Scintillation in the White-Noise Paraxial Regime

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January 28, 2014

Abstract

In this paper the white-noise paraxial wave model is considered. This model describes for instance the propagation of laser beams in the atmosphere in some typical scaling regimes. The closed-form equations for the second- and fourth-order moments of the field are solved in two particular situations. The first situation corresponds to a random medium with a transverse correlation radius smaller than the beam radius. This is the spot-dancing regime: the beam shape spreads out as in a homogeneous medium and its center is randomly shifted according to a Gaussian process whose variance grows like the third power of the propagation distance. The second situation corresponds to a plane-wave initial condition, a small amplitude for the medium fluctuations, and a large propagation distance. This is the scintillation regime: the normalized variance of the intensity converges to one exponentially with the propagation distance, corresponding to strong intensity fluctuations and in agreement with the conjecture that the statistics of the field becomes complex Gaussian.

AMS subject classifications. 60H15, 35R60, 74J20.

Key words. Waves in random media, parabolic approximation, scintillation.

1 Introduction

We will consider wave propagation through time-independent media with a complex spatially varying or "cluttered" velocity. It is then convenient to model the fluctuations in the velocity or index of refraction as a random field. Typically we cannot expect to know this parameter pointwise, but we may be able to characterize its statistics and we are interested in how the statistics of the medium affect the statistics of the wave field. In its most common form, the analysis of wave propagation in random media consists in studying the field u solution of the scalar time-harmonic wave or Helmholtz equation

$$\Delta u + k_0^2 n^2(z, \boldsymbol{x}) u = 0, \qquad (z, \boldsymbol{x}) \in \mathbb{R}^{1+d} \tag{1.1}$$

where k_0 is the free space homogeneous wavenumber and n is a randomly heterogeneous index of refraction. Since the index of refraction n is a random process, the field u is also a random process whose statistical behavior can be characterized by the calculations of its moments. Even though the scalar wave equation is simple and linear, the relation between

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the statistics of the index of refraction and the statistics of the field is highly nontrivial and nonlinear. For instance, the scintillation problem is a well-known paradigm, related to the observation that the irradiance of a star fluctuates due to the light propagation through the turbulent atmosphere. This common observation is far from being fully understood mathematically. However, experimental observations indicate that the statistical distribution of the irradiance is exponential, with the irradiance being the square magnitude of the complex wave field. Indeed it is a well-accepted conjecture that the statistics of the complex field becomes circularly symmetric complex Gaussian when the wave propagates through the turbulent atmosphere [33, 35] (so that the irradiance is the sum of the squares of the two independent Gaussian random variables, which upto a scaling has χ-square distribution with two degrees of freedom, that is an exponential distribution). However, so far there is no mathematical proof of this conjecture, except in randomly layered media [12, Chapter 9]. One of the main contributions of this paper is to show that the normalized variance of the irradiance converges to one (which is the normalized variance of the exponential distribution) as the wave propagates through a random medium in a fairly general situation. Here, with "normalized variance" we mean the variance of the field over the square of its expected value. This quantity is often referred to as the *scintillation index*. A better understanding of the structure of the scintillation is important in order to design adaptive optics and other schemes aimed at mitigating the effects that atmospheric turbulence has on a propagating laser beams, this is an active area of research [22, 31].

There are various approaches to model wave propation in random media in order to compute the moments of the field: Born or Rytov approximations, extended Huygens-Fresnel principle and the white-noise paraxial model in particular. Here we will use the white-noise paraxial model, but comment first on the other models.

The Born approximation looks for an expansion of the solution of the wave equation as a sum of terms $u = u_0 + u_1 + u_2 + \cdots$, where u_0 represents the unperturbed or unscattered wave (solution of (1.1) with $n^2 \equiv 1$) while u_1 and u_2 represent the first-order and second-order scattered waves [20, Sec. 17.2]. The perturbations terms u_1 and u_2 are obtained by substitution of a low-order approximation of the solution in the right-hand side of the Lippmann-Schwinger equation (which is an integral formulation of (1.1)):

$$u(z,\boldsymbol{x}) = u_0(z,\boldsymbol{x}) + k_0^2 \int_{\mathbb{R}^{1+d}} G(z,\boldsymbol{x},z',\boldsymbol{x}')\big(n^2(z',\boldsymbol{x}')-1\big)u(z',\boldsymbol{x}')d\boldsymbol{x}'dz',$$

where G is the homogeneous Green's function of the scalar wave equation. The Rytov approximation looks for an expansion of the solution in a multiplicative form: $u = u_0 \exp(\psi_1 + \psi_2 + \cdots)$, where ψ_1 and ψ_2 are first-order and second-order complex phase perturbations [20, Sec. 17.2]. It is possible to relate the Rytov perturbation terms to the Born approximation terms. These two methods give explicit representation formulas of the field u and allow the computation of all moments, but their validity is restricted to weak fluctuation regimes and do not allow one to explore regimes in which the statistics of the wave is strongly affected by the random medium [1].

The extended Huygens-Fresnel principle is a heuristic technique that states that the field $u(z,\boldsymbol{x})$ is given by the following extension of the Huygens-Fresnel formula (in a $1+d=3$-dimensional medium) [20, Sec. 20.18]:

$$u(z,\boldsymbol{x}) = \frac{ik_0}{2\pi z}\exp(ik_0 z)\int_{\mathbb{R}^d} u_0(0,\boldsymbol{x}')\exp\Big(i\frac{k_0|\boldsymbol{x}-\boldsymbol{x}'|^2}{2z} + \psi(z,\boldsymbol{x},\boldsymbol{x}')\Big)d\boldsymbol{x}',$$

where $\psi(z, \boldsymbol{x}, \boldsymbol{x}')$ is the random complex phase associated with a spherical wave emanating from $(0, \boldsymbol{x}')$ evaluated at $(z, \boldsymbol{x})$ and which contains the first- and second-order perturbations as in Rytov's method. This method allows one to get a closed-form expression of the field u and to compute its moments, but its range of validity is unknown [1].

The white-noise paraxial wave equation describes the propagation of waves along a privileged axis, say the z-direction, with negligible backscattering. This model is interesting for three main reasons. First it appears as a very natural model in many applications where the correlation length of the medium is smaller than the propagation distance. This is the case in many situations, in laser beam propagation [27], time reversal in random media [5, 24], underwater acoustics [28], or migration problems in geophysics [6]. Second it can be derived rigorously from the wave equation in random media (1.1) by a separation of scales technique in the high-frequency regime (see [2] in the case of a randomly layered medium and [16, 17, 18] in the case of a $1 + d$-dimensional random medium). Third it allows for the use of Itô's stochastic calculus, which in turn enables the closure of the hierarchy of moment equations [13, 20]. The analysis of important wave propagation problems, such as the star scintillation due to atmospheric turbulence then seems tractable [30]. Unfortunately, even though the equation for the second-order moments can be solved, the equation for the fourth-order moments is very difficult and only approximations or numerical solutions are available (see [10, 19, 29, 32, 34] and [20, Sec 20.18]). A formal high-frequency asymptotic expansion of the moment equations gives the result that the n'th moment of the intensity should be asymptotically $n!$ [35]. By using additional hypotheses on decorrelation properties of the moments one can establish that the stationary solutions to the moment equations correspond to the exponential distribution [13]. Note that scintillation cannot be universal as self-averaging properties of the solution to the white-noise paraxial wave equation can be obtained in some specific regimes [4, 3, 8, 25]. Here, with a self-averaging asymptotic regime we mean that the scintillation index vanishes in the limit. This is in particular the case with relative rapid decorrelation of the medium fluctuations (in both depth and lateral coordinates). As shown in [3] the self-averaging also depends on the initial data and can be lost for very rough initial data even with a high lateral diversity as considered there. In [21, 26] the authors also consider a situation with rapidly fluctuating random medium fluctuations and a regime in which the Wigner transform itself is weakly statistically stable or self-averaging. They then consider the moment hierarchy of the rescaled Wigner transform fluctuations. In this paper we focus rather on the moments of wave field itself in regimes that are not self-averaging.

In our paper we compute the second- and fourth-order moments of the field in a rigorous way in the white-noise paraxial model. We address two different situations using rigorous methods. In the so-called spot-dancing regime we show that the transverse beam profile has the same shape as in a homogeneous medium, but its center experiences random transversal shifts that can be described in terms of a Gaussian process whose variance grows like the third power of the distance. The variance of the intensity then grows with the propagation distance as a power law as well, and therefore scintillation does not occur. In this regime it is possible to give a probabilistic representation of the field in terms of a Gaussian process that correctly reproduces all the finite-order moments, thus giving a convenient representation of the solutions of the complicated partial differential equations that define the field moment hierarchy. In the so-called scintillation regime we show that the normalized variance of the intensity converges exponentially to one. This is the first mathematical proof of this result. We give moreover the full structure of the second-order moments of the field and of the intensity distribution.

The paper is organized as follows. In Sections 2 and 3 respectively, we introduce the Itô-Schrödinger equation and the moment equations. In Sections 2 and 4 we describe the two main regimes of propagation that we consider in this paper. As a step towards the computation of the scintillation index we describe the second-order moments in Section 5. The main results on the fourth-order moments and scintillation index are presented in Section 6.

2 The White-Noise Paraxial Model

The white-noise paraxial model is widely used in the physical literature. It simplifies the full wave equation (1.1) by replacing it with an initial value-problem in the half-space $z > 0$ with $u(z = 0, \boldsymbol{x})$ given. It was studied mathematically in [7]. The proof of its derivation from the three-dimensional wave equation in randomly scattering medium is given in [17]. We describe in the next proposition this model.

Proposition 2.1. *The field $(u(z, \boldsymbol{x}))_{z\in[0,\infty),\boldsymbol{x}\in\mathbb{R}^d}$ is the solution of the Itô-Schrödinger diffusion model*

$$du(z, \boldsymbol{x}) = \frac{i}{2k_0}\Delta_{\boldsymbol{x}} u(z, \boldsymbol{x})dz + \frac{ik_0}{2} u(z, \boldsymbol{x}) \circ dB(z, \boldsymbol{x}), \tag{2.1}$$

with the initial condition in the plane $z = 0$:

$$u(z = 0, \boldsymbol{x}) = u_{\rm ic}(\boldsymbol{x}).$$

The symbol $\circ$ stands for the Stratonovich stochastic integral, $B(z, \boldsymbol{x})$ is a real-valued Brownian field with covariance

$$\mathbb{E}[B(z_1, \boldsymbol{x}_1)B(z_2, \boldsymbol{x}_2)] = \min\{z_1, z_2\}C(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{2.2}$$

The covariance function C is assumed to decay fast enough at infinity so that it belongs to $L^1(\mathbb{R}^d)$. Its Fourier transform is nonnegative (since it is the power spectral density of the stationary process $\boldsymbol{x} \to B(1, \boldsymbol{x})$). In [7] the existence and uniqueness has been established for the random process $u(z, \boldsymbol{x})$. It is shown that the process $u(z, \boldsymbol{x})$ is a continuous Markov diffusion process in $L^2(\mathbb{R}^d)$.

