Submitted to the Annals of Applied Probability

A GENERAL FRAMEWORK FOR WAVES IN RANDOM MEDIA WITH LONG-RANGE CORRELATIONS

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We consider waves propagating in a randomly layered medium with long-range correlations. An example of such a medium is studied in [19] and leads in particular to an asymptotic travel time described in terms of a fractional Brownian motion. Here we study the asymptotic transmitted pulse under very general assumptions on the long-range correlations. In the framework that we introduce in this paper, we prove in particular that the asymptotic time-shift can be described in terms of non-Gaussian and/or multifractal processes.

1. Introduction. Wave propagation in random media has been extensively studied for many years from both theoretic and applied points of view. In particular, the study of the effective shape of an acoustic pulse propagating through a layered medium has attracted a lot of attention [1, 5, 27]. Recently, applications to time reversal [11] have also attracted a lot of attention. Currently there is also a strong interest in problems related to noise and correlations, [12]. In all these cases the statistical properties of the medium are important since they affect the statistical properties of the wave field.

In [5] the authors consider an acoustic pulse propagating in a one-dimensional random medium with rapidly decaying correlations. They rigorously prove the classical O'Doherty and Anstey's result [20] that establishes that the effective transmitted pulse is characterised by deterministic spreading and a random time-shift. More precisely, the deterministic spreading is expressed as a convolution with a Gaussian density and the random time-shift is described in terms of a Brownian motion.

More recently, wave propagation in random media with long-range correlations and/or defined in terms of fractional Brownian motion [2, 13, 19, 26] has been considered. In [19], we extend the result of [5] to such a framework. Then, the asymptotic description of the transmitted pulse is dramatically different from what happens in a mixing case. Indeed, the pulse keeps its initial shape, and its random time-shift is now described in terms of a fractional Brownian motion whose Hurst index depends on the decay rate of

AMS 2000 subject classifications: 34F05, 34E10, 37H10, 60H20

Keywords and phrases: Waves in random media, Long range dependence, Fractional and multifractional processes

the correlation function of the random fluctuations. We considered in [19] a particular form of a random process describing the medium, such that it was roughly speaking close to a Gaussian process. Thus, it still remains to study more general cases under long-range assumptions. This is the aim of the present work. We establish that under general long-range assumptions on the medium, the effective pulse still keeps its initial shape as observed in [19], but the time-shift can be very different, non-Gaussian for instance, depending on the form of the random fluctuations. Besides, our general result allow us to deal with media with a decay rate correlations varying along the propagation direction. This leads to an effective time-shift, described in terms of a multifractional random process which is, roughly speaking, a fractional Brownian motion with a varying Hurst index, that reflects the non-homogeneity of the propagation medium.

In Section 2 we introduce the problem and review the basic wave decomposition approach. Next, we establish the general technical result (Theorem 3.1) in Section 3 that we apply to non-Gaussian media in Section 4, to multifractal Gaussian media in Section 5 and to multifractal non-Gaussian media in Section 6 where we prove the main result of the paper (Theorem 6.1). Finally, Section 7 is devoted to the derivation of Threorem 3.1.

2. Preliminaries.

2.1. *Wave Decomposition.* The governing equations are the non-dimensionalized Euler equations giving conservation of moments and mass:

$$\rho^\varepsilon(z)\frac{\partial u^\varepsilon}{\partial t}(z,t)+\frac{\partial p^\varepsilon}{\partial z}(z,t) \;=\; 0\,, \tag{2.1}$$

$$\frac{1}{K^\varepsilon(z)}\frac{\partial p^\varepsilon}{\partial t}(z,t)+\frac{\partial u^\varepsilon}{\partial z}(z,t) \;=\; 0\,, \tag{2.2}$$

where t is the time, z is the depth into the medium, p^ε is the pressure and u^ε the particle velocity. The medium parameters are the density ρ^ε and the bulk-modulus K^ε (reciprocal of the compressibility). We assume that ρ^ε is a constant identically equal to one in our non-dimensionalized setting and $1/K^\varepsilon$ is modeled as follows

$$\frac{1}{K^\varepsilon(z)}=\begin{cases} 1+\mu^\varepsilon(z) & \text{for } z\in[0,Z]\,,\\ 1 & \text{for } z\in\mathbb{R}-[0,Z]\,,\end{cases} \tag{2.3}$$

where μ^ε is a centered random process. The number $\varepsilon>0$ is a parameter that all quantities depend on. As we will see below it is introduced to describe the scales of the problem.

We introduce the right- and left-going waves:

$$A^\varepsilon = p^\varepsilon + u^\varepsilon \qquad \text{and} \qquad B^\varepsilon = u^\varepsilon - p^\varepsilon \,. \tag{2.4}$$

The boundary conditions are of the form

$$A^\varepsilon(z=0,t) = f(t/\varepsilon^\tau) \qquad \text{and} \qquad B^\varepsilon(z=Z,t) = 0\,, \tag{2.5}$$

for a positive real number $\tau > 0$ and a source function f. This indicate that the energy entering the medium is of order ε^τ. In order to deduce a description of the transmitted pulse, we open a window of size ε^τ in the neighborhood of the travel time of the homogenized medium and define the processes

$$a^\varepsilon(z,s) = A^\varepsilon(z, z+\varepsilon^\tau s) \qquad \text{and} \qquad b^\varepsilon(z,s) = B^\varepsilon(z, -z+\varepsilon^\tau s)\,. \tag{2.6}$$

Observe that the background or homogenized medium in our scaling has a constant speed of sound equal to unity and that the medium is matched so that in the frame introduced in (2.6) the pulse shape is constant if $\mu^\varepsilon \equiv 0$ or if we consider the homogenized medium, [11]. We introduce next the Fourier transforms $\widehat{a}^\varepsilon$ and $\widehat{b}^\varepsilon$ of a^ε and b^ε respectively:

$$\widehat{a}^\varepsilon(z,\omega) = \int_{-\infty}^{\infty} e^{i\omega s} a^\varepsilon(z,s) ds \qquad \text{and} \qquad \widehat{b}^\varepsilon(z,\omega) = \int_{-\infty}^{\infty} e^{i\omega s} b^\varepsilon(z,s) ds\,,$$

that satisfy

$$\frac{d\widehat{a}^\varepsilon}{dz} = \frac{i\omega}{2}\nu^\varepsilon(z)\left(\widehat{a}^\varepsilon - e^{-2i\omega z/\varepsilon^\tau}\widehat{b}^\varepsilon\right), \qquad \widehat{a}^\varepsilon(0,\omega) = \widehat{f}(\omega)\,, \tag{2.7}$$

$$\frac{d\widehat{b}^\varepsilon}{dz} = \frac{i\omega}{2}\nu^\varepsilon(z)\left(e^{2i\omega z/\varepsilon^\tau}\widehat{a}^\varepsilon - \widehat{b}^\varepsilon\right), \qquad \widehat{b}^\varepsilon(Z,\omega) = 0 \tag{2.8}$$

where we use the notation

$$\nu^\varepsilon = \frac{\mu^\varepsilon}{\varepsilon^\tau}. \tag{2.9}$$

Following [5, 11] we express the previous system of equations in term of propagator $P^\varepsilon_\omega(z)$ which can be written as

$$P^\varepsilon_\omega(z) = \begin{pmatrix} \alpha^\varepsilon_\omega(z) & \overline{\beta^\varepsilon_\omega}(z) \\ \beta^\varepsilon_\omega(z) & \overline{\alpha^\varepsilon_\omega}(z) \end{pmatrix}, \tag{2.10}$$

and that satisfies

$$\frac{dP^\varepsilon_\omega}{dz}(z) = \mathcal{H}^\varepsilon_\omega(\frac{z}{\varepsilon^\tau}, z)P^\varepsilon_\omega(z), \qquad P^\varepsilon_\omega(z=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

with

$$\mathcal{H}^{\varepsilon}_{\omega}(z_1, z_2) = \frac{i\omega}{2}\nu^{\varepsilon}(z_2)\begin{pmatrix} 1 & -e^{-2i\omega z_1} \\ e^{2i\omega z_1} & -1 \end{pmatrix}.$$

Defining next the transmission coefficient T^{ε}_{ω} and the reflection coefficient R^{ε}_{ω} by

$$T^{\varepsilon}_{\omega}(z) = \frac{1}{\overline{\alpha^{\varepsilon}_{\omega}(z)}} \quad \text{and} \quad R^{\varepsilon}_{\omega}(z) = \frac{\beta^{\varepsilon}_{\omega}(z)}{\overline{\alpha^{\varepsilon}_{\omega}(z)}}\,, \tag{2.12}$$

we can write

$$a^{\varepsilon}(Z, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\omega} T^{\varepsilon}_{\omega}(Z)\widehat{f}(\omega)\, d\omega\,, \tag{2.13}$$

and

$$b^{\varepsilon}(0, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\omega} R^{\varepsilon}_{\omega}(Z)\widehat{f}(\omega)\, d\omega\,. \tag{2.14}$$

Hence we shall study the asymptotics of the propagator P^{ε}_{ω} in order to characterize a^{ε} and b^{ε} as ε goes to 0.

