= \left[\frac{(\beta-1)l_I(\bar{e}^*)+\bar{e}^*}{\sqrt{\bar{e}^*}}\right].
\end{aligned} \tag{35}$$

The maximal speed-ups relative to the pure phases are

$$\begin{aligned}
g_1 &= \frac{s(1)}{s(\bar{e}^*)} = \frac{1+\bar{e}^*}{2\sqrt{\bar{e}^*}}, \\
g_2 &= \frac{s(h)}{s(\bar{e}^*)} = \frac{\bar{e}^*+h}{2\sqrt{h\bar{e}^*}}.
\end{aligned} \tag{36}$$

From (33) and (36) we find the bound for G defined in (7):

$$G \leq \mathcal{G}(h,\beta) = \min[g_1, g_2] = \begin{cases} \frac{(1-h)\beta}{2\sqrt{h(1-\beta)(\beta-h)}} & \text{for } \frac{2h}{1+h} \leq \beta \leq \sqrt{h}, \\ \frac{1-h}{2\sqrt{(1-\beta)(\beta-h)}} & \text{for } \sqrt{h} \leq \beta \leq \frac{1+h}{2}, \\ 1 & \text{otherwise,} \end{cases}$$

as stated in (8). The maximum speed-up for a given h is

$$\max_{\beta} \mathcal{G}(h,\beta) = \mathcal{G}(h,\sqrt{h}) = \frac{1+\sqrt{h}}{2h^{1/4}},$$

and then $\bar{e}^* = \sqrt{h}$ and $p = \sqrt{h}/(1+\sqrt{h})$.

3.4. Analysis of the two dimensional isotropic case. We adapt the analysis of the previous section to a two dimensional isotropic medium. With the volume fraction a free parameter, we seek upper bounds for the group velocity. First note that the bounds for α^* are as in (30), only with different bounds for $\varepsilon^*_{\varepsilon_1}$, that is, different l_I and l_{II}. In this case we have

$$\alpha_I(e^*) = l_I(e^*)\gamma + e^*\alpha_2 = \frac{(he^*+1)(e^*-h)}{1-h^2}\gamma + e^*\alpha_2, \tag{37}$$

$$\alpha_{II}(e^*) = l_{II}(e^*)\gamma + e^*\alpha_2 = \frac{(e^*)^2-h^2}{1-h^2}\gamma + e^*\alpha_2. \tag{38}$$

The bounds for $\varepsilon^*_{\varepsilon_1}$ follow by the argument presented in Appendix A, and they correspond to cylinder assemblages realizing the two dimensional Hashin–Shtrikman bounds [11]. The bound for the group velocity itself can now be found by a geometric argument similar to that of the previous section. As above, the speed-up exceeds unity only for $\gamma < 0$. Hence, the maximum group velocity is realized by composites attaining the bound in (37). From (37) and (38) it follows that in this case the analogue of the set marked $\mathcal{A}$ in Figure 8 is not convex. As we show in Appendix C, this implies that $\mathcal{G}$, the speed-up, exceeds unity only for $h < e^* < .432$. Let $\bar{e}^*$ be the effective dielectric permittivity for the composite associated with the maximum speed-up. By an argument as in the previous section we find that $\bar{e}^*$ now solves

$$\beta = \beta(\bar{e}^*) = \frac{3h(\bar{e}^*)^2 + h}{3h(\bar{e}^*)^2 + (1-h^2)\bar{e}^* + h}. \tag{39}$$

This defines the set of β values $(\beta(.432), \beta(h))$ that corresponds to the bound on the group velocity exceeding unity. This set is marked by dots in Figure 2. Let g_1 and g_2 be the maximal speed-up relative to the two phases; then we find

$$g_1(\bar{e}^*) = \frac{3(\bar{e}^*)^2+1}{2\sqrt{\bar{e}^*}((\bar{e}^*)^2+1)}, \tag{40}$$

$$g_2(h,\bar{e}^*) = \frac{3h(\bar{e}^*)^2 + (1-h^2)\bar{e}^* + h}{2\sqrt{h\bar{e}^*}((\bar{e}^*)^2+1)},$$

and $g_1 = g_2$ for $\bar{e}^*$ solving

$$\bar{e}^* = \frac{c(h) - \sqrt{c(h)^2-12}}{6} \equiv f(h),$$

$$c(h) = \frac{(1+h)(1+\sqrt{h})}{\sqrt{h}}.$$

Moreover, $dg_1/d\bar{e}^* = 0$ for $\bar{e}^* \in \{\pm 1, \pm 1/\sqrt{3}\}$, and $dg_2/d\bar{e}^* = 0$ for $\bar{e}^* \in \{\pm 1/\sqrt{3}, h, -1/h\}$. From this it follows that $\mathcal{G}_2$ in (13) is

$$\mathcal{G}_2(h,\beta) = \begin{cases} g_1(e(\beta)) & \text{for } \beta(.432) \le \beta \le \beta(f(h)), \\ g_2(h, e(\beta)) & \text{for } \beta(f(h)) \le \beta \le \beta(h), \\ 1 & \text{otherwise,} \end{cases} \tag{41}$$

with $e(\cdot)$ being the smaller of the roots of (39) when solved with respect to $\bar{e}^*$, and $\beta(\cdot)$ being defined by (39). The maximum speed-up is obtained with $\beta = \beta(f(h))$. For h small we find

$$\max_\beta \mathcal{G}(h,\beta) \sim \mathcal{G}(h,\sqrt{h}) \sim \frac{1}{2h^{1/4}};$$

then $\bar{e}^* \sim \sqrt{h}$ and $p \sim 2\sqrt{h}$.