Remark. The model (2.1) can be obtained from the $1 + d$-dimensional scalar wave equation (1.1) by a separation of scales technique in which the $1+d$-dimensional fluctuations of the index of refraction $n(z, \boldsymbol{x})$ are described by a zero-mean stationary random process $\nu(z, \boldsymbol{x})$ with mixing properties: $n^2(z, \boldsymbol{x}) = 1 + \nu(z, \boldsymbol{x})$. The covariance function $C(\boldsymbol{x})$ in (2.2) is then given in terms of the two-point statistics of the random process ν by

$$C(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\nu(z' + z, \boldsymbol{x}' + \boldsymbol{x})\nu(z', \boldsymbol{x}')]dz. \tag{2.3}$$

Note that, if σ is the standard deviation of the fluctuations of the index of refraction, if l_z (resp. l_x) is the longitudinal (resp. transverse) correlation length of the fluctuations of the index of refraction, then $C(\boldsymbol{0})$ is of order $\sigma^2 l_z$ and the transverse scale of variation of $C(\boldsymbol{x})$ is of order l_x.

Assume that ε a small dimensionless parameter. Denote by λ_0 the carrier wavelength, by L the propagation distance, and by R_0 the radius of the inital transverse beam. The two regimes that we consider in this paper are:

- *The spot-dancing regime.* The random medium fluctuations are relatively strong, so that $\sigma^2 l_z L/\lambda_0^2$ is of order $1/\varepsilon^2$, the initial beam support is small, so that R_0/l_x is of order ε, and the propagation distance is such that $L\lambda_0/R_0^2$ is of order one. This will lead to a picture where the wave field center "dances" according to a random frame.
- *The scintillation regime.* The random medium fluctuations are relatively weak, so that $\sigma^2 l_z L/\lambda_0^2$ is of order 1, the initial beam support is broad, so that R_0/l_x is of order 1 (or even larger), while the propagation distance is relatively long, so that $L\lambda_0/l_x^2$ is of order $1/\varepsilon$. This will lead to a picture consistent with random and Gaussian intensity fluctuations.

We will consider the situation with plane wave initial data in the scintillation regime in Section 6.3 which strictly should be considered as localized plane waves via a slowly decaying modulation function (with large radius R_0).

3 The General Moment Equations

The main tool for describing wave statistics are the finite-order moments. We show in this section that in the context of the Itô-Schrödinger equation (2.1) the moments of the field satisfy a closed system at each order [20, 13]. For $p, q \in \mathbb{N}$, we define

$$M_{p,q}\big(z, (\boldsymbol{x}_j)_{j=1}^p, (\boldsymbol{y}_l)_{l=1}^q\big) = \mathbb{E}\Big[\prod_{j=1}^p u(z,\boldsymbol{x}_j) \prod_{l=1}^q \overline{u(z,\boldsymbol{y}_l)}\Big], \qquad (\boldsymbol{x}_j)_{j=1}^p \in \mathbb{R}^{dp},\ (\boldsymbol{y}_l)_{l=1}^q \in \mathbb{R}^{dq}. \tag{3.1}$$

Using the stochastic equation (2.1) and Itô's formula for Hilbert space valued processes [23], we find that the function $M_{p,q}$ satisfies the Schrödinger-type system:

$$\frac{\partial M_{p,q}}{\partial z} = \frac{i}{2k_0}\Big(\sum_{j=1}^p \Delta_{\boldsymbol{x}_j} - \sum_{l=1}^q \Delta_{\boldsymbol{y}_l}\Big) M_{p,q} + \frac{k_0^2}{4} U_{p,q}\big((\boldsymbol{x}_j)_{j=1}^p, (\boldsymbol{y}_l)_{l=1}^q\big) M_{p,q}, \tag{3.2}$$

$$M_{p,q}(z=0) = \prod_{j=1}^p u_{\rm ic}(\boldsymbol{x}_j) \prod_{l=1}^q \overline{u_{\rm ic}(\boldsymbol{y}_l)}, \tag{3.3}$$

with the generalized potential

$$\begin{aligned} U_{p,q}\big((\boldsymbol{x}_j)_{j=1}^p, (\boldsymbol{y}_l)_{l=1}^q\big) &= \sum_{j=1}^p \sum_{l=1}^q C(\boldsymbol{x}_j - \boldsymbol{y}_l) - \frac{1}{2}\sum_{j,j'=1}^p C(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) - \frac{1}{2}\sum_{l,l'=1}^q C(\boldsymbol{y}_l - \boldsymbol{y}_{l'}) \\ &= \sum_{j=1}^p \sum_{l=1}^q C(\boldsymbol{x}_j - \boldsymbol{y}_l) - \sum_{1\le j<j'\le p} C(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) - \sum_{1\le l<l'\le q} C(\boldsymbol{y}_l - \boldsymbol{y}_{l'}) - \frac{p+q}{2} C(\boldsymbol{0}). \end{aligned} \tag{3.4}$$

We introduce the Fourier transform

$$\begin{aligned} \hat{M}_{p,q}\big(z, (\boldsymbol{\xi}_j)_{j=1}^p, (\boldsymbol{\zeta}_l)_{l=1}^q\big) &= \iint M_{p,q}\big(z, (\boldsymbol{x}_j)_{j=1}^p, (\boldsymbol{y}_l)_{l=1}^q\big) \\ &\quad \times \exp\Big(-i\sum_{j=1}^p \boldsymbol{x}_j\cdot\boldsymbol{\xi}_j + i\sum_{l=1}^q \boldsymbol{y}_l\cdot\boldsymbol{\zeta}_l\Big) d\boldsymbol{x}_1\cdots d\boldsymbol{x}_p d\boldsymbol{y}_1\cdots d\boldsymbol{y}_q. \end{aligned} \tag{3.5}$$

It satisfies

$$\frac{\partial \hat{M}_{p,q}}{\partial z} = -\frac{i}{2k_0}\Big(\sum_{j=1}^{p}|\boldsymbol{\xi}_j|^2 - \sum_{l=1}^{q}|\boldsymbol{\zeta}_l|^2\Big)\hat{M}_{p,q} + \frac{k_0^2}{4}\hat{\mathcal{U}}_{p,q}\hat{M}_{p,q}, \tag{3.6}$$

$$\hat{M}_{p,q}(z=0) = \prod_{j=1}^{p}\hat{u}_{\rm ic}(\boldsymbol{\xi}_j)\prod_{l=1}^{q}\overline{\hat{u}_{\rm ic}(\boldsymbol{\zeta}_l)}, \tag{3.7}$$

where $\hat{u}_{\rm ic}$ is the Fourier transform of the initial field:

$$\hat{u}_{\rm ic}(\boldsymbol{\xi}) = \int u_{\rm ic}(\boldsymbol{x})\exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})d\boldsymbol{x}.$$

The operator $\hat{\mathcal{U}}_{p,q}$ is defined by

$$\hat{\mathcal{U}}_{p,q}\hat{M}_{p,q} = \frac{1}{(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[\sum_{j=1}^{p}\sum_{l=1}^{q}\hat{M}_{p,q}(\boldsymbol{\xi}_j-\boldsymbol{k},\boldsymbol{\zeta}_l-\boldsymbol{k}) - \sum_{1\leq j<j'\leq p}\hat{M}_{p,q}(\boldsymbol{\xi}_j-\boldsymbol{k},\boldsymbol{\xi}_{j'}+\boldsymbol{k}) - \sum_{1\leq l<l'\leq q}\hat{M}_{p,q}(\boldsymbol{\zeta}_l-\boldsymbol{k},\boldsymbol{\zeta}_{l'}+\boldsymbol{k}) - \frac{p+q}{2}\hat{M}_{p,q}\Big]d\boldsymbol{k}, \tag{3.8}$$

where we only write the arguments that are shifted. It turns out that the equation for the Fourier transform $\hat{M}_{p,q}$ is easier to solve. In particular it can be integrated readily if the medium is homogeneous. In Equation (3.1) we define the moments relative to the field at fixed depth z. In applications this is typically the quantity of interest since one wants to characterize the wave field in distribution as it emerges at a particular depth and how the microstructure affects the spreading and decorrelation of the field laterally. However, the stochastic equation (2.1) can also be used (with Itô's formula) to derive the equations that give the moments of the wave fields when evaluated at different depths. The problem giving the moments of the wave field when evaluated at different depths involve Schrödinger systems of the above type, but with a potential term and a dispersive operator that change at the set of depths were the field terms are being evaluated.

Remark. As a first application, we can consider the equation satisfied by the first-order moment $M_{1,0}(z,\boldsymbol{x}) = \mathbb{E}[u(z,\boldsymbol{x})]$:

$$\frac{\partial M_{1,0}}{\partial z} = \frac{i}{2k_0}\Delta_{\boldsymbol{x}}M_{1,0} - \frac{k_0^2}{8}C(\mathbf{0})M_{1,0}, \tag{3.9}$$

starting from $M_{1,0}(z=0,\boldsymbol{x}) = u_{\rm ic}(\boldsymbol{x})$. Compared to the homogeneous case, the random medium is responsible for an exponential damping of the wave solution. More exactly the first-order moment is given by the solution of the homogeneous Schrödinger equation multiplied by the damping factor $\exp(-k_0^2C(\mathbf{0})z/8)$.

4 Regimes of Propagation

4.1 The Spot-Dancing Regime

In this subsection we review the results that can be found in [1, 7, 14, 15] and put them in a convenient form for the forthcoming analysis. We consider the spot-dancing regime described at the end of Section 2. In this regime the covariance function C^ε is of the form:

$$C^\varepsilon(\boldsymbol{x}) = \varepsilon^{-2} C(\varepsilon \boldsymbol{x}), \tag{4.1}$$

for a small dimensionless parameter ε. We want to study the asymptotic behavior of the moments of the field in this regime, that we call spot-dancing regime for reasons that will become clear in the analysis.

In the spot-dancing regime we assume that the power spectral density $\hat{C}(\boldsymbol{k})$ decays fast enough so that $\int |\boldsymbol{k}|^2 \hat{C}(\boldsymbol{k}) d\boldsymbol{k}$ is finite. This means that the covariance function $C(\boldsymbol{x})$ is at least twice differentiable at $\boldsymbol{x} = \boldsymbol{0}$, which corresponds to a smooth random medium. For simplicity, we also assume that the random fluctuations are isotropic in the transverse directions, in the sense that the covariance function $C(\boldsymbol{x})$ depends only on $|\boldsymbol{x}|$. We denote

$$\gamma = \frac{1}{d(2\pi)^d} \int |\boldsymbol{k}|^2 \hat{C}(\boldsymbol{k}) d\boldsymbol{k} = -\frac{1}{d} \Delta C(\boldsymbol{0}). \tag{4.2}$$

