WHITE-NOISE PARAXIAL APPROXIMATION FOR A GENERAL RANDOM HYPERBOLIC SYSTEM

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Abstract. In this paper we consider a general hyperbolic system subjected to random perturbations which models wave propagation in random media. We consider the paraxial white-noise regime, which is the regime in which the propagation distance is much larger than the diameter of the input beam or source, which is itself much larger than the typical wavelength, and in which the correlation length of the medium is of the same order as the diameter of the input beam or source. We introduce a general framework that allows to consider the general random hyperbolic system. Using invariant imbedding and asymptotic analysis we show how to derive the system of Itô-Schrödinger equations driven by Brownian fields that govern the wave propagation. The form of the diffraction operator and the covariance matrix of the Brownian fields are computed from the eigenvalues and eigenvectors of the unperturbed system and from the two-point statistics of the random fluctuations of the medium. Applications are given for acoustic, elastic, and electromagnetic waves.

Key words. Wave propagation, random media, multiscale analysis.

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1. Introduction.

The paraxial (or parabolic) approximation for wave propagation in homogeneous and heterogeneous media is a model used in a vast number of applications, for instance in communication and imaging [42]. It usually has the form of a Schrödinger equation that describes waves propagating along a privileged propagation axis determined by the input beam or source. It is very simple compared to the full wave equation, both from the theoretical and numerical viewpoints, and it enables the analysis of many important phenomena, such as laser beam propagation [41], light propagation through the atmosphere for astronomy [44], tropospheric electromagnetic wave propagation [38], time reversal in random media [5, 7, 9, 39], broadband communications [17], passive imaging [15], virtual source imaging [23, 24], underwater acoustics [42, 21], or migration problems in geophysics [10].

The paraxial approximation for waves in homogeneous media is rather well-understood and used when the radius of the input beam or source is large compared to the wavelength, but small compared to the propagation distance. The white-noise paraxial approximation for waves in random media is used when the medium fluctuations have weak amplitudes and are slow compared to the wavelength, but rapid compared to the propagation distance. In these conditions backscattering is neglected and the medium fluctuations are approximated by a white-noise term. The main motivations for considering the white-noise paraxial wave equation are (i) it appears as a very natural model in many situations where the correlation length of the randomly heterogeneous medium is relatively small, (ii) it allows for the use of Itô's stochastic calculus, which in turn enables the closure of the hierarchy of moment equations and the statistical analysis of important wave propagation problems, such as scintillation [3, 4, 20, 31, 44] or other questions related to applications in imaging and communication (see [1] and references therein).

When the white-noise paraxial approximation can be justified for scalar waves, then the effective equation takes the form of the random Schrödinger equation driven

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by a Brownian field studied in particular in [12]. In [14] paraxial equations were discussed in the context of general rough coefficients. The proof of the convergence of the solution of the wave equation in a random medium to the solution of the white-noise paraxial equation was obtained for stratified weakly fluctuating media in [2] and recently for three-dimensional random media in the context of acoustic waves in [26]. In [8] a coupled system of paraxial equations is derived to describe the mode amplitudes in three-dimensional random waveguides, when the waves are trapped by top and bottom boundaries and the medium is unbounded in the two other directions. The white-noise paraxial approximation can also be justified for vector waves. So far the problem has been studied for linear elastic waves in isotropic solids [27] and for electromagnetic waves in isotropic media [28].

In this paper we consider general wave propagation in open media with statistically isotropic and homogeneous fluctuations. Our goal is to introduce a unified framework that allows to address the general wave propagation problem, so the results scattered in the literature that address specific wave equations can be recovered in a simple manner and can then be extended. It is not straightforward to extend the paraxial approximation to arbitrary waves because different wavenumbers may coexist for plane waves in a given direction in a homogeneous medium and because nonpropagating waves may exist. The main problem is to capture the effective behavior of the coupling terms between wave modes in homogeneous and in heterogeneous media.