3.5. Analysis of the three dimensional isotropic case. The analysis of the three dimensional case follows the same steps as those presented above for a two dimensional medium. We summarize the modifications of the involved expressions. The bounds for α^* are

$$\alpha_I = l_I(e^*)\gamma + e^*\alpha_2 = \frac{(e^*-h)(e^*h+2)}{(1-h)(2+h)}\gamma + e^*\alpha_2, \tag{42}$$
$$\alpha_{II} = l_{II}(e^*)\gamma + e^*\alpha_2 = \frac{(e^*-h)(e^*+2h)}{(1-h)(1+2h)}\gamma + e^*\alpha_2.$$

The bounds correspond to sphere assemblages realizing the three dimensional Hashin–Shtrikman bounds [11]. We derive the bounds for $\varepsilon^*_{\varepsilon_1}$ in Appendix A, based on those presented in [2] for the fixed volume case. In this case $\mathcal{G}_3$, the speed-up, exceeds unity only for $h < e^* < .756$ and $\gamma < 0$; see Appendix C. Denote by $\bar{e}^*$ the effective dielectric permittivity for the composite with maximal group velocity; then

$$\beta = \beta(\bar{e}^*) = \frac{3h(\bar{e}^*)^2 + h\bar{e}^* + 2h}{3h(\bar{e}^*)^2 + (2-h^2)\bar{e}^* + 2h}, \tag{43}$$

with $h < \bar{e}^* < .756$. The maximal speed-ups relative to the pure phases are

$$g_1(\bar{e}^*) = \frac{3(\bar{e}^*)^2 + \bar{e}^* + 2}{\sqrt{\bar{e}^*}2((\bar{e}^*)^2+2)},$$
$$g_2(h,\bar{e}^*) = \frac{3h(\bar{e}^*)^2 + (2-h^2)\bar{e}^* + 2h}{2\sqrt{h\bar{e}^*}((\bar{e}^*)^2+2)},$$

with $dg_1/d\bar{e}^* = 0$ for $\bar{e}^* \in \{-2, \pm\sqrt{2/3}, 1\}$, and $dg_2/d\bar{e}^* = 0$ for $\bar{e}^* \in \{\pm\sqrt{2/3}, h, -2/h\}$. We find that $g_1 = g_2$ for $\bar{e}^*$ solving

$$\bar{e}^* = \frac{c(h) - \sqrt{c(h)^2 - 24}}{6} \equiv f(h),$$
$$c(h) = \frac{h^{3/2} + h + \sqrt{h} + 2}{\sqrt{h}}.$$

From this it follows that $\mathcal{G}_3$ in (13) is

$$\mathcal{G}_3(h,\beta) = \begin{cases} g_1(e(\beta)) & \text{for } \beta(.756) \le \beta \le \beta(f(h)), \\ g_2(h, e(\beta)) & \text{for } \beta(f(h)) \le \beta \le \beta(h), \\ 1 & \text{otherwise,} \end{cases} \tag{44}$$

with $e(\cdot)$ being the smaller of the roots of (43) when solved with respect to $\bar{e}^*$, and $\beta(\cdot)$ being defined by (43). The maximum speed-up is obtained for $\beta = \beta(f(h))$. For h small we find

$$\max_\beta \mathcal{G}(h,\beta) \sim \mathcal{G}(h,\sqrt{h}) \sim \frac{1}{2h^{1/4}},$$

and then $\bar{e}^* \sim \sqrt{h}$ and $p \sim 3\sqrt{h}/2$.

4. Upper bound with a fixed volume fraction. So far we have derived bounds for the group velocity when the volume fraction is considered as a free parameter. We chose the volume fraction to obtain a maximum speed-up for the composite. In this section we derive bounds for the speed-up of the composite when the volume fraction is fixed. In this case the speed-up is defined relative to those geometries that are associated with the upper and lower bounds for the dielectric permittivity of the composite for the given volume fraction.

4.1. Anisotropic media. We present first the bounds in the general anisotropic case. The two isotropic constituents of the composite are defined by (6) as before. Define the scaled slowness of the composite as in (28):

$$s = \frac{c}{v_g\sqrt{\varepsilon_1}} = \sqrt{e^*} + \frac{\alpha^*}{\sqrt{e^*}}.$$

The bounds for e^* in this expression are

$$\begin{aligned} e^* &\geq h_1(p,h) = \frac{h}{ph + (1-p)}, \\ e^* &\leq h_2(p,h) = p + (1-p)h, \end{aligned} \tag{45}$$

with p being the volume fraction of phase 1. These extremal values for e^* correspond to a laminated composite with lamination parallel (h_1), respectively orthogonal, to the direction of propagation. Let the set of realizable values for the dispersion parameter α^* be enveloped by α_I and α_{II} as in (30):

$$\begin{aligned} \alpha_I(e^*) &= l_I(e^*)\gamma + e^*\alpha_2, \\ \alpha_{II}(e^*) &= l_{II}(e^*)\gamma + e^*\alpha_2. \end{aligned}$$