The operator $\hat{\mathcal{U}}^\varepsilon_{p,q}$ has then the form

$$\begin{aligned}\hat{\mathcal{U}}^\varepsilon_{p,q} \hat{M}_{p,q} = \frac{\varepsilon^{-2}}{(2\pi)^d} \int \hat{C}(\boldsymbol{k}) \Big[\sum_{j=1}^p \sum_{l=1}^q \hat{M}_{p,q}(\boldsymbol{\xi}_j - \varepsilon \boldsymbol{k}, \boldsymbol{\zeta}_l - \varepsilon \boldsymbol{k}) - \sum_{1 \leq j < j' \leq p} \hat{M}_{p,q}(\boldsymbol{\xi}_j - \varepsilon \boldsymbol{k}, \boldsymbol{\xi}_{j'} + \varepsilon \boldsymbol{k}) \\ - \sum_{1 \leq l < l' \leq q} \hat{M}_{p,q}(\boldsymbol{\zeta}_l - \varepsilon \boldsymbol{k}, \boldsymbol{\zeta}_{l'} + \varepsilon \boldsymbol{k}) - \frac{p+q}{2} \hat{M}_{p,q} \Big] d\boldsymbol{k},\end{aligned} \tag{4.3}$$

and it can be expanded as

$$\begin{aligned}\hat{\mathcal{U}}^\varepsilon_{p,q} \hat{M}_{p,q} = \frac{\gamma}{2} \Big[(q-p+1) \sum_{j=1}^p \Delta_{\boldsymbol{\xi}_j} + (p-q+1) \sum_{l=1}^q \Delta_{\boldsymbol{\zeta}_l} + 2 \sum_{j=1}^p \sum_{l=1}^q \nabla_{\boldsymbol{\xi}_j} \cdot \nabla_{\boldsymbol{\zeta}_l} \\ +2 \sum_{1 \leq j < j' \leq p} \nabla_{\boldsymbol{\xi}_j} \cdot \nabla_{\boldsymbol{\xi}_{j'}} + 2 \sum_{1 \leq l < l' \leq q} \nabla_{\boldsymbol{\zeta}_l} \cdot \nabla_{\boldsymbol{\zeta}_{l'}} \Big] \hat{M}_{p,q} - \frac{(p-q)^2}{2\varepsilon^2} C(\boldsymbol{0}) \hat{M}_{p,q}.\end{aligned} \tag{4.4}$$

As shown in [7], this implies that:
1) if $p \neq q$, then $\hat{M}_{p,q} \to 0$ as $\varepsilon \to 0$ because of the strong damping factor of order ε^{-2}.
2) if $p = q$, then, in the regime $\varepsilon \to 0$, the function $\hat{M}_{p,p}$ satisfies the partial differential equation

$$\begin{aligned}\frac{\partial \hat{M}_{p,p}}{\partial z} &= -\frac{i}{2k_0} \Big(\sum_{j=1}^p |\boldsymbol{\xi}_j|^2 - \sum_{l=1}^p |\boldsymbol{\zeta}_l|^2 \Big) \hat{M}_{p,p} \\ &\quad + \frac{k_0^2 \gamma}{8} \Big(\sum_{j=1}^p \nabla_{\boldsymbol{\xi}_j} + \sum_{l=1}^p \nabla_{\boldsymbol{\zeta}_l} \Big) \cdot \Big(\sum_{j=1}^p \nabla_{\boldsymbol{\xi}_j} + \sum_{l=1}^p \nabla_{\boldsymbol{\zeta}_l} \Big) \hat{M}_{p,p}, \qquad (4.5) \\ \hat{M}_{p,p}(z=0) &= \prod_{j=1}^p \hat{u}_{\rm ic}(\boldsymbol{\xi}_j) \prod_{l=1}^p \overline{\hat{u}_{\rm ic}(\boldsymbol{\zeta}_l)}. \qquad (4.6)\end{aligned}$$

Using the Feynman-Kac formula, we find that

$$\hat{M}_{p,p}(z) = \mathbb{E} \Big[\prod_{j=1}^p \hat{u}_{\rm sd}(z, \boldsymbol{\xi}_j) \prod_{l=1}^p \overline{\hat{u}_{\rm sd}(z, \boldsymbol{\zeta}_l)} \Big], \tag{4.7}$$

where

$$\hat{u}_{\rm sd}(z,\boldsymbol{\xi}) = \hat{u}_{\rm ic}\Big(\boldsymbol{\xi} + \frac{k_0\sqrt{\gamma}}{2}\boldsymbol{W}_z\Big)\exp\Big(-\frac{i}{2k_0}\int_0^z \big|\boldsymbol{\xi} + \frac{k_0\sqrt{\gamma}}{2}\boldsymbol{W}_{z'}\big|^2 dz'\Big), \tag{4.8}$$

and $\boldsymbol{W}$ is a standard d-dimensional Brownian motion. Of course, when $\gamma = 0$, we recover the expression of the field in the homogeneous case.

In the regime $\varepsilon \to 0$, the representation formula (4.8) allows us to compute all the moments of the field in which there is an equal number of field components and complex-conjugated field components. Remember that moments for which the number p of field components is different from the number q of complex-conjugated field components are zero in this regime (in fact, they are of the order of $\exp(-C(\mathbf{0})(p-q)^2 z/(2\varepsilon^2)))$. Consequently, a representation formula that allows us to compute all possible moments of the form (3.1) is:

$$\hat{u}_{\rm sd}(z,\boldsymbol{\xi}) = \hat{u}_{\rm ic}\Big(\boldsymbol{\xi} + \frac{k_0\sqrt{\gamma}}{2}\boldsymbol{W}_z\Big)\exp\Big(-\frac{i}{2k_0}\int_0^z \big|\boldsymbol{\xi} + \frac{k_0\sqrt{\gamma}}{2}\boldsymbol{W}_{z'}\big|^2 dz'\Big)\exp\big(i\phi_z\big), \tag{4.9}$$

where ϕ_z is a random variable with uniform distribution over $(0, 2\pi)$.

4.2 The Scintillation Regime

In this subsection We consider the scintillation regime described at the end of Section 2. In this regime the covariance function C^ε is of the form:

$$C^\varepsilon(\boldsymbol{x}) = \varepsilon C(\boldsymbol{x}). \tag{4.10}$$

In order to observe a random effect of order one, we need to consider large propagation distances, of the order of ε^{-1}. Thus, in this regime we make the rescaling $z = z'/\varepsilon$ and suppress the "prime" below. For reasons that will become clear in the analysis we call this regime the scintillation regime. The evolution equations (3.6) of the Fourier transforms of the moments now become

$$\frac{\partial \hat{M}_{p,q}}{\partial z} = -\frac{i}{2k_0\varepsilon}\Big(\sum_{j=1}^p |\boldsymbol{\xi}_j|^2 - \sum_{l=1}^q |\boldsymbol{\zeta}_l|^2\Big)\hat{M}_{p,q} + \frac{k_0^2}{4}\hat{\mathcal{U}}_{p,q}\hat{M}_{p,q}, \tag{4.11}$$

which shows the appearance of a rapid phase. The asymptotic behavior as $\varepsilon \to 0$ of the moments is therefore determined by the solutions of partial differential equations with rapid phase terms. Although we were not able to determine these asymptotic behaviors for all moments, a key limit theorem will allow us to get a representation of the fourth-order moments in Section 6 in the asymptotic regime $\varepsilon \to 0$.

5 The Second-Order Moments

The second-order moments play an important role in wave imaging problems and we will need them to compute the scintillation index. We describe them in detail in this section.

5.1 The Wigner Transform

The second-order moments

$$M_{1,1}(z,\boldsymbol{x},\boldsymbol{y}) = \mathbb{E}\big[u(z,\boldsymbol{x})\overline{u(z,\boldsymbol{y})}\big] \tag{5.1}$$

satisfy the system:

$$\frac{\partial M_{1,1}}{\partial z} = \frac{i}{2k_0}\big(\Delta_{\boldsymbol{x}} - \Delta_{\boldsymbol{y}}\big)M_{1,1} + \frac{k_0^2}{4}\big(C(\boldsymbol{x}-\boldsymbol{y}) - C(\boldsymbol{0})\big)M_{1,1}, \tag{5.2}$$

$$M_{1,1}(z=0) = u_{\rm ic}(\boldsymbol{x})\overline{u_{\rm ic}(\boldsymbol{y})}. \tag{5.3}$$

A convenient approach for solving the second-order moment equation is via the Wigner transform. The Wigner transform of the field is defined by

$$W(z,\boldsymbol{x},\boldsymbol{q}) = \int \exp\big(-i\boldsymbol{q}\cdot\boldsymbol{y}\big)\mathbb{E}\Big[u\big(z,\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\big)\overline{u}\big(z,\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\big)\Big]\,d\boldsymbol{y}. \tag{5.4}$$

Using (5.2) we find that it satisfies the closed system

$$\frac{\partial W}{\partial z} + \frac{1}{k_0}\boldsymbol{q}\cdot\nabla_{\boldsymbol{x}}W = \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[W(\boldsymbol{q}-\boldsymbol{k}) - W(\boldsymbol{q})\Big]d\boldsymbol{k}, \tag{5.5}$$

starting from $W(z=0,\boldsymbol{x},\boldsymbol{q}) = W_{\rm ic}(\boldsymbol{x},\boldsymbol{q})$, which is the Wigner transform of the initial field $u_{\rm ic}$. Eq. (5.5) has the form of a radiative transport equation for the angularly-resolved wave energy density W. In this context $k_0^2C(\boldsymbol{0})/4$ is the total scattering cross-section and $k_0^2\hat{C}(\cdot)/[4(2\pi)^d]$ is the differential scattering cross-section that gives the mode conversion rate.

By taking a Fourier transform in $\boldsymbol{q}$ and $\boldsymbol{x}$ of Eq. (5.5), we obtain a transport equation that can be integrated and we find the following integral representation for W:

$$\begin{aligned} W(z,\boldsymbol{x},\boldsymbol{q}) \;&=\; \frac{1}{(2\pi)^d}\iint \exp\Big(i\boldsymbol{\xi}\cdot\big(\boldsymbol{x}-\boldsymbol{q}\frac{z}{k_0}\big) - i\boldsymbol{y}'\cdot\boldsymbol{q}\Big)\hat{W}_{\rm ic}(\boldsymbol{\xi},\boldsymbol{y}') \\ &\quad\times\exp\Big(\frac{k_0^2}{4}\int_0^z C\big(\boldsymbol{y}'+\boldsymbol{\xi}\frac{z'}{k_0}\big) - C(\boldsymbol{0})dz'\Big)d\boldsymbol{\xi}d\boldsymbol{y}', \end{aligned} \tag{5.6}$$

where $\hat{W}_{\rm ic}$ is a partial Fourier transform of the initial field $u_{\rm ic}$:

$$\hat{W}_{\rm ic}(\boldsymbol{\xi},\boldsymbol{y}) = \int \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{x}\big)\mathbb{E}\Big[u_{\rm ic}\big(\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\big)\overline{u}_{\rm ic}\big(\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\big)\Big]\,d\boldsymbol{x}. \tag{5.7}$$

5.2 The Mutual Coherence Function

By taking an inverse Fourier transform the expression (5.6) can indeed be used to compute and discuss the second-order moment of the field (or mutual coherence function):

$$\begin{aligned} \Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) \;&=\; \mathbb{E}\Big[u\Big(z,\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\Big)\overline{u}\Big(z,\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\Big)\Big] \\ &=\; \frac{1}{(2\pi)^d}\int \exp\big(i\boldsymbol{\xi}\cdot\boldsymbol{x}\big)\hat{W}_{\rm ic}\big(\boldsymbol{\xi},\boldsymbol{y}-\boldsymbol{\xi}\frac{z}{k_0}\big) \\ &\quad\times\exp\Big(\frac{k_0^2}{4}\int_0^z C\big(\boldsymbol{y}-\boldsymbol{\xi}\frac{z'}{k_0}\big) - C(\boldsymbol{0})dz'\Big)d\boldsymbol{\xi}, \end{aligned} \tag{5.8}$$

where $\boldsymbol{x}$ is the mid-point and $\boldsymbol{y}$ is the offset. Let us examine two particular initial conditions, which corresponds to a Gaussian-beam wave and to a plane wave, respectively.