2.2. *A short-range medium.* We recall now what happens in a mixing (or short-range) model when $\tau = 1$ and $\mu^{\varepsilon}(z) = \nu(z/\varepsilon^2)$. We assume that $\nu = \Phi \circ m$ where Φ is a bounded function and m is a centered Markov process with an invariant probability measure whose generator satisfies the Fredholm alternative. This implies that the correlation length σ of the medium is finite:

$$\sigma^2 = \int_0^{\infty} |\mathbb{E}[\nu(0)\nu(z)]|\, dz \in [0, \infty)\,.$$

This property is the mixing property or the short-range property. It is well known [5, 11] that under these assumptions the propagator equations P^{ε}_{ω} converge to a system of stochastic differential equations driven by independent Brownian motions from which we can deduce that $a^{\varepsilon}(Z, s) \longrightarrow \widetilde{a}(Z, s)$ as ε goes to 0 with

$$\widetilde{a}(Z, s) = (f * G)(s - B)\,, \tag{2.15}$$

where G is a centered Gaussian density with variance $\sigma^2 Z/2$ and B a Gaussian random variable that can be expressed in terms of a Brownian motion W as $B = \sigma W(Z)/\sqrt{2}$. Proving this result involves using the Diffusion Approximation Theorem [11] to get an asymptotic propagator from which we can deduce the expression of the limit $\widetilde{a}(Z, s)$. Notice that, whereas the variance of B depends in particular on Φ, the result does not depend qualitatively on Φ in the sense that B remains Gaussian whatever Φ is.

2.3. *A long-range medium.* In [19], the propagation in a long-range medium is investigated. The model considered is defined in terms of a fractional Brownian motion as $\nu(z) = \Phi(m(z))$ for every z where:

- Φ is an odd $\mathcal{C}^\infty-$function.
- m is a Gaussian process, centered, stationary and has a correlation function r_m which has the following asymptotic property as z goes to ∞:

$$r_m(z) = \mathbb{E}[m(0)m(z)] \sim c_m z^{-\gamma}\,, \quad \gamma \in (0,1)\,. \tag{2.16}$$

The property (2.16) implies that the covariance function r_ν of ν is not integrable:

$$\int_0^\infty |r_\nu(z)|\, dz = \infty\,,$$

which means that the correlation length is infinite. This is the so-called long-range property. We mention that a typical example of a process satisfying (2.16) can be constructed as

$$m(z) = W_H(z+1) - W_H(z)\,, \tag{2.17}$$

where B_H is a fractional Brownian motion (fBm in short) with Hurst parameter $H > 1/2$. In this case, we proved that $a^\varepsilon(Z,s) \longrightarrow \widetilde{a}(Z,s)$ with

$$\widetilde{a}(Z,s) = f(s - B)\,, \tag{2.18}$$

where B a Gaussian random variable. We can write B as $B = \sigma_H W_H(Z)$ where W_H is a fractional Brownian motion with Hurst parameter $H = (2-\gamma)/2$ and σ_H is a positive constant that depends on H and Φ.

3. Medium assumptions and main technical result. The results presented above show that statistical properties of ν strongly affect the asymptotic behavior of the pulse shape $a^\varepsilon(Z,s)$. In Sections 4 and 5 we carry out the analysis of the particular long-range media that we consider in this paper. To facilitate this analysis we establish in this section a theorem under the following general assumptions on $\nu^\varepsilon = \mu^\varepsilon/\varepsilon^\tau$. Let $\lambda > 0$ and define:

- Assumption $\mathbf{A_1}$: As ε goes to 0, the finite-dimensional distributions of the process $\{\int_0^z \nu^\varepsilon(z')\,dz'\}_z$ converge to those of a process $V = \{V(z)\}_z$ with finite second-order moments.
- Assumption $\mathbf{A_2}(\lambda)$: There exist two symmetric, continuous and two-variable functions $\gamma : [0,Z]^2 \to [\gamma_-,\gamma_+] \subset (0,1)$ and $R : [0,Z]^2 \to \mathbb{R}_+$

such that for every $\delta > 0$, there exists $z_\delta > 0$ sufficiently large such that for every z_1, z_2 and ε satisfying $|z_1 - z_2| > \varepsilon^\lambda z_\delta$,

$$|\mathbb{E}[\nu^\varepsilon(z_1)\nu^\varepsilon(z_2)] - R(z_1, z_2)|z_1 - z_2|^{-\gamma(z_1,z_2)}| \le \delta R(z_1, z_2)|z_1 - z_2|^{-\gamma(z_1,z_2)}.$$

- Assumption $\mathbf{A_3}(\lambda)$. For every $\rho > 0$ there exist $C_\rho > 0$ and $\gamma_\rho \in (0, 1)$ such that $\mathbb{E}[\nu^\varepsilon(z_1)\nu^\varepsilon(z_2)] \le C_\rho|z_1 - z_2|^{-\gamma_\rho}$ for every $\varepsilon > 0$ and $|z_1 - z_2|/\varepsilon^\lambda < \rho$.

Assumption $\mathbf{A_1}$ is merely the convergence of the travel-time. Assumptions $\mathbf{A_2}(\lambda)$ and $\mathbf{A_3}(\lambda)$ are long-range assumptions for non-stationary processes. They describe how the long-range property varies with the propagation distance. In particular, these enable us to apply the next theorem to multifractal media (Sections 5 and 6), which are non-homogeneous.

Here we give the main technical result of this paper.

THEOREM 3.1. *Assume that there exists $\lambda > 0$ such that $\mathbf{A_1}$, $\mathbf{A_2}(\lambda)$ and $\mathbf{A_3}(\lambda)$ are satisfied. Then, as ε goes to 0, $\{a^\varepsilon(Z, s)\}_s$ converges in distribution in the space of continuous functions endowed with the uniform topology to the random process $\{\widetilde{a}(Z, s)\}_s$ that can be written as*

$$\widetilde{a}(Z, s) = f\left(s - \frac{1}{2}V(Z)\right) . \tag{3.1}$$

Theorem 3.1 establishes that, under general long-range assumptions, if the travel-time converges then the asymptotic pulse keeps its initial shape but its time shift is described in terms of the asymptotic travel-time. As recalled in Subsection 2.3 this fact was observed in a particular case in [19]. In fact, the result of [19] follows from Theorem 3.1. Indeed, the model presented in Subsection 2.3 satisfies $\mathbf{A_1}$, $\mathbf{A_2}(2)$ and $\mathbf{A_3}(2)$. In particular, the finite-dimensional distributions of $\{\int_0^z \nu^\varepsilon(z')\,dz'\}_z$ converge to those of the process $\{\sigma_H W_H(z)\}_z$, so that the asymptotic pulse is of the form (2.18).

Theorem 3.1 is next used in Sections 4 and 5 to establish the asymptotic pulse shape respectively in non-Gaussian and multifractal media.