In this case with the volume fraction fixed, l_I and l_{II}, the bounds for $\varepsilon^*_{\varepsilon_1}$, are given by

$$\begin{aligned} l_I(e^*) &= e^* - \frac{h}{1-p}\left(\frac{1-e^*}{1-h}\right)^2, \\ l_{II}(e^*) &= \frac{1}{p}\left(\frac{e^*-h}{1-h}\right)^2. \end{aligned} \tag{46}$$

These bounds are derived in [20]. The set enveloped by α_I and α_{II}, the analogue of the set marked $\mathcal{A}$ in Figure 8, is therefore convex and lens-shaped. The analysis can now proceed as above. Again we need $\gamma < 0$, and now the relative dispersion parameter β that corresponds to a maximum speed-up with $e^* = \bar{e}^*$ is

$$\begin{aligned} \beta = \frac{1+\alpha_1}{1+\alpha_2} &= 1 - \frac{\bar{e}^*}{2l_I'(\bar{e}^*)\bar{e}^* - l_I(\bar{e}^*)} \\ &= 1 - \frac{\bar{e}^*}{a(1 + 2\bar{e}^* - 3(\bar{e}^*)^2) + \bar{e}^*}, \end{aligned} \tag{47}$$

where

$$a = \frac{h}{(1-p)(1-h)^2}.$$

From (45) and (47), the set of values for the parameters β and h that are associated with a positive speed-up can be found. We show this set for three different values of p in Figure 9.

We next find the bounds for the speed-up that derives from the "optimal" composite with $e^* = \bar{e}^*$. The speed-up is defined relative to the composites corresponding to the bounds for e^* in (45). Thus, we do *not* define the speed-ups by comparison with the *pure* phases as above, but instead by comparison with the laminate geometries.

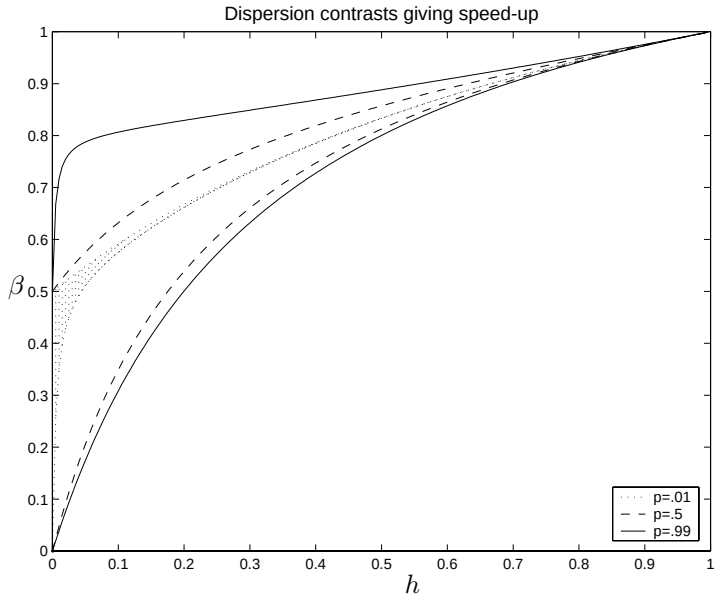

FIG. 9. *The figure shows the range of β and h values that, for a given volume fraction, is associated with a maximal speed-up relative to the laminated geometries exceeding unity. The sets are those enveloped by the solid, dashed, and dotted lines, corresponding to $p = .99$, $.5$, and $.01$, respectively.*

With β chosen as in (47), denote by g_i the resulting maximal speed-up relative to the composite with $e^* = h_i(p,h)$. Then

$$g_i = \sqrt{\frac{h_i}{\bar{e}^*}}\left[\frac{1+\bar{e}^*(h_i+h_i^{-1})-3(\bar{e}^*)^2}{2(1-(\bar{e}^*)^2)}\right]. \tag{48}$$

Large speed-ups are obtained for small values of h. In this regime it follows from (45), (47), and (48) that the speed-up is bounded as follows.

LEMMA 4.1. *Under assumptions* (i)–(iv) *stated above,*

$$G \le \max_{\beta} \mathcal{G}(h,\beta,p) \sim \begin{cases} \frac{1}{2h^{1/4}}[p(1-p)]^{1/4} & \text{for } h = o(1) \ll p \ll 1, \\ \frac{1+\sqrt{K+1}}{2(K+1)^{1/4}} & \text{for } p = Kh = o(1), \\ 1 & \text{for } p \ll h = o(1), \end{cases} \tag{49}$$

with $\mathcal{G} = \min_i[g_i]$.

In Figure 10 we show the maximal speed-up $\max_\beta \mathcal{G}$ as a function of p and h. The solid lines correspond to the exact bounds, and the dashed lines to the approximation (49).

The bound (49) for the speed-up as we have defined it in this section is close to the bound (11). In (11) the speed-up is defined to be relative to the *pure* phases, and we optimize also with respect to the volume fraction. Next, we look at the possible speed-up when we constrain the volume fraction but define the speed-up to be relative to the pure phases. We ask the question: If material i is doped with the other phase such that $p_i \ge 1-\bar{p}$, where $\bar{p}$ is a small parameter which constrains the maximum amount of doping material used, what is the possible speed-up for the composite relative to phase i, v^*/v_i? Recall that p_i is the volume fraction of material i. By a