If the input beam spatial profile is Gaussian with radius $r_{\rm ic}$:

$$u_{\rm ic}(\boldsymbol{x}) = \exp\Big(- \frac{|\boldsymbol{x}|^2}{2r_{\rm ic}^2}\Big), \tag{5.9}$$

then we have

$$\hat{W}_{\rm ic}(\boldsymbol{\xi},\boldsymbol{y}) = \big(\pi r_{\rm ic}^2\big)^{d/2} \exp\Big(- \frac{r_{\rm ic}^2|\boldsymbol{\xi}|^2}{4} - \frac{|\boldsymbol{y}|^2}{4r_{\rm ic}^2}\Big), \tag{5.10}$$

and we find from (5.8) that the second-order moment of the field has the form

$$\begin{aligned} \Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) &= \Big(\frac{r_{\rm ic}^2}{4\pi}\Big)^{d/2} \int \exp\Big(- \frac{1}{4r_{\rm ic}^2}\big|\boldsymbol{y}-\boldsymbol{\xi}\frac{z}{k_0}\big|^2 - \frac{r_{\rm ic}^2|\boldsymbol{\xi}|^2}{4} + i\boldsymbol{\xi}\cdot\boldsymbol{x}\Big) \\ &\quad \times \exp\Big(\frac{k_0^2}{4}\int_0^z C\big(\boldsymbol{y}-\boldsymbol{\xi}\frac{z'}{k_0}\big) - C(\boldsymbol{0}) dz'\Big) d\boldsymbol{\xi}. \end{aligned} \tag{5.11}$$

If the initial condition is a plane wave with unit amplitude (which can be viewed as a limit of the Gaussian-beam wave situation in which $r_{\rm ic}\to\infty$), then $\hat{W}_{\rm ic}(\boldsymbol{\xi},\boldsymbol{y}) = (2\pi)^d\delta(\boldsymbol{\xi})$ and

$$\Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) = \exp\Big(\frac{k_0^2 z}{4}\big(C(\boldsymbol{y})-C(\boldsymbol{0})\big)\Big), \tag{5.12}$$

which depends only on the offset $\boldsymbol{y}$ as the field is statistically homogeneous in the transverse direction.

5.3 The Spot-Dancing Regime

In the spot-dancing regime (when the covariance function of the medium fluctuations is of the form (4.1)) with the covariance function $C(\boldsymbol{x})$ depending only on $|\boldsymbol{x}|$ we have

$$C^\varepsilon(\boldsymbol{y}) - C^\varepsilon(\boldsymbol{0}) = -\frac{\gamma}{2}|\boldsymbol{y}|^2 + o(1), \tag{5.13}$$

as $\varepsilon\to 0$, and therefore we find from (5.11) that the second-order moment of the field for a Gaussian beam-wave initial condition is

$$\begin{aligned} \Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) &= \Big(\frac{r_{\rm ic}^2}{4\pi}\Big)^{d/2} \int \exp\Big(- \frac{1}{4r_{\rm ic}^2}\big|\boldsymbol{y}-\boldsymbol{\xi}\frac{z}{k_0}\big|^2 - \frac{r_{\rm ic}^2|\boldsymbol{\xi}|^2}{4} + i\boldsymbol{\xi}\cdot\boldsymbol{x}\Big) \\ &\quad \times \exp\Big(- \frac{k_0^2\gamma}{8}\big(|\boldsymbol{y}|^2 z - \boldsymbol{y}\cdot\boldsymbol{\xi}\frac{z^2}{k_0} + |\boldsymbol{\xi}|^2\frac{z^3}{3k_0^2}\big)\Big) d\boldsymbol{\xi}. \end{aligned} \tag{5.14}$$

We remark that here and below, with some abuse of notation, the equality sign represents also the asymptotic limit as $\varepsilon\to 0$.

By computing the integral in $\boldsymbol{\xi}$ we obtain that the second-order moment of the field has the Gaussian form

$$\Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) = \Big(\frac{r_{\rm ic}}{r_z^{(2)}}\Big)^d \exp\Big(- \frac{|\boldsymbol{x}|^2}{(r_z^{(2)})^2} - \frac{|\boldsymbol{y}|^2}{4(\rho_z^{(2)})^2} + i\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{(\chi_z^{(2)})^2}\Big). \tag{5.15}$$

The beam radius $r_z^{(2)}$, the correlation radius $\rho_z^{(2)}$, and the parameter $\chi_z^{(2)}$ are given by

$$r_z^{(2)} = r_{\rm ic}\Big(1+\frac{z^2}{k_0^2 r_{\rm ic}^4}+\frac{\gamma z^3}{6r_{\rm ic}^2}\Big)^{1/2}, \tag{5.16}$$

$$\rho_z^{(2)} = r_z^{(2)}\Big(1+\frac{k_0^2 r_{\rm ic}^2\gamma z}{2}+\frac{\gamma z^3}{6r_{\rm ic}^2}+\frac{k_0^2\gamma^2 z^4}{48}\Big)^{-1/2}, \tag{5.17}$$

$$\chi_z^{(2)} = r_z^{(2)}\Big(\frac{z}{k_0 r_{\rm ic}^2}+\frac{k_0\gamma z^2}{4}\Big)^{-1/2}. \tag{5.18}$$

Note in particular that the beam radius $r_z^{(2)}$ increases at the anomalous rate $z^{3/2}$ (which was first obtained in the physical literature in Ref. [11] and confirmed mathematically in Ref. [9]). Furthermore, the lateral correlation radius $\rho_z^{(2)}$ decays to zero, which means that the beam becomes partially coherent.
Note also that the energy conservation relation (the conservation of the L^2-norm of the field) is preserved: the mean intensity $\mathbb{E}[|u(z,\boldsymbol{x})|^2] = \Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{0})$ indeed satisfies

$$\int \mathbb{E}[|u(z,\boldsymbol{x})|^2]d\boldsymbol{x} = \int |u_{\rm ic}(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

Remark 1. In the case of a plane wave, i.e., in the limit $r_{\rm ic}\to\infty$, we find that

$$\Gamma^{(2)}(z,\boldsymbol{x},\boldsymbol{y}) = \exp\Big(-\frac{k_0^2\gamma z|\boldsymbol{y}|^2}{8}\Big), \tag{5.19}$$

which shows that the correlation radius is $2\sqrt{2}/(k_0\sqrt{\gamma z})$. This can be obtained from (5.12) as well (the spot-dancing limit $\varepsilon\to 0$ and the plane wave limit $r_{\rm ic}\to\infty$ are exchangeable).

Remark 2. The previous results can also be obtained using the representation formula (4.8), which gives after an inverse Fourier transform in $\boldsymbol{\xi}$:

$$|u_{\rm sd}(z,\boldsymbol{x})|^2 = \Big(\frac{r_{\rm ic}}{r_z^{(0)}}\Big)^d \exp\Big(-\frac{|\boldsymbol{x}-\boldsymbol{X}_z|^2}{(r_z^{(0)})^2}\Big), \tag{5.20}$$

where

$$r_z^{(0)} = r_{\rm ic}\Big(1+\frac{z^2}{k_0^2 r_{\rm ic}^4}\Big)^{1/2} \tag{5.21}$$

is the radius of the beam in the homogeneous medium and

$$\boldsymbol{X}_z = \frac{\sqrt{\gamma}}{2}\Big(z\boldsymbol{W}_z - \int_0^z \boldsymbol{W}_{z'}dz'\Big) = \frac{\sqrt{\gamma}}{2}\int_0^z z' d\boldsymbol{W}_{z'} \tag{5.22}$$

is the random center of the beam (remember that $\boldsymbol{W}$ is a standard d-dimensional Brownian motion), that is a $\mathbb{R}^d$-valued Gaussian process with mean zero and covariance

$$\mathbb{E}\big[\boldsymbol{X}_z\boldsymbol{X}_{z'}^T\big] = \frac{\gamma(z\wedge z')^3}{12}\mathbf{I}. \tag{5.23}$$

This representation (5.20) justifies the name ”spot-dancing regime”: the beam has the same transverse profile as in a homogeneous medium, but its center is randomly shifted by the Gaussian process $\boldsymbol{X}_z$.

5.4 The Scintillation Regime

In the scintillation regime (when the covariance function of the medium fluctuations is of the form (4.10) and the propagation distance is z/ε), the behavior of the second-order moment for a plane-wave initial condition is given by (5.12) for any ε, and a fortiori in the limit $\varepsilon \to 0$.

6 The Fourth-Order Moments

We consider the fourth-order moment $M_{2,2}$ of the field, which is the main quantity of interest in this paper, and parameterize the four points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2$ in (3.1) in the special way:

$$\boldsymbol{x}_1 = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, \qquad \boldsymbol{y}_1 = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{q}_1 - \boldsymbol{q}_2}{2},$$
$$\boldsymbol{x}_2 = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, \qquad \boldsymbol{y}_2 = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 - \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}.$$

In particular $\boldsymbol{r}_1/2$ is the barycenter of the four points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2$:

$$\boldsymbol{r}_1 = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{y}_1 + \boldsymbol{y}_2}{2}, \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{y}_1 - \boldsymbol{y}_2}{2},$$
$$\boldsymbol{r}_2 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{y}_1 - \boldsymbol{y}_2}{2}, \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2 - \boldsymbol{y}_1 + \boldsymbol{y}_2}{2}.$$

In these new variables the function $M_{2,2}$ satisfies the system:

$$\frac{\partial M_{2,2}}{\partial z} = \frac{i}{k_0}\big(\nabla_{\boldsymbol{r}_1} \cdot \nabla_{\boldsymbol{q}_1} + \nabla_{\boldsymbol{r}_2} \cdot \nabla_{\boldsymbol{q}_2}\big) M_{2,2} + \frac{k_0^2}{4} U_{2,2}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) M_{2,2}, \tag{6.1}$$

with the generalized potential

$$\begin{aligned} U_{2,2}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) \quad = \quad & C(\boldsymbol{q}_2 + \boldsymbol{q}_1) + C(\boldsymbol{q}_2 - \boldsymbol{q}_1) + C(\boldsymbol{r}_2 + \boldsymbol{q}_1) + C(\boldsymbol{r}_2 - \boldsymbol{q}_1) \\ & - C(\boldsymbol{q}_2 + \boldsymbol{r}_2) - C(\boldsymbol{q}_2 - \boldsymbol{r}_2) - 2C(\boldsymbol{0}). \end{aligned} \tag{6.2}$$

Note in particular that the generalized potential does not depend on $\boldsymbol{r}_1$ as the medium is statistically homogeneous.