4. Non-Gaussian asymptotics.

In this section we study the case where ν^ε has the form

$$\nu^\varepsilon(z) = \varepsilon^{\kappa-\tau}\nu\left(\frac{z}{\varepsilon^2}\right) \text{ for } z \in [0, Z]$$

where $\kappa > 0$ and ν is a process that is assumed to have the form

$$\nu(z) = \Phi(m(z))$$

for every z where:

- Φ is a continuous function such that $\Phi(\sigma_0 \times \cdot)$ has a Hermite index equal to $K \in \mathbb{N}^*$, where $\sigma_0^2 = \mathbb{E}[m(0)^2]$.
- m is a continuous Gaussian process, centered, stationary and has a correlation function r_m which has the following asymptotic property as z goes to ∞:

$$r_m(z) = \mathbb{E}[m(0)m(z)] \sim c_m z^{-\gamma} \tag{4.1}$$

  where $0 < \gamma < 1/K$.

We denote the $K-$th Hermite coefficient of $\Phi(\sigma_0 \times \cdot)$

$$J(K) = \mathbb{E}[\Phi(\sigma_0 X) P_K(X)]$$

where $X \sim \mathcal{N}(0,1)$ and P_K is the $K-$th Hermite polynomial. Applying Theorem 3.1 we get the following result.

THEOREM 4.1. *Assume that $\tau - \kappa = \gamma K$. Then, as ε goes to 0, $\{a^\varepsilon(Z,s)\}_s$ converges in distribution in the space of continuous functions endowed with the uniform topology to the random process $\{\widetilde{a}(Z,s)\}_s$ that can be written as*

$$\widetilde{a}(Z,s) = f\left(s - \frac{1}{2} W_H^K(Z)\right), \tag{4.2}$$

where W_H^K is the $K-$th Hermite process of index $H = (2-\gamma K)/2 \in (1/2,1)$ defined for every z by

$$W_H^K(z) = \frac{1}{\sigma_0} \int_{\mathbb{R}^K} \mathcal{G}_{H,K}(z, x_1, \cdots, x_K) \prod_{k=1}^K \widehat{B}(dx_k) \tag{4.3}$$

with

$$\mathcal{G}_{H,K}(z, x_1, \cdots, x_K) = \frac{J(K) e^{-iz\sum_{j=1}^K x_j} - 1}{K! C(H)^K \sum_{j=1}^K x_j} \prod_{k=1}^K \frac{x_k}{|x_k|^{(H-1)/K+3/2}}$$

where $\widehat{B}(dx)$ is the Fourier transform of a Brownian measure,

$$C(H)^2 = \int_{-\infty}^{\infty} \frac{|e^{-ix}-1|^2}{|x|^{2H+1}} \, dx = \frac{\pi}{H\Gamma(2H)\sin(\pi H)}.$$

and the multiple stochastic integral is in the sense of [9].

For $H \in (1/2, 1)$ and $K \in \mathbb{N}^*$ given, the Hermite process defined by (4.3) was introduced independently in [10] and [28]. Its increments are stationary and its covariance is

$$\mathbb{E}[W_H^K(z_1) W_H^K(z_2)] = \frac{1}{2}(|z_1|^{2H} + |z_2|^{2H} - |z_1 - z_2|^{2H}).$$

It is selfsimilar and $H-$Hölder. It is Gaussian if and only if $K = 1$, thus, it is a fractional Brownian motion if and only if $K = 1$. As a consequence, the result of [19] corresponds to the case of $K = 1$ in Theorem 4.1. Moreover, this result is in dramatic contrast to the short range case where the asymptotics does not depend qualitatively on Φ.

PROOF. Following [10] or [28], we find that the finite-dimensional distributions of the antiderivative of ν^ε converge to those of W_H^K, therefore, $\mathbf{A}_1$ is satisfied. Next we show that $\mathbf{A}_2(2)$ and $\mathbf{A}_3(2)$ hold. Because of the stationarity of m it is enough to show that

$$\mathbb{E}[\nu(0)\nu(z)] \sim c_\nu z^{-K\gamma} \qquad \text{as } z \to \infty \tag{4.4}$$

for some constant $c_\nu > 0$. In view of (4.4) we can write

$$\nu(z) = \Phi\left(\sigma_0 \frac{m(z)}{\sigma_0}\right) = \sum_{k=K}^{\infty} \frac{J(k)}{k!} P_k\left(\frac{m(z)}{\sigma_0}\right).$$

Using the properties of the Hermite polynomials we get

$$\begin{aligned} \mathbb{E}[\nu(0)\nu(z)] &= \sum_{k=K}^{\infty} \frac{J(k)^2}{(k!)^2} \mathbb{E}\left[P_k\left(\frac{m(0)}{\sigma_0}\right) P_k\left(\frac{m(z)}{\sigma_0}\right)\right] \\ &= \sum_{k=K}^{\infty} \frac{J(k)^2}{k!\sigma_0^{2k}} r_m(z)^k. \qquad (4.5) \end{aligned}$$

Therefore, we need to study the limit of

$$z^{\gamma K} \mathbb{E}[\nu(0)\nu(z)] = \sum_{k=K}^{\infty} \frac{J(k)^2}{k!\sigma_0^{2k}} z^{\gamma K} r_m(z)^k.$$

Observe that for $k = K$ we have $z^{\gamma K} r_m(z) \sim c$ as $z \to \infty$ and for $k > K$ we have $z^{\gamma K} r_m(z)^k \to 0$. Moreover, we have the uniform upper bound for z sufficiently large:

$$\frac{J(k)^2}{k!\sigma_0^{2k}} z^{\gamma K} |r_m(z)|^k \le \frac{J(k)^2}{k!}.$$

Using the fact that $\sum_{k=1}^{\infty} \frac{J(k)^2}{k!} < \infty$, (4.4) follows from the uniform convergence theorem. $\square$

5. Application to multifractal media. In this section we study the case where the asymptotic medium is described in terms of a multifractional process. In all the situations described above, the media were asymptotically expressed in terms of fractional processes. A drawback of fractional processes for applications is the strong homogeneity of their properties, which are described by their (constant) Hurst index. Therfore, multifractional processes have attracted much attention [3, 24]. Multifractional processes have locally the same properties as fractional processes. Their properties are governed by a $(0,1)-$valued function h which is called the multifractional function. Some of the main properties are that multifractional processes are locally self-similar and their pointwise Hölder exponents vary along their trajectory. In particular, multifractional processes are relevant in order to describe non-homogeneous media. Before stating the main result of this section, we mention that the most famous multifractional process is the multifractional Brownian motion. It was independently introduced in [3, 24] and can be defined from the harmonizable representation of fractional Brownian motion for every z:

$$W_H(z) = \frac{1}{C(H)} \int_{-\infty}^{\infty} \frac{e^{-izx} - 1}{|x|^{H+1/2}} \widehat{B}(dx), \tag{5.1}$$

where $\widehat{B}$ is the Fourier transform of a real Gaussian measure B and the constant $C(H)$ is a renormalisation constant and can be written as

$$C(H)^2 = \int_{-\infty}^{\infty} \frac{|e^{-ix} - 1|^2}{|x|^{2H+1}} \, dx = \frac{\pi}{H\Gamma(2H)\sin(\pi H)}.$$

Now we consider a $(0,1)-$valued function h and we substitute H by $h(z)$ for every z to obtain

$$W_h(z) = \frac{1}{\widetilde{C}(z)} \int_{-\infty}^{\infty} \frac{e^{-izx} - 1}{|x|^{h(z)+1/2}} \widehat{B}(dx), \tag{5.2}$$

where the constant $\widetilde{C}(z)$ is a renormalisation function.