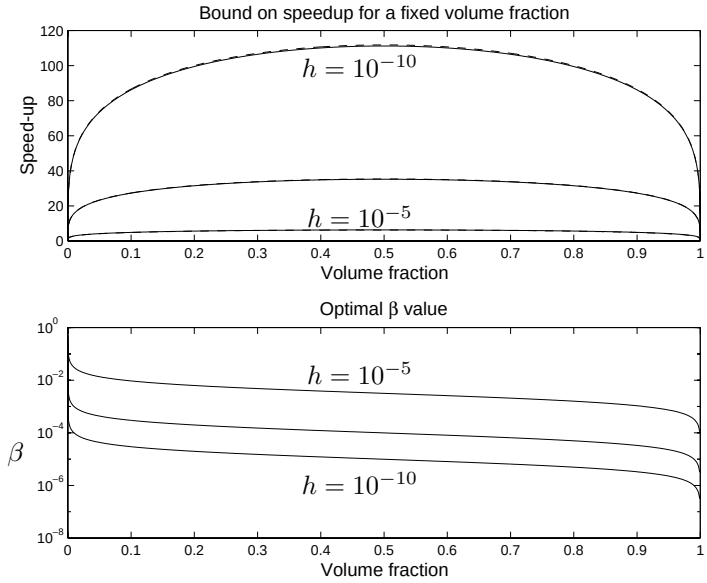

FIG. 10. *The figure shows the optimal speed-up for a given value of the volume fraction of material one, as a function of this volume fraction and for* $h \in \{10^{-10}, 10^{-8}, 10^{-5}\}$, *with the largest speed-up corresponding to the smallest* h*-value. Note that the speed-up is relative to the laminated geometries corresponding to extremal values for the effective dielectric permittivity. The bottom plot shows the corresponding optimal value for the relative dispersion parameter* β.

modification of the above analysis, we find that the speed-up is then bounded by

$$G \leq \frac{\sqrt{\bar{p}}}{\sqrt{h}}. \tag{50}$$

Note a higher rate of the speed-up here with respect to h than in (11) and (49). The higher rate is obtained since we compare only with the doped phase rather than with both constituents. The bounds correspond to laminated geometries with $p_i = 1 - \bar{p}$.

4.2. Isotropic media. In this section we consider the group velocity of a dielectric composite when the volume fraction is known and fixed and when, in addition, we constrain the composite to be isotropic.

In the isotropic fixed volume case, Axell [2] derives the bounds

$$\begin{aligned} \varepsilon^*_{\varepsilon_1} &\leq -c_3^{(d)} + c_4^{(d)} e^* - c_5^{(d)} (e^*)^2 = l_I^{(d)}(e^*), \\ \varepsilon^*_{\varepsilon_1} &\geq c_0^{(d)} - c_1^{(d)} e^* + c_2^{(d)} (e^*)^2 = l_{II}^{(d)}(e^*), \end{aligned} \tag{51}$$

with $c_j^{(d)} = c_j^{(d)}(h,p) > 0$ being listed in Appendix D. The superscript $d \in \{2,3\}$ denotes the space dimension. In two space dimensions the bounds (51) are realized by a composite defined in terms of doubly coated cylinders [15]. In three space dimensions the bound l_I derives from Bergman's bound [4] for e^*. This bound *may or may not* be optimal. It has been shown to be attained for five different values of e^* corresponding to five different geometries; see, for example, [1]. The bound l_{II} is realized by a geometry defined in terms of doubly coated spheres [1] but does not play a role in what follows. Using (51) in (30) gives the appropriate bounds for α^*, and the analysis can proceed as before. Now $h_1^{(d)}(h,p) \leq e^* \leq h_2^{(d)}(h,p)$, with h_i being the

(normalized) Hashin–Shtrikman bounds [11], which are given in Appendix D. The set defined by the bounds for α^* is convex and lens-shaped. As before, no relative speed-up is possible for $\gamma > 0$; see Appendix B. With $\gamma < 0$ we find that the maximal speed-up is characterized by

$$\beta(\bar{e}^*) = 1 - \frac{\bar{e}^*}{c_3^{(d)} + c_4^{(d)}\bar{e}^* - 3c_5^{(d)}(\bar{e}^*)^2},$$

which with $h_1^{(d)}(h,p) \le \bar{e}^* \le h_2^{(d)}(h,p)$ determines the range of values of β for which $\mathcal{G} > 1$. The maximal speed-ups relative to the sphere assemblages associated with the Hashin–Shtrikman bounds, $g_i = s(h_i^{(d)})/s(\bar{e}^*)$, are now

$$g_i = \frac{\sqrt{h_i^{(d)}}\left(c_3^{(d)} + \bar{e}^*\left(c_3^{(d)}/h_i^{(d)} + c_5^{(d)}h_i^{(d)}\right) - 3c_5^{(d)}(\bar{e}^*)^2\right)}{2\sqrt{\bar{e}^*}\left(c_3^{(d)} - c_5^{(d)}(\bar{e}^*)^2\right)}, \tag{52}$$

with $h_1^{(d)}(h,p) \le \bar{e}^* \le h_2^{(d)}(h,p)$ and g_i the speed-up relative to the sphere assemblage associated with $h_i^{(d)}$.

As above, large speed-ups are obtained for h small. We find the following result for the maximal speed-up.

LEMMA 4.2. *Under assumptions* (i)–(iv) *stated above,*

$$\max_{\beta} \mathcal{G}_d(h,\beta,p) \sim \frac{1}{2h^{1/4}}[p(1-p)]^{1/4}\left[\frac{d-1}{(d-p)(1+p(d-1))}\right]^{1/4} \quad \text{as } h \downarrow 0. \tag{53}$$

These bounds are only slightly ($\approx 10\%$) smaller than those of the anisotropic case given in (49).