In this paper we consider a general randomly perturbed hyperbolic system. Originally motivated by elastic problems in geophysics, we assume that the privileged propagation axis is the vertical direction. This choice explains the terminology "downward-going" waves and "upward-going" waves that we use in the context of the wave decomposition that we set forth. Our goal is to provide a derivation of the paraxial wave equations for the general hyperbolic system in homogeneous media and in random media. This allows us:
- to unify the different results previously known,
- to obtain original results for other physically relevant systems (such as electromagnetic waves in anisotropic media that we address as an application in this paper),
- to identify the general conditions under which the paraxial approximation is valid.

1) *In the case of a homogeneous medium*, we consider the distinguished limit in which the propagation distance is much larger than the initial beam width, which is itself much larger than the typical wavelength. We apply an invariant imbedding theorem and an averaging theorem for rapidly oscillating differential equations in order to prove the convergence of the solution of the wave equation to the solution of a system of Schrödinger equations. The invariant imbedding allows us to transform the boundary value problem (with radiation conditions) into an initial value problem. The averaging theorem allows us to obtain effective paraxial wave equations (see Proposition 5.1). In particular, the coupling terms between downward-going waves and nonpropagating and upward-going waves have rapid phases. By using an averaging theorem for highly oscillatory differential equations, we show that these coupling terms average out to give non-zero effective terms (in the form of Lie brackets). It turns out that the computation of the Lie brackets for classical physical systems gives simple forms for the paraxial approximations of the wave modes.

2) *In the case of a random medium*, we assume additionally that the correlation length of the random medium is of the same order as the initial beam width, and that the amplitude of the random fluctuations is small. By applying diffusion-

approximation theorems we prove the convergence of the solution of the random hyperbolic system to the solution of a system of Itô-Schrödinger equations (see Proposition 5.2). These Schrödinger equations are driven by a set of correlated Brownian fields, whose covariance function depends on the two-point statistics of the fluctuations of the parameters of the medium. This result shows that it is possible to justify both the paraxial approximation and the white-noise approximation in the distinguished limit considered in this paper. The limit system permits easy numerical simulations on the one hand, and exact computations of moments using Itô's formula on the other hand.

The paper is organized as follows. In Section 2 we describe the white-noise paraxial wave model from a physical perspective. In Section 3 we describe the general hyperbolic system that we want to address. In Section 4 we introduce the fundamental wave mode decomposition and we formulate the propagation problem in terms of the wave mode amplitudes. In Section 5 we present the main results, i.e. the random paraxial wave equations. The proofs of the results are given in Section 6 when the medium is homogeneous and in Section 7 when the medium is randomly heterogeneous. In Section 8 we apply the general results to classical physical systems.

2. The Itô-Schrödinger Equations. In this section we introduce the Itô-Schrödinger equation from a physical perspective and discuss how this relates to the framework set forth in this paper. This section can be seen as a review. In the subsequent sections we will provide a mathematical derivation of this model. We will moreover generalize it to general symmetric hyperbolic systems.

2.1. The Paraxial Approximation. We consider in this section scalar waves and assume the governing equation:

$$(\partial_{x_d}^2 + \Delta_{\boldsymbol{x}_\perp})u - \frac{n^2(\boldsymbol{x}_\perp, x_d)}{c_0^2}\partial_t^2 u = 0, \tag{2.1}$$

for $\boldsymbol{x} = (\boldsymbol{x}_\perp, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, x_d corresponding to the propagation axis and $\boldsymbol{x}_\perp$ being the lateral spatial coordinates. Here $n(\boldsymbol{x}_\perp, x_d)$ is the local index of refraction and we assume radiation conditions at infinity. It is convenient to Fourier transform in time:

$$\check{u}(\omega, \boldsymbol{x}_\perp, x_d) = \int_{-\infty}^{\infty} u(t, \boldsymbol{x}_\perp, x_d) \exp\big(i\omega t\big) \mathrm{d}t. \tag{2.2}$$