We shall here use a different framework for the multifractal modeling that is convenient for the asymptotic analysis and describe this next. We assume that ν^ε has the form

$$\nu^\varepsilon(z) = \varepsilon^{\kappa(z)-\tau} \nu\left(\frac{z}{\varepsilon^2}, z\right) \text{ for } z \in [0, Z]$$

where κ is a positive function and ν is a field that is written as $\nu(z_1, z_2) = \Phi(m(z_1, h(z_2)))$ for every z_1 and z_2 where:

- Φ is a continuous function with Hermite index 1.
- h is a continuous function taking values in $[h_-, h_+] \subset (1/2, 1)$.
- $m = \{m(z,H)\}_{z,H}$ is a centered and continuous Gaussian field such that $\mathbb{E}[m(z,H)^2] = 1$ for every z and H and that satisfies:
  - For every $M > 0$ the map
  $$(z_1, z_2, H_1, H_2) \longmapsto \mathbb{E}[m(z_1, H_1)m(z_2, H_2)]$$
  is bounded on $\{(z_1, z_2) \in \mathbb{R}_+^2, |z_1 - z_2| \leq M\} \times [h_-, h_+]^2$.
  - There exists a continuous function $R : [h_-, h_+]^2 \to (0, \infty)$ (that we call the asymptotic covariance of m) such that
  $$\lim_{z_1 - z_2 \to \infty} \sup_{(H_1, H_2)} |R(H_1, H_2) - (z_1 - z_2)^{2-H_1-H_2}\mathbb{E}[m(z_1, H_1)m(z_2, H_2)]| = 0.$$

These assumptions describe that the field m has the long-range property with respect to the variable z. They also express that for each H, the process $m(\cdot, H)$ is stationary and asymptotically fractional because it satisfies the classical invariance principle. As established in [6] this field enable us to define a process that is asymptotically multifractional.

Applying Theorem 3.1 we now get:

THEOREM 5.1. *Let $\gamma(z) := \tau - \kappa(z)$ and assume $h(z) = (2 - \gamma(z))/2$. Then, as ε goes to 0, $\{a^\varepsilon(Z, s)\}_s$ converges in distribution in the space of continuous functions endowed with the uniform topology to the random process $\{\widetilde{a}(Z, s)\}_s$ that can be written as*

$$\widetilde{a}(Z, s) = f\left(s - \frac{1}{2}S_h(Z)\right), \tag{5.3}$$

where S_h is a centered Gaussian process with covariance given for $z_1, z_2 \geq 0$ by:

$$\mathbb{E}[S_h(z_1)S_h(z_2)] = J(1)^2 \int_0^{z_1} du_1 \int_0^{z_2} du_2\, \widetilde{\mathcal{R}}(u_1, u_2), \tag{5.4}$$

where

$$\widetilde{\mathcal{R}}(u_1, u_2) = \mathcal{R}(u_1, u_2; h(u_1), h(u_2))|u_1 - u_2|^{h(u_1)+h(u_2)-2}$$

with

$$\mathcal{R}(z_1, z_2; H_1, H_2) = R(H_1, H_2)1_{z_1 \geq z_2} + R(H_2, H_1)1_{z_1 < z_2}. \tag{5.5}$$

The process S_h was introduced in [6]. This process is continuous and multifractional in the sense that its pointwize Hölder exponent is $h(t_0)$ at the point t_0:

$$\sup\left\{H, \lim_{\varepsilon\to 0}\frac{S_h(t_0+\varepsilon)-S_h(t_0)}{|\varepsilon|^H}=0\right\}=h(t_0).$$

Notice that in the case of h is constant Theorem 5.1 corresponds to the result of [19].

PROOF. By the same procedure as in proving (4.4), we get from the asymptotic assumptions for $\{m(z,H)\}$ that

$$\lim_{z_1-z_2\to\infty}\sup_{(H_1,H_2)\in[h_-,h_+]^2}\left|(z_1-z_2)^{2-H_1-H_2}\mathbb{E}[\nu(z_1,H_1)\nu(z_2,H_2)]-J(1)^2R(H_1,H_2)\right|=0.$$

If we denote respectively v^ε and v_1^ε the antiderivatives of $z\mapsto\nu^\varepsilon(z)$ and $z\mapsto\varepsilon^{2h(z)-2}m\left(z/\varepsilon^2,h(z)\right)$, then, by using the same argument as above we also get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|v^\varepsilon(z)-J(1)v_1^\varepsilon(z)|^2\right]=0, \tag{5.6}$$

which implies that the convergence of the finite dimensional distributions of v^ε can be reduced to those of v_1^ε. Hence, without loss of generality we can assume that Φ =Id. Following [6], the finite-dimensional distributions of the antiderivative of ν^ε converges to those of S_h, thus $\mathbf{A}_1$ is fulfilled. Now we check $\mathbf{A}_2(2)$. We can write

$$\nu^\varepsilon(z)=\varepsilon^{2h(z)-2}m\left(\frac{z}{\varepsilon^2},h(z)\right).$$

We let $\delta>0$ and thanks to the asymptotic assumption on m, there exists z_δ such that for every z_1, z_2 and ε satisfying $|z_1-z_2|>\varepsilon^2 z_\delta$ we have

$$\sup_{(H_1,H_2)}\left|\left|\frac{z_1-z_2}{\varepsilon^2}\right|^{2-H_1-H_2}\mathbb{E}\left[m\left(\frac{z_1}{\varepsilon^2},H_1\right)m\left(\frac{z_2}{\varepsilon^2},H_2\right)\right]-\mathcal{R}\left(\frac{z_1}{\varepsilon^2},\frac{z_2}{\varepsilon^2};H_1,H_2\right)\right|<\delta.$$

Then, noting that $\mathcal{R}(z_1/\varepsilon^2,z_2/\varepsilon^2,H_1,H_2)=\mathcal{R}(z_1,z_2,H_1,H_2)$ and substituting (H_1,H_2) by $(h(z_1),h(z_2))$ we get

$$\left|\left|\frac{z_1-z_2}{\varepsilon^2}\right|^{2-h(z_1)-h(z_2)}\mathbb{E}\left[m\left(\frac{z_1}{\varepsilon^2},h(z_1)\right)m\left(\frac{z_2}{\varepsilon^2},h(z_2)\right)\right]-\mathcal{R}\left(z_1,z_2;h(z_1),h(z_2)\right)\right|<\delta.$$

Letting $\mathcal{R}^*(z_1, z_2) := \mathcal{R}(z_1, z_2; h(z_1), h(z_2))$ and noticing that $\sup(1/\mathcal{R}^*) < \infty$ (because $\inf \mathcal{R}^* > 0$) we obtain

$$\begin{aligned}\Big|\mathbb{E}\left[\nu^\varepsilon(z_1)\nu^\varepsilon(z_2)\right] - \mathcal{R}^*(z_1, z_2)\left|z_1 - z_2\right|^{h(z_1)+h(z_2)-2}\Big| \\ < \delta\mathcal{R}^*(z_1, z_2)\left|z_1 - z_2\right|^{h(z_1)+h(z_2)-2}\sup(1/\mathcal{R}^*),\end{aligned}$$

which proves $\mathbf{A_2}(2)$. It remains to check $\mathbf{A_3}(2)$. Let $\rho > 0$. Because of the boundedness assumption on m, there exists a constant $C_1(\rho) > 0$ so that for every z_1, z_2 and ε satisfying $|z_1 - z_2|/\varepsilon^2 < \rho$, we have

$$\left|\mathbb{E}\left[m\left(\frac{z_1}{\varepsilon^2}, H_1\right) m\left(\frac{z_2}{\varepsilon^2}, H_2\right)\right]\right| \leq C_1(\rho).$$

Thus,

$$\begin{aligned}\left|\mathbb{E}[\nu^\varepsilon(z_1)\nu^\varepsilon(z_2)]\right| &\leq C_1(\rho)\varepsilon^{2h(z_1)+2h(z_2)-4} \\ &= C_1(\rho)\left|z_1 - z_2\right|^{h(z_1)+h(z_2)-2}\left|\frac{z_1 - z_2}{\varepsilon^2}\right|^{2-h(z_1)-h(z_2)} \\ &= C_1(\rho)\left|z_1 - z_2\right|^{h(z_1)+h(z_2)-2}\rho^{2-h(z_1)-h(z_2)} \\ &\leq C_2(\rho)\left|z_1 - z_2\right|^{h(z_1)+h(z_2)-2}\end{aligned}$$

where $C_2(\rho)$ can be chosen such that $C_1(\rho)\rho^{2-h(z_1)-h(z_2)} \leq C_2(\rho)$. So $\mathbf{A_3}(2)$ is fulfilled and the proof can be concluded by applying Theorem 3.1. □