5. A lower bound on the group velocity.

5.1. Results. In this section we present a *lower* bound for the group velocity for one dimensional wave propagation in the dielectric composite. The bounds presented here concern general anisotropic media. Tight bounds for an *isotropic* medium can be obtained by modifications analogous to those in section 3.4.

The composite is defined as before, as a dielectric composite with two isotropic phases. The two isotropic phases are characterized by the parameters (6). Let $\{v_1, v_2, v_*\}$ be the group velocities for, respectively, the phases and the composite, and define

$$J = \min\left[\frac{v_1}{v_*}, \frac{v_2}{v_*}\right].$$

The lower bound for the group velocity is obtained by maximizing this quantity with respect to the *geometry* of the composite and the *volume fraction*. We find that the bound for the slow-down is characterized by the following.

THEOREM 5.1. *Under assumptions* (i)–(iv) *stated above,*

$$J \le \mathcal{J}(h,\beta) = \begin{cases} \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)h(1-h)}} & \text{for } \max[0, \frac{3h-1}{2h}] \le \beta \le \sqrt{h}, \\ \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)(1-h)\beta}} & \text{for } \sqrt{h} \le \beta \le \frac{2}{3-h}, \\ 1 & \text{otherwise.} \end{cases} \tag{54}$$

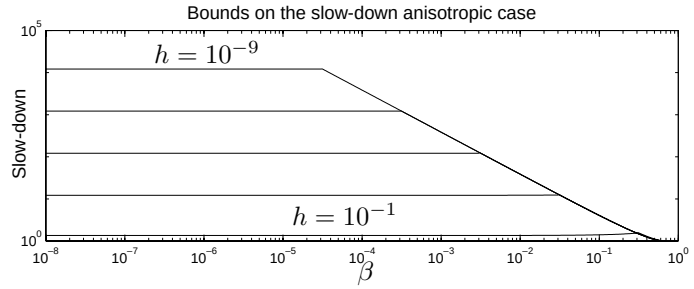

FIG. 11. *The figure shows the optimal slow-down as a function of β for a range of values of h. The set of solid lines corresponds to $h \in \{10^{-9}, 10^{-7}, 10^{-5}, 10^{-3}, 10^{-1}\}$, with the largest slow-downs being obtained for small h values.*

h and β are defined in (9), and the wave is polarized so that its electric field is directed normal to the layers. The bound is again realized by propagation in a laminated medium, but now with lamination *parallel* to the direction of propagation, and we need $\gamma < 0$. The bound, when maximized also with respect to the value of β, becomes

$$\max_{\beta} \mathcal{J}(h,\beta) = \mathcal{J}(h,\sqrt{h}) = \frac{2(1+\sqrt{h}+h)^{3/2}}{\sqrt{h}(1+\sqrt{h})3^{3/2}} \sim \frac{2}{3^{3/2}\sqrt{h}} = \frac{v_{p,2}/v_{p,1}}{3^{3/2}/2} \qquad \text{as } h \downarrow 0. \tag{55}$$

At the maximal slow-down, $p \sim 1 - 2h$ and $v_1 = v_2$. Note that a large slow-down can be obtained for h small; then the volume fraction p for the "optimal" composite is close to unity. Thus, contrary to the maximal speed-up case, a large slow-down is obtained when material one is doped with material two, rather than vice versa.

In Figure 11 we plot $\mathcal{J}(h,\beta)$ for a range of parameter values. Large slow-downs are obtained when h is small and β is close to zero. This corresponds to the dielectric permittivity of material two's being much smaller than that of material one; moreover, it implies that the relative dispersion in material two is much larger than the relative dispersion in material one.

5.2. Illustration. The above results suggest how to construct a composite such that its group velocity is small relative to that of the phases. To accomplish this task we can use the *same* phases as in section 3.2. We again discuss the anisotropic case. To obtain a large reduction in the group velocity, we now let the medium be laminated with its layers orthogonal to the electric field, i.e., with the direction of propagation parallel to the layers. Recall that the phases are defined by the Lorentzian models

$$\varepsilon_1(\omega) = 1 + \frac{\Delta^{-1}}{5/4 - \omega^2},$$
$$\varepsilon_2(\omega) = 1 + \frac{\sqrt{\Delta}}{1/4 + \sqrt{\Delta} - \omega^2}.$$

In this case we choose the volume fraction of phase 1 to be $p = 1 - 2h = 1 - 2(\varepsilon_2(\omega_0)/\varepsilon_1(\omega_0))$, and we use $\Delta = 10^{-3}$. The effective dielectric permittivity of the composite is

$$\frac{1}{\varepsilon^*} = \frac{p}{\varepsilon_1} + \frac{1-p}{\varepsilon_2}.$$