The Fourier transform (in $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{r}_1$, and $\boldsymbol{r}_2$) of the fourth-order moment is defined by:

$$\begin{aligned} \hat{M}_{2,2}(z, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) = \iint & M_{2,2}(z, \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) \\ & \times \exp\big(- i\boldsymbol{q}_1 \cdot \boldsymbol{\xi}_1 - i\boldsymbol{r}_1 \cdot \boldsymbol{\zeta}_1 - i\boldsymbol{q}_2 \cdot \boldsymbol{\xi}_2 - i\boldsymbol{r}_2 \cdot \boldsymbol{\zeta}_2\big) d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{q}_1 d\boldsymbol{q}_2. \end{aligned} \tag{6.3}$$

It satisfies

$$\begin{aligned} \frac{\partial \hat{M}_{2,2}}{\partial z} + \frac{i}{k_0}\big(\boldsymbol{\xi}_1 \cdot \boldsymbol{\zeta}_1 + \boldsymbol{\xi}_2 \cdot \boldsymbol{\zeta}_2\big) \hat{M}_{2,2} = \frac{k_0^2}{4(2\pi)^d} \int \hat{C}(\boldsymbol{k}) \Big[& \hat{M}_{2,2}(\boldsymbol{\xi}_1 - \boldsymbol{k}, \boldsymbol{\xi}_2 - \boldsymbol{k}, \boldsymbol{\zeta}_2) \\ + \hat{M}_{2,2}(\boldsymbol{\xi}_1 - \boldsymbol{k}, \boldsymbol{\xi}_2, \boldsymbol{\zeta}_2 - \boldsymbol{k}) + \hat{M}_{2,2}(\boldsymbol{\xi}_1 + \boldsymbol{k}, \boldsymbol{\xi}_2 - \boldsymbol{k}, \boldsymbol{\zeta}_2) & + \hat{M}_{2,2}(\boldsymbol{\xi}_1 + \boldsymbol{k}, \boldsymbol{\xi}_2, \boldsymbol{\zeta}_2 - \boldsymbol{k}) \\ - 2\hat{M}_{2,2}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\zeta}_2) - \hat{M}_{2,2}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 - \boldsymbol{k}, \boldsymbol{\zeta}_2 - \boldsymbol{k}) & - \hat{M}_{2,2}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 + \boldsymbol{k}, \boldsymbol{\zeta}_2 - \boldsymbol{k}) \Big] d\boldsymbol{k}. \end{aligned} \tag{6.4}$$

The resolution of this transport equation would give the expression of the fourth-order moment. However, in contrast to the second-order moment, we cannot solve this equation and find a closed-form expression of the fourth-order moment in the general case. Therefore we address in the next two subsections the two particular regimes described in Section 4 in which explicit expressions can be obtained. These two regimes are very different in that the spot-dancing regime that we address in Subsection 6.1 is characterized by a large variance of the intensity distribution, while the scintillation regime that we address in Subsection 6.3 is characterized by a normalized variance that stabilizes to the value one, which is characteristic of complex Gaussian fields.

6.1 Spot-Dancing Regime

In the spot-dancing regime the equation for the Fourier transform of the fourth-order moment can be simplified:

$$\frac{\partial \hat{M}_{2,2}}{\partial z} + \frac{i}{k_0}\big(\boldsymbol{\xi}_1 \cdot \boldsymbol{\zeta}_1 + \boldsymbol{\xi}_2 \cdot \boldsymbol{\zeta}_2\big)\hat{M}_{2,2} = \frac{k_0^2 \gamma}{2}\Delta_{\boldsymbol{\xi}_1}\hat{M}_{2,2}. \tag{6.5}$$

This equation can be solved (by a Fourier transform in $\boldsymbol{\xi}_1$):

$$\hat{M}_{2,2}(z,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = \int \hat{M}_{\rm ic}(\boldsymbol{\xi}_1',\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)\exp\Big(-i\frac{z}{k_0}\boldsymbol{\xi}_1'\cdot\boldsymbol{\zeta}_1 - i\frac{z}{k_0}\boldsymbol{\xi}_2\cdot\boldsymbol{\zeta}_2\Big)\psi(z,\boldsymbol{\xi}_1-\boldsymbol{\xi}_1',\boldsymbol{\zeta}_1)d\boldsymbol{\xi}_1' \tag{6.6}$$

with

$$\psi(z,\boldsymbol{\xi},\boldsymbol{\zeta}_1) = \frac{1}{(2\pi k_0^2\gamma z)^{d/2}}\exp\Big(-\frac{\gamma z^3}{24}|\boldsymbol{\zeta}_1|^2 - i\frac{z}{2k_0}\boldsymbol{\xi}\cdot\boldsymbol{\zeta}_1 - \frac{1}{2k_0^2\gamma z}|\boldsymbol{\xi}|^2\Big), \tag{6.7}$$

and $\hat{M}_{\rm ic}$ is the Fourier transform of the fourth-order moment $M_{2,2}$ of the initial field $u_{\rm ic}$.

In terms of the function $M_{2,2}$, the second-order moment of the intensity defined by

$$\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) = \mathbb{E}\Big[\big|u\big(z,\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\big)\big|^2\big|u\big(z,\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\big)\big|^2\Big] \tag{6.8}$$

is given by

$$\begin{aligned}\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) &= M_{2,2}(z,\mathbf{0},\mathbf{0},\boldsymbol{r}_1=2\boldsymbol{x},\boldsymbol{r}_2=\boldsymbol{y})\\ &= \frac{1}{(2\pi)^{4d}}\iint \hat{M}_{2,2}(z,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)\exp\big(2i\boldsymbol{x}\cdot\boldsymbol{\zeta}_1 + i\boldsymbol{y}\cdot\boldsymbol{\zeta}_2\big)d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\zeta}_1 d\boldsymbol{\zeta}_2.\end{aligned} \tag{6.9}$$

In particular the mean square intensity is

$$\begin{aligned}\mathbb{E}\big[|u(z,\boldsymbol{x})|^4\big] &= \Gamma^{(4)}(z,\boldsymbol{x},\mathbf{0})\\ &= \frac{1}{(2\pi)^{4d}}\iint \hat{M}_{2,2}(z,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)\exp\big(2i\boldsymbol{x}\cdot\boldsymbol{\zeta}_1\big)d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\zeta}_1 d\boldsymbol{\zeta}_2.\end{aligned} \tag{6.10}$$

6.2 Spot-Dancing Regime with a Gaussian-beam Wave Initial Condition

If we assume that the initial condition corresponds to a Gaussian-beam wave (5.9) with width $r_{\rm ic}$, so that

$$\hat{M}_{\rm ic}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = (2\pi r_{\rm ic}^2)^{2d}\exp\Big(-\frac{r_{\rm ic}^2}{2}\big(|\boldsymbol{\xi}_1|^2+|\boldsymbol{\xi}_2|^2+|\boldsymbol{\zeta}_1|^2+|\boldsymbol{\zeta}_2|^2\big)\Big), \tag{6.11}$$

then we find from (6.10) that

$$\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{0}) = \mathbb{E}\big[|u(z,\boldsymbol{x})|^4\big] = \Big(\frac{r_{\rm ic}^2}{r_z^{(0)} r_z^{(4)}}\Big)^d \exp\Big(-\frac{2|\boldsymbol{x}|^2}{(r_z^{(4)})^2}\Big), \tag{6.12}$$

with $r_z^{(0)}$ the radius (5.21) of the beam in the homogeneous medium and

$$r_z^{(4)} = r_{\rm ic}\Big(1+\frac{z^2}{k_0^2 r_{\rm ic}^4}+\frac{\gamma z^3}{3r_{\rm ic}^2}\Big)^{1/2}. \tag{6.13}$$

Note that the radius $r_z^{(4)}$ is of the same order as the radius $r_z^{(2)}$ of the mean intensity up to a factor two in the last term.

Using (6.9) we obtain that the second-order moment of the intensity is given by

$$\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) = \Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{0}) \exp\Big(-\frac{|\boldsymbol{y}|^2}{2(\rho_z^{(4)})^2}\Big), \tag{6.14}$$

with

$$\rho_z^{(4)} = r_{\rm ic}\Big(1+\frac{z^2}{k_0^2 r_{\rm ic}^4}\Big)^{1/2}. \tag{6.15}$$

The correlation radius $\rho_z^{(4)}$ of the intensity is equal to the radius $r_z^{(0)}$ of the beam in a homogeneous medium. This is in agreement with the spot-dancing picture: the beam has the same transverse profile as in a homogeneous medium, but its center is randomly shifted by a Gaussian process whose standard deviation increases as $z^{3/2}$, which gives a radius for the average intensity or the average square intensity that increases as $z^{3/2}$ as well.

Finally, the normalized variance of the intensity (or scintillation index):

$$S(z,\boldsymbol{x}) = \frac{\mathbb{E}\big[|u(z,\boldsymbol{x})|^4\big]-\mathbb{E}\big[|u(z,\boldsymbol{x})|^2\big]^2}{\mathbb{E}\big[|u(z,\boldsymbol{x})|^2\big]^2} \tag{6.16}$$

is given at the center of the beam by:

$$S(z,\boldsymbol{0}) = \Big(\frac{(r_z^{(2)})^2}{r_z^{(0)} r_z^{(4)}}\Big)^d - 1 = \left(1+\frac{\big(\frac{\gamma z^3}{6(r_z^{(0)})^2}\big)^2}{1+\frac{\gamma z^3}{3(r_z^{(0)})^2}}\right)^{d/2} - 1, \tag{6.17}$$

which grows with the propagation distance as

$$S(z,\boldsymbol{0}) \overset{\gamma z^3 \ll (r_z^{(0)})^2}{\simeq} \frac{d}{2}\Big(\frac{\gamma z^3}{6(r_z^{(0)})^2}\Big)^2, \qquad S(z,\boldsymbol{0}) \overset{\gamma z^3 \gg (r_z^{(0)})^2}{\simeq} \Big(\frac{\gamma z^3}{12(r_z^{(0)})^2}\Big)^{d/2}. \tag{6.18}$$

The large scintillation index reflects the spot dancing and a heavy-tailed intensity distribution (a non-central chi-square distribution with two degrees of freedom, also known as the Rice-Nakagami distribution).

In the wings of the beam the scintillation index is even larger:

$$S(z,\boldsymbol{x}) = \left(1+\frac{\big(\frac{\gamma z^3}{6(r_z^{(0)})^2}\big)^2}{1+\frac{\gamma z^3}{3(r_z^{(0)})^2}}\right)^{d/2} \exp\left(\frac{2|\boldsymbol{x}|^2}{(r_z^{(0)})^2}\frac{\frac{\gamma z^3}{6(r_z^{(0)})^2}}{\big(1+\frac{\gamma z^3}{6(r_z^{(0)})^2}\big)\big(1+\frac{\gamma z^3}{3(r_z^{(0)})^2}\big)}\right) - 1, \tag{6.19}$$

which grows with the propagation distance as

$$S(z,\boldsymbol{x}) \overset{\gamma z^3 \ll (r_z^{(0)})^2}{\simeq} \frac{2|\boldsymbol{x}|^2}{(r_z^{(0)})^2}\Big(\frac{\gamma z^3}{6(r_z^{(0)})^2}\Big) + \frac{d}{2}\Big(\frac{\gamma z^3}{6(r_z^{(0)})^2}\Big)^2, \tag{6.20}$$

$$S(z,\boldsymbol{x}) \overset{\gamma z^3 \gg (r_z^{(0)})^2}{\simeq} \Big(\frac{\gamma z^3}{12(r_z^{(0)})^2}\Big)^{d/2} \exp\Big(\frac{6|\boldsymbol{x}|^2}{\gamma z^3}\Big). \tag{6.21}$$

Remember that $(r_z^{(0)})^2$ is the square beam width when $\gamma z^3 \ll (r_z^{(0)})^2$ and that $\gamma z^3/6$ is the square beam width when $\gamma z^3 \gg (r_z^{(0)})^2$. This shows that the scintillation index in the wings of the beam is larger than the scintillation index at the center of the beam. We remark however that this does not mean that the wave field shape is random, the above moments are consistent with self-averaging of this in the sense that the field observed in the random frame is statistically stable or deterministic to leading order.