We finish this subsection by applying Theorem 5.1 to an example that was mentioned in [6]. Let us consider W_H defined as in (5.1). We let

$$m(z, H) = W_H(z + 1) - W_H(z). \tag{5.7}$$

We compute the covariance between $m(z_1, H_1)$ and $m(z_2, H_2)$ for every z_1,z_2,H_1 and H_2:

$$\begin{aligned}\mathbb{E}[m(z_1, H_1)m(z_2, H_2)] &= \frac{1}{2}\frac{C\left(\frac{H_1+H_2}{2}\right)^2}{C(H_1)C(H_2)}\left|z_1 - z_2\right|^{H_1+H_2} \\ &\quad\times\left(\left|1 + \frac{1}{z_1 - z_2}\right|^{H_1+H_2} + \left|1 - \frac{1}{z_1 - z_2}\right|^{H_1+H_2} - 2\right).\end{aligned} \tag{5.8}$$

By the Taylor formula we get that the asymptotic covariance R of $\{m(z, H)\}_{z,H}$ can be written as

$$R(H_1, H_2) = \frac{1}{2}(H_1 + H_2)(H_1 + H_2 - 1)\frac{C\left(\frac{H_1+H_2}{2}\right)^2}{C(H_1)C(H_2)}.$$

Then applying Theorem 5.1 we get that $\{a^\varepsilon(Z,s)\}_s$ converges in distribution to $\widetilde{a}(Z,s) = f\left(s - \frac{1}{2}S_h(Z)\right)$ where

$$S_h(Z) = J(1)\int_{-\infty}^{\infty}\left(\int_0^Z \frac{-ixe^{iux}}{C(h(u))|x|^{h(u)+1/2}}du\right)\widehat{B}(dx). \tag{5.9}$$

As mentioned in Section 6.1 of [6], we also can observe that if we assume that h is differentiable then we can write $S_h(Z)$ as

$$\begin{aligned} S_h(Z) = J(1)\int_{-\infty}^{\infty}\widehat{B}(dx)\Bigg\{&\frac{(e^{iZx}-1)}{C(h(Z))|x|^{h(Z)+1/2}} \\ &-\int_0^Z \frac{(e^{iux}-1)}{|x|^{h(u)+1/2}}\Big(\frac{\log|x|}{C(h(u))} - \frac{C'(h(u))}{C(h(u))^2}\Big)h'(u)\,du\Bigg\}, \end{aligned} \tag{5.10}$$

which means that $S_h(Z)$ is the sum of a multifractional Brownian motion as in (5.2) and of a regular process.

6. A non-Gaussian and multifractal medium. In this section we study the case of a medium that generalizes the media discussed above. We define $\{m(z,H)\}_{z,H}$ for every $z \geq 0$ by

$$m(z,H) = \frac{1}{C(H)}\int_{\mathbb{R}} \exp(izx)\psi(x)|x|^{1/2-H}\widehat{W}(dx) \tag{6.1}$$

where $H \in (1/2,1)$ and ψ is a complex-valued function and $\widehat{W}(dx)$ is the Fourier transform of a real Gaussian measure. We assume that ψ is continuous, $\psi(0) = 1$ and satisfies $|\psi(x)| = \mathcal{O}_{|x|\to\infty}(|x|^{-1})$. Notice that the family of fractional Brownian motion $\{W_H(z)\}_{z,H}$ as previously defined by (5.1) and as in (5.7) is an example of such a process.

Thus $\{m(z,H)\}_{z,H}$ is a centered Gaussian field and its covariance can be written as

$$\mathbb{E}[m(z_1,H_1)m(z_2,H_2)] = \int_{\mathbb{R}} \frac{\exp(i(z_1-z_2)x)|\psi(x)|^2}{C(H_1)C(H_2)|x|^{H_1+H_2-1}}dx. \tag{6.2}$$

Now we consider a fonction h that takes its values in $[h_-,h_+] \subset (1/2,1)$ and a truncation function Φ with Hermite index $K \in \mathbb{N}^*$. We define ν^ε as

$$\nu^\varepsilon(z) = \varepsilon^{\kappa(z)-\tau}\nu\left(\frac{z}{\varepsilon^2}, z\right)$$

where

$$\nu\left(z_1,z_2\right) = \Phi\left(m\left(z_1, \widetilde{h}_K(z_2)\right)\right)$$

with

$$\widetilde{h}_K(z) = \frac{h(z)-1}{K} + 1$$

We can observe that ν^ε satisfies Assumptions $A_2(2)$ and $A_3(2)$. In particular and more precisely

$$\mathbb{E}[\nu^\varepsilon(z_1)\nu^\varepsilon(z_2)] \sim \frac{J(K)^2}{K!} R(h(z_1), h(z_2))|z_1 - z_2|^{h(z_1)+h(z_2)-2} \tag{6.3}$$

when $|z_1 - z_2|/\varepsilon^2$ goes to ∞ if we assume that $\kappa(z) - \tau = 2h(z) - 2$. Therefore, because Theorem 3.1 says that, under long-range assumptions, the asymptotic behavior of $a^\varepsilon(Z,s)$ is essentially given by the limit of $v^\varepsilon(z)$, we can conclude by the following result.

THEOREM 6.1. *As ε goes to 0, $\{a^\varepsilon(Z,s)\}_s$ converges in distribution in the space of continuous functions endowed with the uniform topology to the random process $\{\widetilde{a}(Z,s)\}_s$ that can be written as*

$$\widetilde{a}(Z,s) = f\left(s - \frac{1}{2} S_h^K(Z)\right), \tag{6.4}$$

where S_h^K is a centered process given for every z by:

$$S_h^K(z) = \int_{\mathbb{R}^K} \mathcal{G}_{h,K}(z, x_1, \cdots, x_K) \prod_{k=1}^K \widehat{B}(dx_k) \tag{6.5}$$

where

$$\mathcal{G}_{h,K}(z, x_1, \cdots, x_K) = \int_0^z \frac{J(K) e^{-iu\sum_{k=1}^K x_k}}{K! C(h(u))^K} \prod_{k=1}^K \frac{-ix_k}{|x_k|^{\widetilde{h_K}(u)+1/2}} \, du.$$

Notice that the process S_h^K is equal (in distribution) to W_H^K of Section 4 if h is a constant equal to H, and is equal to S_h of Section 5 if $K = 1$. Because of these facts, S_h^K is in general non-Gaussian and multifractional. This shows that under general long-range assumptions the asymptotic time-shift is neither Gaussian, nor homogeneous. This is in dramatic contrast to the short-range case where the time shift is a Brownian motion, which is homogeneous and Gaussian.