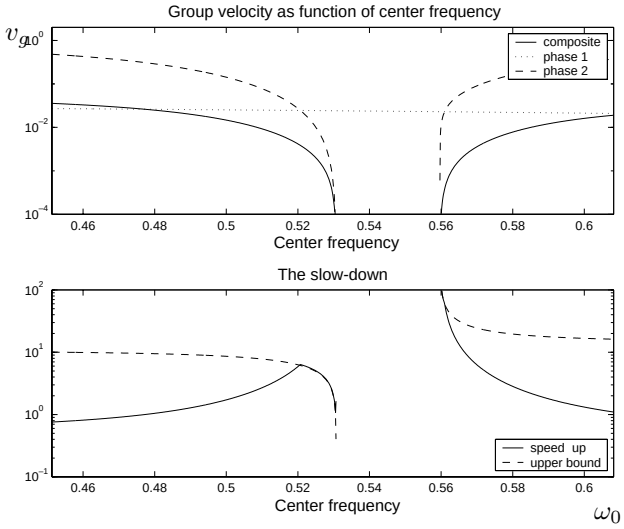

FIG. 12. *The top plot shows the group velocities as a function of the center frequency for a composite of materials defined by the Lorentzian model. The dotted, dashed, and solid lines correspond, respectively, to materials one, two, and the composite. In the bottom plot we show the slow-down of the composite relative to the phases. The dashed line is the upper bound on the slow-down for the given ratio of the dielectric permittivities.*

The top plot in Figure 12 shows the group velocity for the pure phases and the composite. We show the group velocity as a function of the center frequency of the pulse ω_0. The displayed frequency range is close to the resonance frequency of phase 2. The bottom plot shows the actual slow-down for the composite relative to both of the phases. Note that by combining the same phases as in section 3.2 we are able to construct a composite with a comparatively small group velocity. Moreover, note that the realized slow-down is approximately equal to the bound when the group velocities of the phases coincide. For $\omega_0 \approx 0.5605$ the realized slow-down is approximately 80!.

5.3. Outline of the analysis. The derivation of the lower bound is but a slight modification of the analysis presented in section 3.3. Now we seek an *upper* bound for

$$s = \frac{c}{v_g \varepsilon_1} = \sqrt{e^*} + \frac{\alpha^*}{\sqrt{e^*}}, \tag{56}$$

with $h < e^* \leq 1$ and α^* restricted to a set as shown in Figure 8. It is easy to show that the slow-down J exceeds unity only for $\gamma < 0$ and that in this case

$$\beta = \beta(\bar{e}^*) = \frac{3\bar{e}^* - 1}{3\bar{e}^* - h}, \tag{57}$$

with $\bar{e}^*$ being the value of the effective dielectric permittivity of the composite that corresponds to the maximal slow-down. Now $\bar{e}^*$ lies in the range $\max[1/3, h] < \bar{e}^* < 1$. From (57) it follows that $\mathcal{J} > 1$ for

$$\max\left[0, \frac{3h-1}{2h}\right] < \beta < \frac{2}{3-h}. \tag{58}$$

Let $g_i^{-1} = v_i/v^*$; then, using (57), we find

$$g_1^{-1} = \frac{2(\bar{e}^*)^{3/2}}{3\bar{e}^* - 1}, \tag{59}$$

$$g_2^{-1} = \frac{2(\bar{e}^*)^{3/2}}{\sqrt{h}(3e^* - h)},$$

and

$$J \le \mathcal{J}(h,\beta) = \begin{cases} \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)h(1-h)}} & \text{for } \max[0, \frac{3h-1}{2h}] \le \beta \le \sqrt{h}, \\ \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)(1-h)\beta}} & \text{for } \sqrt{h} \le \beta \le \frac{2}{3-h}, \\ 1 & \text{otherwise,} \end{cases}$$

as stated in (54). Moreover, we find that

$$\max_\beta \mathcal{J}(h,\beta) \le \mathcal{J}(h,\sqrt{h}) = \frac{2(1+\sqrt{h}+h)^{3/2}}{\sqrt{h}(1+\sqrt{h})3^{3/2}}$$

at this optimum $\bar{e}^* = (1+\sqrt{h}+h)/3$, and the volume fraction is

$$p = \frac{1+\sqrt{h}-2h}{(1-h)(1+\sqrt{h}+h)} \sim 1-2h \qquad \text{as } h \downarrow 0.$$

Appendix A. Bounds for $\varepsilon_{\varepsilon_1}$ in the isotropic case. We seek bounds for $\varepsilon^*_{\varepsilon_1}$ in the two ($d=2$) and three ($d=3$) dimensional isotropic cases. For a fixed volume fraction, (51) gives

$$\varepsilon^*_{\varepsilon_1} \le l_I^{(d)}(e^*;p), \tag{60}$$

$$\varepsilon^*_{\varepsilon_1} \ge l_{II}^{(d)}(e^*;p), \tag{61}$$

with $h_1^{(d)}(h,p) < e^* < h_2^{(d)}(h,p)$ and $h_i^{(d)}$ the Hashin–Shtrikman bounds (68). We seek expressions for the boundary of the union of the lens-shaped regions enveloped by $l_I^{(d)}$ and $l_{II}^{(d)}$ when the union is taken with respect to the volume fraction. As we show, the boundary curves are those traced out by the endpoints of the lens-shaped regions, that is, $l_I^{(d)}(e^*;p_i^{(d)}(e^*))$, with $p_i^{(d)}(e^*)$ obtained by solving (68) with respect to p for a given e^* that attains the Hashin–Shtrikman bound. The lower envelope is defined by $l_I^{(d)}(e^*;p_1^{(d)}(e^*))$. Note that the bounds $h_i^{(d)}$ in (68) are *monotonic* in p. It then follows by a geometric argument that the envelopes are defined in the way described if

$$\frac{\partial l_I^{(d)}(e^*;p)}{\partial p} < 0 \quad \text{for } p > p_2^{(d)}(e^*), \tag{62}$$

$$\frac{\partial l_{II}^{(d)}(e^*;p)}{\partial p} < 0. \tag{63}$$

Consider the three dimensional case and (62). We find

$$\begin{aligned} \frac{\partial l_I^{(3)}(e^*;p)}{\partial p}\frac{(1-h)^3}{3h} &= -\frac{\partial\{[(p+(1-p)h-e^*)^2]/(p(1-p))\}}{\partial_p} \\ &= -\frac{(1-e^*)^2}{(1-p)^2} + \frac{(e^*-h)^2}{p^2} \equiv \mathcal{H}(p,e^*). \end{aligned}$$