6.3 Plane-Wave Initial Condition

In the spot-dancing regime the beam is relatively narrow. Here we do not assume anymore the spot-dancing regime and we consider the case with a plane-wave initial condition. We will see in the next section that this is the situation that leads to a scintillation regime and this is the motivation for analyzing the plane-wave configuration in detail. If the initial condition corresponds to a plane wave with unit amplitude, which implies that the initial condition for $M_{2,2}$ is equal to one and thus does not depend on $\boldsymbol{r}_1$, then the fourth-order moment $M_{2,2}$ remains independent on $\boldsymbol{r}_1$ for any z. Moreover, the subfamily of fourth-order moments for $\boldsymbol{q}_1 = \boldsymbol{0}$ satisfies a closed-form system. Let us denote

$$N(z,\boldsymbol{q},\boldsymbol{r}) = \mathbb{E}\Big[u\big(z,\frac{\boldsymbol{r}+\boldsymbol{q}}{2}\big)u\big(z,\frac{-\boldsymbol{r}-\boldsymbol{q}}{2}\big)\overline{u}\big(z,\frac{\boldsymbol{r}-\boldsymbol{q}}{2}\big)\overline{u}\big(z,\frac{-\boldsymbol{r}+\boldsymbol{q}}{2}\big)\Big]. \tag{6.22}$$

Then this function satisfies

$$\frac{\partial N}{\partial z} = \frac{i}{k_0}\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{q}} N + \frac{k_0^2}{4}U(\boldsymbol{r},\boldsymbol{q})N, \tag{6.23}$$

with the initial condition $N(z=0,\boldsymbol{q},\boldsymbol{r}) = 1$ and the generalized potential

$$U(\boldsymbol{r},\boldsymbol{q}) = 2C(\boldsymbol{q}) + 2C(\boldsymbol{r}) - C(\boldsymbol{q}+\boldsymbol{r}) - C(\boldsymbol{q}-\boldsymbol{r}) - 2C(\boldsymbol{0}). \tag{6.24}$$

Note that, if we take $\boldsymbol{q} = \boldsymbol{0}$, then

$$N(z,\boldsymbol{q}=\boldsymbol{0},\boldsymbol{r}=\boldsymbol{y}) = \mathbb{E}\Big[\big|u\big(z,\frac{\boldsymbol{y}}{2}\big)\big|^2\big|u\big(z,-\frac{\boldsymbol{y}}{2}\big)\big|^2\Big] = \Gamma^{(4)}(z,\boldsymbol{0},\boldsymbol{y}) \tag{6.25}$$

turns out to be the second-order moment of the intensity (with offset $\boldsymbol{y}$). Note that here the field and intensity are statistically homogeneous in the transverse direction and so $\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) = \Gamma^{(4)}(z,\boldsymbol{0},\boldsymbol{y})$. Therefore (6.25) shows that the computation of the subfamily N of fourth-order moments of the field gives the second-order moment of the intensity.

Let us take a Fourier transform in $\boldsymbol{q}$ and $\boldsymbol{r}$:

$$\hat{N}(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = \iint N(z,\boldsymbol{r},\boldsymbol{q})\exp\big(-i\boldsymbol{q}\cdot\boldsymbol{\xi} - i\boldsymbol{r}\cdot\boldsymbol{\zeta}\big)d\boldsymbol{r}d\boldsymbol{q}. \tag{6.26}$$

It satisfies

$$\frac{\partial \hat{N}}{\partial z} + \frac{i}{k_0}\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\hat{N} = \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[2\hat{N}(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta}) + 2\hat{N}(\boldsymbol{\xi},\boldsymbol{\zeta}-\boldsymbol{k}) - 2\hat{N}(\boldsymbol{\xi},\boldsymbol{\zeta}) - \hat{N}(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k}) - \hat{N}(\boldsymbol{\xi}+\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})\Big]d\boldsymbol{k}, \qquad (6.27)$$

starting from $\hat{N}(z=0,\boldsymbol{\xi},\boldsymbol{\zeta}) = (2\pi)^{2d}\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta})$.

The new function $\tilde{N}$ defined by

$$\tilde{N}(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = \hat{N}(z,\boldsymbol{\xi},\boldsymbol{\zeta})\exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z}{k_0}\Big) \qquad (6.28)$$

satisfies

$$\frac{\partial \tilde{N}}{\partial z} = \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[2\tilde{N}(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta})e^{i\frac{z}{k_0}\boldsymbol{k}\cdot\boldsymbol{\zeta}} + 2\tilde{N}(\boldsymbol{\xi},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{k_0}\boldsymbol{k}\cdot\boldsymbol{\xi}} - 2\tilde{N}(\boldsymbol{\xi},\boldsymbol{\zeta}) - \tilde{N}(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta})-|\boldsymbol{k}|^2)} - \tilde{N}(\boldsymbol{\xi}+\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta})+|\boldsymbol{k}|^2)}\Big]d\boldsymbol{k}, \qquad (6.29)$$

starting from $\tilde{N}(z=0,\boldsymbol{\xi},\boldsymbol{\zeta}) = (2\pi)^{2d}\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta})$.

6.4 Scintillation Regime with a Plane-Wave Initial Condition

We now assume a plane-wave initial condition and we moreover consider the scintillation regime, that is to say, the fluctuations of the medium are small, of the order of ε, as in (4.10), and the propagation distance is large, of the order of ε^{-1}. In this regime the rescaled function $\tilde{N}^\varepsilon$ defined by

$$\tilde{N}^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = \tilde{N}\big(\frac{z}{\varepsilon},\boldsymbol{\xi},\boldsymbol{\zeta}\big) \qquad (6.30)$$

satisfies the equation with fast phases

$$\frac{\partial \tilde{N}^\varepsilon}{\partial z} = \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[2\tilde{N}^\varepsilon(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta})e^{i\frac{z}{\varepsilon k_0}\boldsymbol{k}\cdot\boldsymbol{\zeta}} + 2\tilde{N}^\varepsilon(\boldsymbol{\xi},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}\boldsymbol{k}\cdot\boldsymbol{\xi}} - 2\tilde{N}^\varepsilon(\boldsymbol{\xi},\boldsymbol{\zeta}) - \tilde{N}^\varepsilon(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta})-|\boldsymbol{k}|^2)} - \tilde{N}^\varepsilon(\boldsymbol{\xi}+\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta})+|\boldsymbol{k}|^2)}\Big]d\boldsymbol{k}, \qquad (6.31)$$

starting from $\tilde{N}^\varepsilon(z=0,\boldsymbol{\xi},\boldsymbol{\zeta}) = (2\pi)^{2d}\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta})$. Our goal is now to study the asymptotic behavior of $\tilde{N}^\varepsilon$ as $\varepsilon \to 0$. We have the following result:

Proposition 6.1. *The distribution $\tilde{N}^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta})$ can be expanded as*

$$\begin{aligned}\tilde{N}^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) &= \phi(z,\boldsymbol{\xi})\delta(\boldsymbol{\zeta}) + \phi(z,\boldsymbol{\zeta})\delta(\boldsymbol{\xi}) + K(z)\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta}) \\ &\quad + \overline{Q^\varepsilon(z,\boldsymbol{\xi})}\delta(\boldsymbol{\xi}+\boldsymbol{\zeta}) + Q^\varepsilon(z,\boldsymbol{\xi})\delta(\boldsymbol{\xi}-\boldsymbol{\zeta}) + R^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}),\end{aligned} \qquad (6.32)$$

where the functions ϕ and K are defined by

$$\phi(z,\boldsymbol{\xi}) = (2\pi)^d\int \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{x}\big)\Big\{\exp\Big(\frac{k_0^2}{2}[C(\boldsymbol{x})-C(\boldsymbol{0})]z\Big) - \exp\Big(-\frac{k_0^2}{2}C(\boldsymbol{0})z\Big)\Big\}d\boldsymbol{x}, \qquad (6.33)$$

$$K(z) = (2\pi)^{2d}\exp\Big(-\frac{k_0^2}{2}C(\boldsymbol{0})z\Big), \qquad (6.34)$$

and the functions $R^\varepsilon \in L^\infty([0,\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *and* $Q^\varepsilon \in L^\infty([0,\infty), L^1(\mathbb{R}^d))$ *satisfy* $\|R^\varepsilon\|_{L^\infty([0,Z],L^1(\mathbb{R}^d\times\mathbb{R}^d))} \to 0$ *and* $\|Q^\varepsilon\|_{L^\infty([0,Z],L^1(\mathbb{R}^d))} \to 0$ *as* $\varepsilon \to 0$*, for any* $Z > 0$*.*

As a corollary, by taking an inverse Fourier transform in $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we obtain that, in the regime $\varepsilon \to 0$, the rescaled fourth-order moment N^ε defined by

$$N^\varepsilon(z,\boldsymbol{q},\boldsymbol{r}) = N\big(\frac{z}{\varepsilon},\boldsymbol{q},\boldsymbol{r}\big) \tag{6.35}$$

is given by

$$\begin{aligned} N^\varepsilon(z,\boldsymbol{q},\boldsymbol{r}) &= \frac{1}{(2\pi)^{2d}} \iint \big[\phi(z,\boldsymbol{\xi})\delta(\boldsymbol{\zeta}) + \phi(z,\boldsymbol{\xi})\delta(\boldsymbol{\xi}) + K(z)\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta})\big] \exp\big(i\boldsymbol{q}\cdot\boldsymbol{\xi} + i\boldsymbol{r}\cdot\boldsymbol{\zeta}\big) d\boldsymbol{\xi} d\boldsymbol{\zeta} \\ &= \frac{1}{(2\pi)^{2d}} \big[\check{\phi}(z,\boldsymbol{q}) + \check{\phi}(z,\boldsymbol{r}) + K(z)\big], \end{aligned} \tag{6.36}$$

up to terms that go to zero in ε uniformly in $\boldsymbol{q},\boldsymbol{r}$, where

$$\begin{aligned} \check{\phi}(z,\boldsymbol{q}) &= \int \phi(z,\boldsymbol{\xi}) \exp\big(i\boldsymbol{q}\cdot\boldsymbol{\xi}\big) d\boldsymbol{\xi} \\ &= (2\pi)^{2d}\Big\{ \exp\Big(\frac{k_0^2}{2}[C(\boldsymbol{q}) - C(\mathbf{0})]z\Big) - \exp\Big(-\frac{k_0^2}{2}C(\mathbf{0})z\Big)\Big\}. \end{aligned} \tag{6.37}$$

Consequently, using (6.25) the second-order moment of the intensity (6.8) has the following form in the regime $\varepsilon \to 0$:

$$\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) = 1 + \exp\Big(\frac{k_0^2}{2}[C(\boldsymbol{y}) - C(\mathbf{0})]z\Big) - \exp\Big(-\frac{k_0^2}{2}C(\mathbf{0})z\Big). \tag{6.38}$$