PROOF. We let

$$v^\varepsilon(z) = \int_0^z \nu^\varepsilon(u)\, du = \int_0^z du\, \varepsilon^{2h(u)-2} \Phi\left(m\left(\frac{u}{\varepsilon^2}, \widetilde{h_K}(u)\right)\right)$$

and

$$v_1^\varepsilon(z) \quad = \quad \int_0^z du\, \varepsilon^{2h(u)-2} P_K\left(m\left(\frac{u}{\varepsilon^2}, \widetilde{h_K}(u)\right)\right).$$

Using the same arguments as for the beginning of the proof of Theorem 5.1, using the fact that the Hermite index of Φ is K, we get

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|v^\varepsilon(z) - \frac{J(K)}{K!} v_1^\varepsilon(z)\right|^2\right] = 0. \tag{6.6}$$

Then using the formula (see [14] for instance)

$$P_K\left(\int_{\mathbb{R}} \phi(x)\widehat{B}(dx)\right) = \int_{\mathbb{R}^K} \prod_{k=1}^K \phi(x_k)\widehat{B}(dx_k)$$

for every $\phi \in L^2(\mathbb{R})$ we get

$$\begin{aligned} v_1^\varepsilon(z) \quad &= \quad \int_0^z du\, \frac{\varepsilon^{2h(u)-2}}{C(h(u))^K} \int_{\mathbb{R}^K} e^{-iu\sum_{j=1}^K x_j/\varepsilon^2} \prod_{k=1}^K \frac{\psi(x_k)}{|x_k|^{\widetilde{h_K}(u)-1/2}} \widehat{B}(dx_k) \\ &= \quad \int_{\mathbb{R}^K} \int_0^z du \prod_{k=1}^K \frac{\psi(x_k)}{|x_k|^{\widetilde{h_K}(u)-1/2}} \widehat{B}(dx_k) \frac{\varepsilon^{2h(u)-2}}{C(h(u))^K} e^{-iu\sum_{j=1}^K x_j/\varepsilon^2} \end{aligned}$$

Then we make the substitution $x_k \to \varepsilon^2 x_k$ for every k:

$$\begin{aligned} v_1^\varepsilon(z) \quad &= \quad \int_{\mathbb{R}^K} \int_0^z du \prod_{k=1}^K \frac{\psi(\varepsilon^2 x_k)}{|\varepsilon^2 x_k|^{\widetilde{h_K}(u)-1/2}} \widehat{B}(\varepsilon^2 dx_k) \frac{\varepsilon^{2h(u)-2}}{C(h(u))^K} e^{-iu\sum_{j=1}^K x_j} \\ &= \quad \varepsilon^{-K} \int_{\mathbb{R}^K} \int_0^z du \prod_{k=1}^K \frac{\psi(\varepsilon^2 x_k)}{|x_k|^{\widetilde{h_K}(u)-1/2}} \widehat{B}(\varepsilon^2 dx_k) \frac{1}{C(h(u))^K} e^{-iu\sum_{j=1}^K x_j}. \end{aligned}$$

We let

$$v_2^\varepsilon(z) \quad = \quad \int_{\mathbb{R}^K} \int_0^z du \prod_{k=1}^K \frac{\psi(\varepsilon^2 x_k)}{|x_k|^{\widetilde{h_K}(u)-1/2}} \widehat{B}(dx_k) \frac{1}{C(h(u))^K} e^{-iu\sum_{j=1}^K x_j}.$$

The selfsimilarity of the Brownian motion gives that $\widehat{B}(\varepsilon^2 dx_k)$ is equal in distribution to $\varepsilon \widehat{B}(dx_k)$, then we get that

$$v_1^\varepsilon \quad =^{\text{f.d.d.}} \quad v_2^\varepsilon$$

where $=^{\text{f.d.d.}}$ means the equality of the finite dimensional distributions. Then, using the assumptions on ψ, we obtain the convergence a.s. of the finite dimensional margins of $\frac{J(K)}{K!}v_2^\varepsilon$ to those of S_h^K, and thus the convergence of the finite dimensional distributions of v^ε to those of S_h^K, so A_1 is satisfied. Now, as observed at the beginning of this section, using (6.2) and by the same procedure as in the proof of Theorem 5.1 we show that $A_2(2)$ and $A_3(2)$ hold. We conclude by Theorem 3.1. □

7. Proof of Theorem 3.1. We first give an outline of the proof. As recalled in Section 2 the process $\{a^\varepsilon(Z,s)\}_s$ can be written in terms of the propagator P_ω^ε, and thus the study of the convergence of $\{a^\varepsilon(Z,s)\}_s$ can be analyzed via asymptotic properties of P_ω^ε. The propagator P_ω^ε satisfies the equation

$$\frac{dP_\omega^\varepsilon}{dz}(z) = \mathcal{H}_\omega^\varepsilon\left(\frac{z}{\varepsilon^\tau}, z\right) P_\omega^\varepsilon(z)\,,$$

that we can write in the form

$$dP_\omega^\varepsilon(z) = \frac{i\omega}{2}\sum_{j=1}^{3} F_j P_\omega^\varepsilon(z)\, dv_j^\varepsilon(z)\,, \tag{7.1}$$

where

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } F_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and v_1^ε, v_2^ε and v_3^ε are three processes of bounded variation that we can write as

$$\begin{aligned} v_1^\varepsilon(z) &= \int_0^z \nu^\varepsilon\left(z'\right)\, dz'\,, \\ v_2^\varepsilon(z) &= \int_0^z \nu^\varepsilon\left(z'\right)\cos\left(2\omega\frac{z'}{\varepsilon^\tau}\right)\, dz'\,, \\ v_3^\varepsilon(z) &= \int_0^z \nu^\varepsilon\left(z'\right)\sin\left(2\omega\frac{z'}{\varepsilon^\tau}\right)\, dz'\,. \end{aligned}$$

Thanks to T. Lyons' rough paths theory for which we recall some tools in the Appendix we shall see that the convergence of P_ω^ε can be reduced for a convenient topology to the convergence of the process $\mathbf{v}^\varepsilon$ defined as

$$\mathbf{v}^\varepsilon := (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)\,.$$

Hence, we first prove the convergence of $\mathbf{v}^\varepsilon$, then by Theorem A.1 (see Appendix) we deduce the convergence of P_ω^ε in Section 7.1, and thanks to (2.13) we finally conclude by the convergence of $\{a^\varepsilon(Z,s)\}_s$ in Section 7.2.

7.1. *Convergence of the Propagator.* Using Theorem A.1 and the expression (7.1), the asymptotic study of the propagator is reduced to finding the limit in a rough path space of $\mathbf{v}^\varepsilon := (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)$. This is the aim of the following lemma.

LEMMA 7.1. *There exists $\gamma_* \in (0,1)$ such that for every $p > 2/(2-\gamma_*)$, as ε goes to 0, the increments of $\mathbf{v}^\varepsilon$ converge in Ω_p to those of $\mathbf{V}$ which can be written as*

$$\boldsymbol{V} = (V, 0, 0).$$

The proof of Lemma 7.1 is based on establishing several technical lemmas that we do next.

LEMMA 7.2. *There exist C and γ_* so that*

$$|\mathbb{E}[\nu^\varepsilon(x)\nu^\varepsilon(y)]| \leq C|x-y|^{-\gamma_*}$$

for every x and y.

PROOF. The assumptions of Theorem 3.1 imply that for every $\delta > 0$ there exists $z_\delta > 0$ such that for $|x-y| > \varepsilon^\lambda z_\delta$ we have

$$(1-\delta)R(x,y)|x-y|^{-\gamma(x,y)} \leq r_{\nu^\varepsilon}(x,y) \leq (1+\delta)R(x,y)|x-y|^{-\gamma(x,y)}.$$

Hence, taking $\delta = 1$ we get that for $|x-y| > \varepsilon^\lambda z_1$ we have

$$0 \leq r_{\nu^\varepsilon}(x,y) \leq C|x-y|^{-\gamma_+}.$$

Moreover, thanks to the assumptions of Theorem 3.1, we know that there exist C_{z_1} and γ_{z_1} so that for $|x-y| \leq \varepsilon^\lambda z_1$ we have

$$0 \leq |r_{\nu^\varepsilon}(x,y)| \leq C_{z_1}|x-y|^{-\gamma_{z_1}}.$$

By choosing $\gamma_* := \max(\gamma_+, \gamma_{z_1})$ we get that there exists γ_* so that

$$|\mathbb{E}[\nu^\varepsilon(x)\nu^\varepsilon(y)]| \leq C|x-y|^{-\gamma_*}$$

for every x and y. □

LEMMA 7.3. *For every $z \in [0, Z]$, as ε goes to 0 the sequences $v_2^\varepsilon(z)$ and $v_3^\varepsilon(z)$ converge to 0.*