Note that $\mathcal{H}(p, e^*) < 0$ for $p > (e^* - h)/(1-h)$. Since $p_2^{(3)}(e^*) = 3(e^*-h)/((1-h)(e^*+2))$, we have shown (62) for $d = 3$ and thus obtain the upper bound by substituting the expression for $p_2^{(3)}(e^*)$ into (60). This gives

$$\frac{(e^* - h)(e^* h + 2)}{(1-h)(2+h)},$$

which is the function denoted by l_I in (42).

Next consider (63). This inequality follows since

$$\frac{\partial l_{II}^{(3)}(e^*; p)}{\partial p} \frac{(1-h)^2(2+h)}{3(1+h)} = \frac{\partial \left\{ \frac{(h-e^*)^2}{p} \right\}}{\partial_p} = -\frac{(h-e^*)^2}{p^2} < 0.$$

Substituting the expression for $p_1^{(3)}(e^*)$ into (60), we thus get the lower bound

$$\frac{(e^* - h)(e^* + 2h)}{(1-h)(1+2h)},$$

denoted by l_{II} in (42).

An analogous argument for $d = 2$ gives us the bounds used in (37) and (38).

Appendix B. A condition for speed-up. For the bound on the speed-up, $\mathcal{G}$, to exceed unity, we need $\gamma < 0$. We show this for an isotropic composite with fixed volume fraction. We find by an argument similar to the one in section 3.3 that $\gamma > 0$ and $\mathcal{G} > 1$ imply that the $\bar{e}^*$ given by

$$\begin{aligned} 1 - \beta = -\frac{\gamma}{1+\alpha_2} &= \frac{\bar{e}^*}{2l'_{II}(\bar{e}^*)\bar{e}^* - l_{II}(\bar{e}^*)} \\ &= \frac{\bar{e}^*}{3c_2^{(d)}(\bar{e}^*)^2 - c_1^{(d)}\bar{e}^* - c_0^{(d)}} \end{aligned} \tag{64}$$

must satisfy $h_1^{(d)} < \bar{e}^* < h_2^{(d)}$, with $h_i^{(d)}$ being the Hashin–Shtrikman bounds. The $c_j^{(d)}$ and $h_j^{(d)}$ parameters are functions of h and p and are listed in Appendix D. As above, $d \in \{2, 3\}$ is the spatial dimension. By our assumptions, $\alpha_2 > 0$; hence $\gamma > 0$ implies $1 - \beta < 0$. However, as we show next, the expression in (64) is strictly positive for $h_1^{(d)} < \bar{e}^* < h_2^{(d)}$. Note first that $g(\cdot)$ defined by

$$\frac{\bar{e}^*}{1-\beta} = -\frac{\bar{e}^*(1+\alpha_2)}{\gamma} = 3c_2^{(d)}(\bar{e}^*)^2 - c_1^{(d)}\bar{e}^* - c_0^{(d)} \equiv g(\bar{e}^*)\bar{e}^*$$

is monotone and increasing. Thus we need only show that $g(h_1^{(d)}) \geq 0$ to conclude that $\mathcal{G} \equiv 1$ if $\gamma > 0$. Consider the three dimensional case. We can write the Hashin–Shtrikman bound as

$$\begin{aligned} h_1^{(3)} &= h(1+b_1), \\ \frac{1}{h_1^{(3)}} &= \frac{1}{h}(1-b_2), \end{aligned} \tag{65}$$

with

$$b_1 \equiv \frac{3p(1-h)}{3h+(1-p)(1-h)},$$
$$b_2 \equiv \frac{3p(1-h)}{3h+(1+2p)(1-h)}.$$

Then we have

$$\begin{aligned} g\big(h_1^{(3)}\big)\frac{(1-h)(2+h)}{h} &= -6-3b_1+2b_2+\frac{3(1+h)}{p(1-h)}(3b_1+b_2) \\ &> -6+\frac{3(1+h)}{p(1-h)}(2b_1+b_2) \\ &> -6+3\left(\frac{6}{3h+(1-p)(1-h)}+\frac{3}{3h+(1+2p)(1-h)}\right) > 0, \end{aligned}$$

as we set out to show.

A similar argument applies to the two dimensional case.

Appendix C. Restriction on the effective dielectric permittivity ε^*. Consider the effective dielectric permittivity associated with a maximum speed-up $\bar{e}^*$. We show that in the isotropic case with the volume fraction a free parameter, $\bar{e}^* \in (h, .342)$ for $d = 2$. A similar argument shows that $\bar{e}^* \in (h, .756)$ for $d = 3$.