In particular,

Proposition 6.2. *In the scintillation regime the normalized variance of the intensity (or scintillation index) (6.16) is*

$$S(z,\boldsymbol{x}) = \frac{\mathbb{E}\big[|u(z,\boldsymbol{x})|^4\big] - \mathbb{E}\big[|u(z,\boldsymbol{x})|^2\big]^2}{\mathbb{E}\big[|u(z,\boldsymbol{x})|^2\big]^2} = 1 - \exp\Big(-\frac{k_0^2}{2}C(\mathbf{0})z\Big). \tag{6.39}$$

Thus, $S(z,\boldsymbol{x}) \sim 1$ for $k_0^2 C(\mathbf{0})z \gg 1$. Moreover, when z is large (in the sense that $k_0^2 C(\mathbf{0})z \gg 1$), then

$$\Gamma^{(4)}(z,\boldsymbol{x},\boldsymbol{y}) \simeq 1 + \exp\Big(-\frac{k_0^2\gamma z}{4}|\boldsymbol{y}|^2\Big) \tag{6.40}$$

and therefore we have

Proposition 6.3. *The correlation function of the intensity for* $k_0^2 C(\mathbf{0})z \gg 1$ *is*

$$\frac{\mathbb{E}\big[|u(z,\mathbf{0})|^2|u(z,\boldsymbol{y})|^2\big] - \mathbb{E}\big[|u(z,\mathbf{0})|^2\big]\mathbb{E}\big[|u(z,\boldsymbol{y})|^2\big]}{\mathbb{E}\big[|u(z,\mathbf{0})|^2|\big]\mathbb{E}\big[u(z,\boldsymbol{y})|^2\big]} \simeq \exp\Big(-\frac{k_0^2\gamma z}{4}|\boldsymbol{y}|^2\Big). \tag{6.41}$$

This shows that the correlation radius of the intensity distribution is $2/(k_0\sqrt{\gamma z})$, which is of the same order as the correlation radius of the field as seen from (5.19). We remark that this picture is consistent with the one in which the field looses its coherence and attains the form of Gaussian fluctuations. This is in contrast to a scaling regime with localized initial data and where one looks at smoothed versions of the intensity which may become statistically stable in the situation with additional later diversity in the random potential [25].

7 Conclusion

In this paper we have considered the white-noise paraxial wave model and computed the second-order moment of the field distribution and of the intensity distribution. We have identified a regime with a plane wave initial condition in which the normalized variance of the intensity converges to one with the propagation distance. This is consistent with the scintillation conjecture, that states that the field should acquire complex Gaussian statistics when it propagates through a three-dimensional scattering medium. The full proof requires analysis of all the moments while we here considered only moments up to fourth order. In practical applications however the moments up to order four are the most important ones. Moreover, we started with the Ito-Schrödinger equation rather than the full wave equation. An interesting and challenging next step would be to derive these results for the fourth moment starting with the wave equation itself.

Acknowledgements

This work is partly supported by AFOSR grant # FA9550-11-1-0176, NSF ARRA grant DMS 0908274, and by ERC Advanced Grant Project MULTIMOD-267184.

A Proof of Proposition 6.1

The strategy is to choose the function Q^ε in such a way that the first terms in the right-hand side of (6.32) capture all the singular terms, that the remainder R^ε belongs to L^1, satisfies an equation with zero initial condition and small source terms, and that the singular terms with Q^ε and the absolutely continuous remainder R^ε can be shown to vanish as $\varepsilon \to 0$ by a Gronwall-type argument.

First note that the functions ϕ and K defined in (6.33-6.34) are solution of

$$\frac{\partial \phi}{\partial z}(z,\boldsymbol{\xi}) = \frac{k_0^2}{2(2\pi)^d}\int \hat{C}(\boldsymbol{k})\big[\phi(z,\boldsymbol{\xi}-\boldsymbol{k})-\phi(z,\boldsymbol{\xi})\big]d\boldsymbol{k} + \frac{k_0^2}{2(2\pi)^d}K(z)\hat{C}(\boldsymbol{\xi}), \quad \text{(A.1)}$$

$$\frac{\partial K}{\partial z}(z) = -\frac{k_0^2}{2}C(\mathbf{0})K(z), \quad \text{(A.2)}$$

starting from $\phi(0,\boldsymbol{\xi}) = 0$ and $K(0) = (2\pi)^{2d}$. The function ϕ has nice properties.

Lemma A.1. *The function ϕ is nonnegative real-valued and it belongs to $\mathcal{C}^0([0,\infty), L^1(\mathbb{R}^d))$.*

Proof. We introduce the function $\tilde{\phi}(z,\boldsymbol{\xi}) = \phi(z,\boldsymbol{\xi})\exp(k_0^2 C(\mathbf{0})z/2)$. It satisfies the differential equation

$$\frac{\partial \tilde{\phi}}{\partial z}(z,\boldsymbol{\xi}) = \frac{k_0^2}{2(2\pi)^d}\int \hat{C}(\boldsymbol{k})\tilde{\phi}(z,\boldsymbol{\xi}-\boldsymbol{k})d\boldsymbol{k} + \frac{k_0^2}{2}(2\pi)^d\hat{C}(\boldsymbol{\xi}), \quad \text{(A.3)}$$

with the initial condition $\tilde{\phi}(z=0,\boldsymbol{\xi}) = 0$, which shows that $\tilde{\phi}$ is increasing in z and nonnegative for all $\boldsymbol{\xi}$. Taking a Fourier transform in $\boldsymbol{\xi}$ gives an equation that can be integrated and gives the formula (6.33). Furthermore, by integrating (A.3) in $\boldsymbol{\xi}$ one finds that the L^1-norm satisfies:

$$\frac{\partial \|\tilde{\phi}\|_{L^1}}{\partial z} = \frac{k_0^2}{2}C(\mathbf{0})\|\tilde{\phi}\|_{L^1} + \frac{k_0^2}{2}(2\pi)^{2d}C(\mathbf{0}),$$

which shows that the L^1-norm of ϕ is bounded and given by

$$\|\phi(z,\cdot)\|_{L^1} = (2\pi)^{2d}\Big(1-\exp\Big(-\frac{k_0^2}{2}C(\mathbf{0})z\Big)\Big),$$

as confirmed by a direct integration of (6.33). It also gives the fact that ϕ is Lipschitz-continuous:

$$\|\phi(z_2,\cdot)-\phi(z_1,\cdot)\|_{L^1} \le (2\pi)^{2d}|z_2-z_1|(k_0^2C(\mathbf{0})/2).$$

□

Let us define the function $Q^\varepsilon(z,\boldsymbol{\xi})$ as the solution of the differential equation

$$\begin{aligned}\frac{\partial Q^\varepsilon}{\partial z}(z,\boldsymbol{\xi}) &= -\frac{k_0^2}{2}C(\mathbf{0})Q^\varepsilon(z,\boldsymbol{\xi}) - \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})Q^\varepsilon(z,\boldsymbol{\xi}-\boldsymbol{k})e^{i\frac{z}{k_0\varepsilon}(2\boldsymbol{k}\cdot\boldsymbol{\xi}-|\boldsymbol{k}|^2)}d\boldsymbol{k} \\ &\quad -(2\pi)^d\frac{k_0^2}{4}\hat{C}(\boldsymbol{\xi})e^{-\frac{k_0^2}{2}C(\mathbf{0})z}e^{i\frac{z}{k_0\varepsilon}|\boldsymbol{\xi}|^2}, \qquad \text{(A.4)}\end{aligned}$$

starting from $Q^\varepsilon(0,\boldsymbol{\xi})=0$. Then, using (6.31), the function $R^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta})$ defined by

$$\begin{aligned}R^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) &= \tilde{N}^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta})(z,\boldsymbol{\xi},\boldsymbol{\zeta}) - \phi(z,\boldsymbol{\xi})\delta(\boldsymbol{\zeta}) - \phi(z,\boldsymbol{\zeta})\delta(\boldsymbol{\xi}) - K(z)\delta(\boldsymbol{\xi})\delta(\boldsymbol{\zeta}) \\ &\quad -\overline{Q^\varepsilon(z,\boldsymbol{\xi})}\delta(\boldsymbol{\xi}+\boldsymbol{\zeta}) - Q^\varepsilon(z,\boldsymbol{\xi})\delta(\boldsymbol{\xi}-\boldsymbol{\zeta}) \qquad \text{(A.5)}\end{aligned}$$

is solution of

$$\frac{\partial R^\varepsilon}{\partial z}(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = [\mathcal{F}_z^\varepsilon R^\varepsilon(z,\cdot,\cdot)](\boldsymbol{\xi},\boldsymbol{\zeta}) + G^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta})\exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z}{k_0\varepsilon}\Big), \qquad \text{(A.6)}$$

starting from $R^\varepsilon(0,\boldsymbol{\xi},\boldsymbol{\zeta})=0$, with

$$\begin{aligned}[\mathcal{F}_z^\varepsilon R](\boldsymbol{\xi},\boldsymbol{\zeta}) = \frac{k_0^2}{4(2\pi)^d}\int \hat{C}(\boldsymbol{k})\Big[&2R(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta})e^{i\frac{z}{\varepsilon k_0}\boldsymbol{k}\cdot\boldsymbol{\zeta}} + 2R(\boldsymbol{\xi},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}\boldsymbol{k}\cdot\boldsymbol{\xi}} - 2R(\boldsymbol{\xi},\boldsymbol{\zeta}) \\ &-R(\boldsymbol{\xi}-\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta})-|\boldsymbol{k}|^2)} - R(\boldsymbol{\xi}+\boldsymbol{k},\boldsymbol{\zeta}-\boldsymbol{k})e^{i\frac{z}{\varepsilon k_0}(\boldsymbol{k}\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta})+|\boldsymbol{k}|^2)}\Big]d\boldsymbol{k}, \qquad \text{(A.7)}\end{aligned}$$

and

$$\begin{aligned}G^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) &= G_1(z,\boldsymbol{\xi},\boldsymbol{\zeta}) + G_2^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}), \qquad \text{(A.8)} \\ G_1(z,\boldsymbol{\xi},\boldsymbol{\zeta}) &= \frac{k_0^2}{4(2\pi)^d}\Big\{2\hat{C}(\boldsymbol{\xi})\phi(z,\boldsymbol{\zeta}) + 2\hat{C}(\boldsymbol{\zeta})\phi(z,\boldsymbol{\xi}) \\ &\qquad -[\hat{C}(\boldsymbol{\xi})+\hat{C}(\boldsymbol{\zeta})][\phi(z,\boldsymbol{\zeta}-\boldsymbol{\xi})+\phi(z,\boldsymbol{\zeta}+\boldsymbol{\xi})]\Big\}, \qquad \text{(A.9)} \\ G_2^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) &= \frac{k_0^2}{4(2\pi)^d}\Big\{2\hat{C}(\boldsymbol{\xi}+\boldsymbol{\zeta})\big[\overline{Q^\varepsilon(z,\boldsymbol{\xi})}e^{i\frac{z}{k_0\varepsilon}|\boldsymbol{\xi}|^2} + \overline{Q^\varepsilon(z,\boldsymbol{\zeta})}e^{i\frac{z}{k_0\varepsilon}|\boldsymbol{\zeta}|^2}\big] \\ &\qquad +2\hat{C}(\boldsymbol{\xi}-\boldsymbol{\zeta})\big[Q^\varepsilon(z,\boldsymbol{\xi})e^{-i\frac{z}{k_0\varepsilon}|\boldsymbol{\xi}|^2} + Q^\varepsilon(z,\boldsymbol{\zeta})e^{-i\frac{z}{k_0\varepsilon}|\boldsymbol{\zeta}|^2}\big] \\ &\qquad -2^{-d}\hat{C}\big(\frac{\boldsymbol{\xi}+\boldsymbol{\zeta}}{2}\big)\overline{Q^\varepsilon\big(z,\frac{\boldsymbol{\xi}-\boldsymbol{\zeta}}{2}\big)}e^{i\frac{z}{4k_0\varepsilon}|\boldsymbol{\zeta}-\boldsymbol{\xi}|^2} \\ &\qquad -2^{-d}\hat{C}\big(\frac{\boldsymbol{\xi}-\boldsymbol{\zeta}}{2}\big)Q^\varepsilon\big(z,\frac{\boldsymbol{\xi}+\boldsymbol{\zeta}}{2}\big)e^{-i\frac{z}{4k_0\varepsilon}|\boldsymbol{\zeta}+\boldsymbol{\xi}|^2}\Big\}. \qquad \text{(A.10)}\end{aligned}$$

Here we have used the fact that $Q^\varepsilon(z,-\boldsymbol{\xi}) = Q^\varepsilon(z,\boldsymbol{\xi})$. Lemmas A.2 and A.4 are the two results that give the proof of the proposition.