PROOF. Without loss of generality we present the proof only for $v_2^\varepsilon(z)$ and with $2\omega = 1$. We have

$$\begin{aligned}\mathbb{E}[v_2^\varepsilon(z)^2] &= \int_0^z dx \int_0^z dy \cos\left(\frac{x}{\varepsilon^\tau}\right)\cos\left(\frac{y}{\varepsilon^\tau}\right) r_{\nu^\varepsilon}(x,y) \\ &= I_1^\varepsilon(z) + I_2^\varepsilon(z),\end{aligned}$$

with

$$\begin{aligned}I_1^\varepsilon(z) &= \int_0^z dx \int_0^z dy \cos\left(\frac{x}{\varepsilon^\tau}\right)\cos\left(\frac{y}{\varepsilon^\tau}\right) R(x,y)\,|x-y|^{-\gamma(x,y)}, \\ I_2^\varepsilon(z) &= \int_0^z dx \int_0^z dy \cos\left(\frac{x}{\varepsilon^\tau}\right)\cos\left(\frac{y}{\varepsilon^\tau}\right) \left(r_{\nu^\varepsilon}(x,y) - R(x,y)\,|x-y|^{-\gamma(x,y)}\right).\end{aligned}$$

Let $\delta > 0$, because of the assumptions of Theorem 3.1, we have that for $|x-y| > \varepsilon^\lambda z_\delta$ (with z_δ sufficiently large) $|r_{\nu^\varepsilon}(x,y) - R(x,y)|x-y|^{-\gamma(x,y)}| \le \delta R(x,y)|x-y|^{-\gamma(x,y)}$ for every ε. Combining this with Lemma 7.2 we obtain

$$|I_2^\varepsilon(z)| \quad \le \quad \delta \int_0^z dx \int_0^z dy R(x,y)\,|x-y|^{-\gamma(x,y)} + C_\delta \int_0^z dx \int_0^z dy |x-y|^{-\gamma^*} 1_{|x-y|\le \varepsilon^\lambda z_\delta}$$

so that

$$\limsup_{\varepsilon\to 0} |I_2^\varepsilon(z)| \le \delta \int_0^z dx \int_0^z dy |x-y|^{-\gamma(x,y)}.$$

The inequality above is valid for every $\delta > 0$ and we conclude

$$\lim_{\varepsilon\to 0} I_2^\varepsilon(z) = 0.$$

We can deal with $I_1^\varepsilon(z)$ using a Riemann type result. Indeed, the function $\tilde{R} : (x,y) \mapsto R(x,y)|x-y|^{-\gamma(x,y)}$ is integrable on $\Delta_z = [0,z]^2$, so we can approximate it by a sequence of constant by step functions $(R_N)_N$ such that

$$\lim_{N\to\infty} \int_0^z dx \int_0^z dy |\tilde{R}(x,y) - R_N(x,y)| = 0.$$

Besides, we can write

$$|I_1^\varepsilon(z)| \le \left|\int_0^z dx \int_0^z dy \cos\left(\frac{x}{\varepsilon^\tau}\right)\cos\left(\frac{y}{\varepsilon^\tau}\right) R_N(x,y)\right| + \int_0^z dx \int_0^z dy |\tilde{R}(x,y) - R_N(x,y)|$$

for every ε and N. We easily see that

$$\lim_{\varepsilon\to 0} \int_0^z dx \int_0^z dy \cos\left(\frac{x}{\varepsilon^\tau}\right)\cos\left(\frac{y}{\varepsilon^\tau}\right) R_N(x,y) = 0$$

so that

$$\limsup_{\varepsilon\to 0} |I_1^\varepsilon(z)| \le \int_0^z dx \int_0^z dy |\tilde{R}(x,y) - R_N(x,y)|$$

for every N. This finally shows

$$\lim_{\varepsilon\to 0} I_1^\varepsilon(z) = 0\,,$$

and then

$$\lim_{\varepsilon\to 0} \mathbb{E}[v_2^\varepsilon(z)^2] = 0\,,$$

which concludes the proof. □

Now we deal with a technical lemma regarding the increments of $\mathbf{v}^\varepsilon$.

LEMMA 7.4. *There exist $C > 0$ and $\gamma_* \in (0,1)$ such that for every z, ζ and $\varepsilon > 0$ we have*

$$\mathbb{E}[\|\mathbf{v}^\varepsilon(z) - \mathbf{v}^\varepsilon(\zeta)\|^2] \leq C|z-\zeta|^{2-\gamma^*}.$$

PROOF. Because of Lemma 7.2 there exists γ_* so that $|\mathbb{E}[\nu^\varepsilon(x)\nu^\varepsilon(y)]| \leq C|x-y|^{-\gamma_*}$ for every x and y. Then, for every $j = 1,2,3$, we have (taking $z > \zeta$)

$$\begin{aligned}
\mathbb{E}[|v_j^\varepsilon(z) - v_j^\varepsilon(\zeta)|^2] &\leq \int_\zeta^z dx \int_\zeta^z dy\, |\mathbb{E}\left[\nu^\varepsilon(x)\,\nu^\varepsilon(y)\right]| \\
&\leq C\int_\zeta^z dx \int_\zeta^z dy\, |x-y|^{-\gamma^*} \\
&\leq \frac{2C'}{(1-\gamma^*)(2-\gamma^*)}\,|z-\zeta|^{2-\gamma^*}\,,
\end{aligned}$$

which concludes the proof. □

In the sequel we shall use the notation $H_* := (2-\gamma_*)/2$. Using the above lemmas we next deduce the following lemma which deals with identification of the limit.

LEMMA 7.5. *The process $\mathbf{V}$ defined in Lemma 7.1 is a.s. continous (up to a modification). Moreover, as ε goes to 0, $\mathbf{v}^\varepsilon$ converges to $\mathbf{V}$ in the space of continuous functions endowed with the uniform norm.*

PROOF. Assumptions and lemmas 7.3 give the convergence of finite dimensional distributions of $\mathbf{v}^\varepsilon$ to those of $\mathbf{V}$. Using then the Kolmogorov criterion, [4], Lemma 7.4 and the fact that $2H_* > 1$ we get the tightness of $(\mathbf{v}^\varepsilon)_\varepsilon$ in the space of continuous functions endowed with the uniform norm which establishes the proof. □

Thanks to Lemma 7.5 we conclude with the proof of Lemma 7.1 by establishing the tightness in a rough paths sense.

LEMMA 7.6. *For every $p > 1/H_*$, the sequence $(\mathbf{v}^\varepsilon)_\varepsilon$ is tight in Ω_p and the process* $\mathbf{V}$ *is a.s. of finite $p-$variation.*

PROOF. *(Lemmas 7.1 and 7.6)* Let $q \in (1/H_*, p)$. In view of Lemmas A.1 and 7.5 it is enough to prove

$$\lim_{A\to+\infty} \sup_{\varepsilon>0} \mathbb{P}[V_q(\mathbf{v}^\varepsilon) > A] = 0\,. \tag{7.2}$$

Using Tchebychev's inequality, the fact that $q < 2$, Lemma A.2, the Hölder inequality and Lemma 7.5 we find

$$\begin{aligned}
\mathbb{P}[V_q(\mathbf{v}^\varepsilon) > A] &\leq \frac{1}{A^q}\mathbb{E}[V_q(\mathbf{v}^\varepsilon)^q] \\
&\leq \frac{C}{A^q}\sum_{n=1}^{+\infty} n^C \sum_{k=1}^{2^n} \mathbb{E}[\|\mathbf{v}^\varepsilon(z_k^n) - \mathbf{v}^\varepsilon(z_{k-1}^n)\|^q] \\
&\leq \frac{C}{A^q}\sum_{n=1}^{+\infty} n^C \sum_{k=1}^{2^n} \mathbb{E}[\|\mathbf{v}^\varepsilon(z_k^n) - \mathbf{v}^\varepsilon(z_{k-1}^n)\|^2]^{q/2} \\
&\leq \frac{C'}{A^q}\sum_{n=1}^{+\infty} n^C \sum_{k=1}^{2^n} \left(\frac{1}{2^n}\right)^{qH_*} \\
&\leq \frac{C'}{A^q}\sum_{n=1}^{+\infty} n^C \left(\frac{1}{2^n}\right)^{qH_*-1},
\end{aligned}$$

and since $qH_* > 1$ we deduce (7.2). □

Finally, we can now derive the following lemma which deals with the convergence of the propagator.