We assume that $\gamma < 0$; only then do we have an actual speed-up of the composite relative to both of the two constituents. When $\gamma < 0$, the curve corresponding to the lower boundary of the set marked $\mathcal{A}$ in Figure 8, α_I, has constant and negative curvature and is given by (37):

$$\alpha_I(e^*) = l_I(e^*)\gamma + e^*\alpha_2.$$

The dashed line in Figure 8 that characterizes the minimum slowness ($\bar{s}$) is

$$\sqrt{e^*}\bar{s} - e^* \tag{66}$$

and has a curvature that decreases monotonically with e^*. The curve may therefore be tangential to the set $\mathcal{A}$ at $e^* = \bar{e}^*$ and *also* intersect the lower boundary for some e^* with $\bar{e}^* < e^* \le 1$. Then we do not have a speed-up relative to the pure phases. For a positive speed-up we need

$$\bar{s} - 1 < \alpha_I(1) = \alpha_1. \tag{67}$$

Next, we translate this constraint into a constraint on e^*. Let $s(h)$ be the slowness of phase 2. From (35) we get

$$s(h) = \sqrt{h}(1+\alpha_2).$$

Recall that g_2 is the speed-up relative to phase 2:

$$g_2 = \frac{s(h)}{\bar{s}} = \frac{\sqrt{h}(1+\alpha_2)}{\bar{s}}.$$

The constraint (67) can thus be written

$$\bar{s} = \frac{\sqrt{h}(1+\alpha_2)}{g_2} < (1+\alpha_1)$$

or

$$\sqrt{h} < \frac{1+\alpha_1}{1+\alpha_2} g_2 = \beta g_2.$$

Using the expressions for β and g_2 given in, respectively, (39) and (40), the above inequality becomes

$$\sqrt{h} < \frac{3h(\bar{e}^*)^2 + (1-h^2)\bar{e}^* + h}{2\sqrt{h\bar{e}^*}((\bar{e}^*)^2+1)} \frac{3h(\bar{e}^*)^2 + h}{3h(\bar{e}^*)^2 + (1-h^2)\bar{e}^* + h}.$$

Defining $x \equiv \sqrt{\bar{e}^*}$, we find from this

$$2x^5 - 3x^4 + 2x - 1 = (x-1)^2(2x^3 + x^2 - 1) < 0.$$

Note that this equation is independent of the parameters of the problem, and the appropriate root of $2x^3 + x^2 - 1 = 0$ leads to $\bar{e}^* \leq .432$.

For $d = 3$ an analogous argument shows that in this case the appropriate root of $2x^3 + x^2 - 2 = 0$ leads to $\bar{e}^* \leq .756$.

Appendix D. Some results used in the analysis. For reference we list some bounds used above.

Hashin and Shtrikman [11] give bounds for the effective dielectric permittivity of an isotropic dielectric composite, given the volume fraction and the dielectric permittivities of the two isotropic phases. The bounds for $e^* = \varepsilon^*/\varepsilon_1$ are

$$\begin{aligned} e^* &\geq h_1^{(d)}(h,p) = h + \frac{dph(1-h)}{dh + (1-p)(1-h)}, \\ e^* &\leq h_2^{(d)}(h,p) = 1 - \frac{d(1-p)(1-h)}{d - p(1-h)}, \end{aligned} \tag{68}$$

with h defined in (9), p being the volume fraction of phase 1, and $d \in \{2,3\}$ being the spatial dimension.

Axell [2] gives bounds for $\varepsilon^*_{\varepsilon_1}$ for two and three dimensional isotropic composites with two isotropic constituents. They are

$$\begin{aligned} \varepsilon^*_{\varepsilon_1} &\geq c_0^{(d)} - c_1^{(d)} e^* + c_2^{(d)} (e^*)^2, \\ \varepsilon^*_{\varepsilon_1} &\leq -c_3^{(d)} + c_4^{(d)} e^* - c_5^{(d)} (e^*)^2, \end{aligned} \tag{69}$$

with

$$\begin{aligned} c_0^{(2)} &= \tfrac{2h^2}{p(1-h)^2} + \tfrac{h^2}{1-h^2}, & c_0^{(3)} &= \tfrac{3h^2(1+h)}{p(1-h)^2(2+h)} + \tfrac{2h^2}{(1-h)(2+h)}, \\ c_1^{(2)} &= \tfrac{4h}{p(1-h)^2}, & c_1^{(3)} &= \tfrac{6h(1+h)}{p(1-h)^2(2+h)} + \tfrac{h}{(1-h)(2+h)}, \\ c_2^{(2)} &= \tfrac{2}{p(1-h)^2} - \tfrac{1}{1-h^2}, & c_2^{(3)} &= \tfrac{3(1+h)}{p(1-h)^2(2+h)} - \tfrac{1}{(1-h)(2+h)}, \\ c_3^{(2)} &= \tfrac{2h}{(1-p)(1-h)^2} - \tfrac{h}{1-h^2}, & c_3^{(3)} &= \tfrac{h}{(1-h)} + \tfrac{3h(p+(1-p)h)^2}{p(1-p)(1-h)^3}, \\ c_4^{(2)} &= 1 + \tfrac{4h}{(1-p)(1-h)^2}, & c_4^{(3)} &= \tfrac{1}{(1-h)} + \tfrac{6h(p+(1-p)h)}{p(1-p)(1-h)^3}, \\ c_5^{(2)} &= \tfrac{2h}{(1-p)(1-h)^2} + \tfrac{h}{1-h^2}, & c_5^{(3)} &= \tfrac{3h}{p(1-p)(1-h)^3}. \end{aligned}$$

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