Lemma A.2. *There exists K such that for any $\varepsilon \in (0,1)$ and for any $z > 0$*

$$\|Q^\varepsilon(z,\cdot)\|_{L^1} \leq K\varepsilon^{\frac{d}{d+2}} z^{\frac{2}{d+2}} \exp\Big(- \frac{k_0^2}{4} C(\mathbf{0}) z\Big). \tag{A.11}$$

Proof. We define

$$\tilde{Q}^\varepsilon(z,\boldsymbol{\xi}) = Q^\varepsilon(z,\boldsymbol{\xi}) \exp\big(\frac{k_0^2}{2} C(\mathbf{0}) z\big).$$

From (A.4) it satisfies the integral equation

$$\tilde{Q}^\varepsilon(z,\boldsymbol{\xi}) = -\frac{k_0^2}{4(2\pi)^d} \int_0^z \int_{\mathbb{R}^d} \hat{C}(\boldsymbol{k}) e^{i\frac{z'}{k_0\varepsilon}(2\boldsymbol{k}\cdot\boldsymbol{\xi} - |\boldsymbol{k}|^2)} \tilde{Q}^\varepsilon(z', \boldsymbol{\xi} - \boldsymbol{k}) d\boldsymbol{k} dz' - W^\varepsilon(z,\boldsymbol{\xi}),$$

where

$$W^\varepsilon(z,\boldsymbol{\xi}) = (2\pi)^d \frac{k_0^2}{4} \hat{C}(\boldsymbol{\xi}) \int_0^z e^{i\frac{z'}{k_0\varepsilon}|\boldsymbol{\xi}|^2} dz' = (2\pi)^d \frac{k_0^2}{4} \hat{C}(\boldsymbol{\xi}) \frac{e^{i\frac{z}{k_0\varepsilon}|\boldsymbol{\xi}|^2} - 1}{i\frac{1}{k_0\varepsilon}|\boldsymbol{\xi}|^2}.$$

This gives

$$\|\tilde{Q}^\varepsilon(z,\cdot)\|_{L^1} \leq \frac{k_0^2 C(\mathbf{0})}{4} \int_0^z \|\tilde{Q}^\varepsilon(z',\cdot)\|_{L^1} dz' + \|W^\varepsilon(z,\cdot)\|_{L^1}. \tag{A.12}$$

We have for any $\delta > 0$:

$$\begin{aligned} \sup_{z\in[0,Z]} \|W^\varepsilon(z,\cdot)\|_{L^1} &\leq (2\pi)^d \frac{k_0^2}{4} \Big[Z \int_{|\boldsymbol{\xi}|\leq\delta} \hat{C}(\boldsymbol{\xi}) d\boldsymbol{\xi} + 2k_0\varepsilon\delta^{-2} \int_{|\boldsymbol{\xi}|>\delta} \hat{C}(\boldsymbol{\xi}) d\boldsymbol{\xi} \Big] \\ &\leq \frac{k_0^2}{4} \Big[Z(4\pi)^d \delta^d \|\hat{C}\|_\infty + 2k_0\varepsilon\delta^{-2} C(\mathbf{0}) \Big], \end{aligned}$$

where $\|\hat{C}\|_\infty \leq \|C\|_{L^1}$. Choosing $\delta = \varepsilon^{\frac{1}{d+2}} Z^{-\frac{1}{d+2}}$, we find

$$\sup_{z\in[0,Z]} \|W^\varepsilon(z,\cdot)\|_{L^1} \leq K\varepsilon^{\frac{d}{d+2}} Z^{\frac{2}{d+2}}.$$

Substituting into (A.12) and using Gronwall's lemma yields:

$$\|\tilde{Q}^\varepsilon(z,\cdot)\|_{L^1} \leq K\varepsilon^{\frac{d}{d+2}} z^{\frac{2}{d+2}} \exp\Big(\frac{k_0^2 C(\mathbf{0})}{4} z \Big),$$

which reads in terms of Q^ε as (A.11). □

Lemma A.3. *Let us denote*

$$\tilde{G}^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = G^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta} \frac{z}{k_0\varepsilon} \Big)$$

For any $Z > 0$ we have

$$\sup_{z\in[0,Z]} \Big\| \int_0^z \tilde{G}^\varepsilon(z',\cdot,\cdot) dz' \Big\|_{L^1} \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{A.13}$$

Proof. We first note from (A.10) and Lemma A.2 that $\sup_{z\in[0,Z]} \|G_2^\varepsilon(z,\cdot,\cdot)\|_{L^1} \to 0$ as $\varepsilon \to 0$. Therefore, if we denote

$$\tilde{G}_j^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) = G_j^\varepsilon(z,\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z}{k_0\varepsilon}\Big), \qquad j=1,2,$$

then the contribution from G_2^ε in (A.13) goes to zero trivially:

$$\sup_{z\in[0,Z]} \Big\| \int_0^z \tilde{G}_2^\varepsilon(z',\cdot,\cdot)dz' \Big\|_{L^1} \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Second we observe that the function G_1 defined by (A.9) belongs to $\mathcal{C}^0([0,Z],L^1)$ and does not depend on ε. Indeed, this follows since $\boldsymbol{\xi} \to \hat{C}(\boldsymbol{\xi})$ is integrable since it is nonnegative and $C(\mathbf{0}) < \infty$, and ϕ has been shown to be $\mathcal{C}^0([0,Z],L^1)$ in Lemma A.1.
For any $\delta > 0$ we introduce the domain of $\mathbb{R}^d \times \mathbb{R}^d$:

$$\Omega_\delta = \big\{(\boldsymbol{\xi},\boldsymbol{\zeta}) \in \mathbb{R}^d\times\mathbb{R}^d\,,\, |\boldsymbol{\xi}\cdot\boldsymbol{\zeta}| \leq \delta\big\}.$$

Since

$$\Big|\int_0^z G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz'\Big| \leq \int_0^z |G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta})|dz',$$

we obtain

$$\sup_{z\in[0,Z]} \int_{\Omega_\delta} \Big|\int_0^z G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz'\Big| d\boldsymbol{\xi}d\boldsymbol{\zeta} \leq \int_0^Z \int_{\Omega_\delta} |G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta})| d\boldsymbol{\xi}d\boldsymbol{\zeta}dz'. \quad \text{(A.14)}$$

For any integer n we have

$$\Big|\int_0^z G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz' - \sum_{k=0}^{n-1} \int_{\frac{k}{n}z}^{\frac{k+1}{n}z} G_1\big(\frac{kz}{n},\boldsymbol{\xi},\boldsymbol{\zeta}\big) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz'\Big|$$
$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}z}^{\frac{k+1}{n}z} \big|G_1\big(z',\boldsymbol{\xi},\boldsymbol{\zeta}\big) - G_1\big(\frac{kz}{n},\boldsymbol{\xi},\boldsymbol{\zeta}\big)\big|dz'.$$

Since

$$\Big|\int_{\frac{k}{n}z}^{\frac{k+1}{n}z} \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz'\Big| = \Big|\frac{\exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z}{nk_0\varepsilon}\Big)-1}{i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{1}{k_0\varepsilon}}\Big| \leq \frac{2k_0\varepsilon}{\delta} \quad \text{if} \quad (\boldsymbol{\xi},\boldsymbol{\zeta}) \not\in \Omega_\delta,$$

we obtain

$$\sup_{z\in[0,Z]} \int_{\Omega_\delta^c} \Big|\int_0^z G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta}) \exp\Big(i\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\frac{z'}{k_0\varepsilon}\Big)dz'\Big| d\boldsymbol{\xi}d\boldsymbol{\zeta}$$
$$\leq \sup_{z\in[0,Z]} \|G_1(z,\cdot,\cdot)\|_{L^1} \frac{2k_0 n\varepsilon}{\delta} + Z \sup_{z_1,z_2\in[0,Z],\,|z_1-z_2|\leq Z/n} \big\|G_1(z_1,\cdot,\cdot) - G_1(z_2,\cdot,\cdot)\big\|_{L^1}. \quad \text{(A.15)}$$

If we sum (A.14) and (A.15) and take the lim sup in ε then we find:

$$\limsup_{\varepsilon\to 0} \sup_{z\in[0,Z]} \Big\|\int_0^z \tilde{G}_1(z',\cdot,\cdot)dz'\Big\|_{L^1}$$
$$\leq \int_0^Z \int_{\Omega_\delta} |G_1(z',\boldsymbol{\xi},\boldsymbol{\zeta})| d\boldsymbol{\xi}d\boldsymbol{\zeta}dz' + Z \sup_{z_1,z_2\in[0,Z],\,|z_1-z_2|\leq Z/n} \big\|G_1(z_1,\cdot,\cdot) - G_1(z_2,\cdot,\cdot)\big\|_{L^1}.$$

We then take the limit $\delta \to 0$ and $n \to \infty$ in the right-hand side to obtain the desired result. $\square$

Lemma A.4. *For any $Z > 0$*

$$\sup_{z \in [0,Z]} \|R^\varepsilon(z,\cdot,\cdot)\|_{L^1} \stackrel{\varepsilon \to 0}{\longrightarrow} 0. \tag{A.16}$$

Proof. From (A.7) we have for any z

$$\big\|\mathcal{F}_z^\varepsilon R^\varepsilon(z,\cdot,\cdot)\|_{L^1} \leq 2k_0^2 C(\mathbf{0}) \big\|R^\varepsilon(z,\cdot,\cdot)\|_{L^1}.$$

Therefore using the integral version of (A.6) we obtain

$$\big\|R^\varepsilon(z,\cdot,\cdot)\|_{L^1} \leq 2k_0^2 C(\mathbf{0}) \int_0^z \big\|R^\varepsilon(z',\cdot,\cdot)\|_{L^1} dz' + \Big\| \int_0^z \tilde{G}^\varepsilon(z',\cdot,\cdot) dz' \Big\|_{L^1}.$$

Using Lemma A.3 and Gronwall's lemma gives the desired result. $\square$

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