LEMMA 7.7. *Let $\{\omega_1, \cdots, \omega_n\}$ to be a collection of frequencies. Then, as ε goes to 0, the propagator vector $(P^\varepsilon_{\omega_1}, \cdots, P^\varepsilon_{\omega_n})$ converges in distribution in the space of continuous functions to $(P_{\omega_1}, \cdots, P_{\omega_n})$ which is the asymptotic propagator P_ω that we can write as*

$$P_\omega(z) = \begin{pmatrix} \exp\left(\frac{i\omega}{2}V(z)\right) & 0 \\ 0 & \exp\left(-\frac{i\omega}{2}V(z)\right) \end{pmatrix}.$$

PROOF. By combining Theorem A.1, (7.1) and Lemma 7.1 we get that, as ε goes to 0, P_ω^ε converges in distribution in the space of continuous functions (endowed with the uniform topology) to the solution P_ω of the following system of equations:

$$dP_\omega(z) = \frac{i\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_\omega(z)\, dV(z)\,.$$

This concludes the proof. □

We remark that the situation here contrasts with the short range case. Indeed, the asymptotic propagator is driven by one process in the long range case whereas it is driven by three processes in the short range case.

7.2. *Conclusion of the proof.* The remaining part of the proof of Theorem 3.1 follows the lines of [5, 11], however, we present it here for completeness. Recall that thanks to the formula (2.13) we can write $a^\varepsilon(Z,s)$ in a Fourier-type formula using the transmission coefficient :

$$a^\varepsilon(Z,s) = \frac{1}{2\pi}\int e^{-is\omega} T_\omega^\varepsilon(Z)\widehat{f}(\omega)\, d\omega\,, \tag{7.3}$$

with the transmission coefficient being a functional of the propagator P_ω^ε. We shall use Lemma 7.7 to deduce the convergence of the transmitted wave.

Let $n \in \mathbb{N}$, $s_1 \le \cdots \le s_n \in [0,\infty)$. We can write :

$$\begin{aligned}\mathbb{E}[a^\varepsilon(Z,s_1)\cdots a^\varepsilon(Z,s_n)] &= \mathbb{E}\Big[\frac{1}{(2\pi)^n}\prod_{j=1}^n \int e^{-is_j\omega} T_\omega^\varepsilon(Z)\widehat{f}(\omega)\, d\omega\Big] \\ &= \frac{1}{(2\pi)^n}\int\cdots\int e^{-i\sum_{j=1}^n s_j\omega_j}\widehat{f}(\omega_1)\cdots\widehat{f}(\omega_n)\mathbb{E}[T_{\omega_1}^\varepsilon(Z)\cdots T_{\omega_n}^\varepsilon(Z)]\, d\omega_1\cdots d\omega_n\,.\end{aligned}$$

Thanks to Lemma 7.7 we have that as $\varepsilon \to 0$

$$\mathbb{E}[T_{\omega_1}^\varepsilon(Z)\cdots T_{\omega_n}^\varepsilon(Z)] \to \mathbb{E}\Big[\exp\Big(\frac{iV(Z)}{2}\sum_{j=1}^n \omega_j\Big)\Big]\,,$$

and then

$$\begin{aligned}
\mathbb{E}[a^\varepsilon(Z,s_1)\cdots a^\varepsilon(Z,s_n)] \to \frac{1}{(2\pi)^n}\int\cdots\int e^{-i\sum_{j=1}^n s_j\omega_j}\widehat{f}(\omega_1)\cdots\widehat{f}(\omega_n) \\
\times\mathbb{E}\Big[\exp\Big(\frac{iV(Z)}{2}\sum_{j=1}^n\omega_j\Big)\Big]\,d\omega_1\cdots d\omega_n \\
= \mathbb{E}\Big[\frac{1}{(2\pi)^n}\prod_{j=1}^n\int e^{-i(s_j-V(Z)/2)\omega}\widehat{f}(\omega)\,d\omega\Big] \\
= \mathbb{E}\Big[\prod_{j=1}^n f(s_j-V(Z)/2)\Big].
\end{aligned}$$

The tightness proof is similar to the proof of Lemma 3.2 in [5] and the convergence of $a^\varepsilon(Z,s)$ follows.

APPENDIX A: DIFFERENTIAL EQUATIONS AND ROUGH PATHS

In this section we fix $p\in[1,2)$ and consider a closed interval $I=[0,Z]$. We define the $p-$variation of a continuous function $w:I\to\mathbb{R}^n$ by

$$V_p(w):=\left(\sup_D\sum_{j=0}^{k-1}\|w(z_{j+1})-w(z_j)\|^p\right)^{1/p},$$

where $\sup_D$ runs over all finite partition $\{0=z_0,...,z_k=Z\}$ of I and where here and below $\|\cdot\|$ refers to the L^2 norm. The space of all continuous functions of bounded variation (1-variation) is endowed with the $p-$variation distance

$$\|w\|_p=V_p(w)+\sup_{z\in[0,1]}|w(z)|\,,$$

and is denoted by Ω_p^∞. The closure of this metric space is called the space of all geometric rough paths and is denoted by Ω_p. One of the most important theorems of rough paths theory is the following :

Theorem A.1. ***(T. Lyons' Continuity Theorem)*** *Let*[1] $G:\mathbb{R}\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R},\mathbb{R}^d)$ *and* $F:\mathbb{R}\times\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d)$ *be two smooth functions. Let* y *be the unique solution of the differential equation*

$$dy(z)=G(z,y(z))\,dz+F(z,y(z))\,dw(z),\ \ y(z=0)=y_0\,,$$

[1]Here $\mathcal{L}(\mathbb{R},\mathbb{R}^d)$ (resp. $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d)$) denotes the space of all linear maps from $\mathbb{R}$ (resp. $\mathbb{R}^n$) to $\mathbb{R}^d$

where w is a bounded variation function. Then the Itô's map $\mathcal{I} : w \mapsto y$ is continuous with respect to the $p-$variation distance from $\Omega_p^\infty(\mathbb{R}^n)$ to $\Omega_p^\infty(\mathbb{R}^d)$. Therefore there exists a unique extension of this map (that we still denote by $\mathcal{I}$) to the space $\Omega_p(\mathbb{R}^n)$

This theorem has been proved by T. Lyons and extensively studied and applied (see [8, 15–17]).

The proof of Theorem 3.1 is based on analysis of the tightness in the space of geometric rough paths. In the context of this we need to compute the $p-$variation for $p > 1$. To this effect we will need the following lemmas of which the first can be found for instance in [16], and the second in [15, 16].

LEMMA A.1. *Let $q \in [1, 2)$ and $(v^\varepsilon)_{\varepsilon>0}$ a family of continuous random processes of finite $q-$variation which is tight in the space of continuous functions on I and satisfying*

$$\lim_{A\to+\infty} \sup_{\varepsilon>0} \mathbb{P}[V_q(v^\varepsilon) > A] = 0\,. \tag{A.1}$$

Then $(v^\varepsilon)_{\varepsilon>0}$ is tight in Ω_p for every $p > q$.

LEMMA A.2. *For every $n \in \mathbb{N}$ and every $k = 0, 1, \ldots, 2^n$, we let $z_k^n := Zk/2^n$. Let $q \in [1, 2)$ and v be a function of finite $q-$variation. Then there exist two positive constants C_1, C_2 which do not depend on v such that*

$$V_q(v)^q \le C_1 \sum_{n=1}^{+\infty} n^{C_2} \sum_{k=1}^{2^n} \|v(z_k^n) - v(z_{k-1}^n)\|^q\,.$$

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