We then obtain the Helmholtz equation:

$$(\partial_{x_d}^2 + \Delta_{\boldsymbol{x}_\perp})\check{u} + \frac{\omega^2}{c_0^2} n^2(\boldsymbol{x}_\perp, x_d)\check{u} = 0, \tag{2.3}$$

with $k = \omega/c_0$ being the free space wavenumber. We assume a source located at the "surface" $x_d = 0$ generating a "down-propagating" wave, propagating into the negative x_d-direction. A particular solution of (2.3) in the case of a homogeneous medium $n \equiv 1$ is a down-propagating plane wave:

$$\check{u} = \exp\Big(- i\frac{\omega x_d}{c_0}\Big).$$

We make the ansatz of a slowly-varying envelope around a plane wave going into the negative x_d-direction:

$$\check{u}(\omega, \boldsymbol{x}_\perp, x_d) = \exp\Big(- i\frac{\omega x_d}{c_0}\Big)\check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d). \tag{2.4}$$

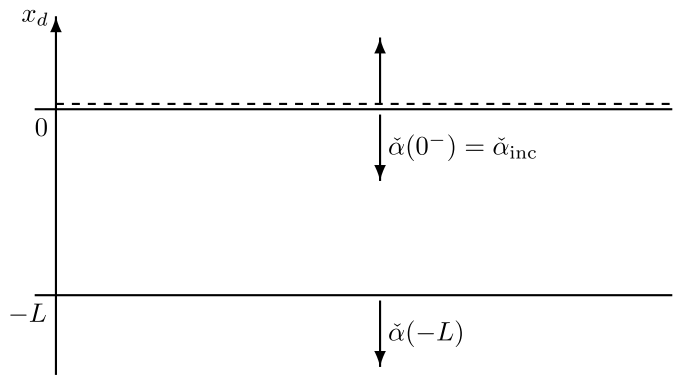

FIG. 2.1. *A source is located at $x_d = 0$ generating a probing down-propagating wave, with the transmitted wave evaluated at $x_d = -L$.*

The amplitude $\breve{\alpha}$ then solves

$$\partial_{x_d}^2 \breve{\alpha} - \frac{2i\omega}{c_0}\partial_{x_d}\breve{\alpha} + \Delta_{\boldsymbol{x}_\perp}\breve{\alpha} + \frac{\omega^2}{c_0^2}\big(n^2(\boldsymbol{x}_\perp, x_d) - 1\big)\breve{\alpha} = 0. \tag{2.5}$$

In many physical configurations there is a wave (beam) propagating in a specific direction with negligible backscattering. As mentioned, we choose a coordinate system so that the probing direction is the negative x_d-direction. We then make the forward or one-way approximation, suppressing the $\partial_{x_d}^2\breve{\alpha}$ term in (2.5). This corresponds to assuming that $\breve{\alpha}$ is relatively slowly varying in x_d and suppressing backscattering so that $\breve{\alpha}$ solves the initial value problem

$$\frac{2i\omega}{c_0}\partial_{x_d}\breve{\alpha} = \Delta_{\boldsymbol{x}_\perp}\breve{\alpha} + \frac{\omega^2}{c_0^2}\big(n^2(\boldsymbol{x}_\perp, x_d) - 1\big)\breve{\alpha}, \tag{2.6}$$

with $\breve{\alpha}(\omega, \boldsymbol{x}_\perp, x_d = 0) = \breve{\alpha}_{\rm inc}(\omega, \boldsymbol{x}_\perp)$ and $\breve{\alpha}_{\rm inc}$ is determined by the probing wave, the source located at the "surface" $x_d = 0$, whose typical transverse width r_0 must be larger than the typical wavelength λ_0 (the typical wavelength is related to the typical frequency ω_0 by $\lambda_0 = 2\pi c_0/\omega_0$) in order to satisfy the slowly-varying envelope assumption.

In the case of a homogeneous medium $n \equiv 1$, we then find that the transmitted wave to $x_d = -L$ is given by

$$\breve{\alpha}(\omega, \boldsymbol{x}_\perp, -L)\,|_{n\equiv 1} = \breve{\alpha}_{\rm homo}(\omega, \boldsymbol{x}_\perp, -L), \tag{2.7}$$

$$\breve{\alpha}_{\rm homo}(\omega, \boldsymbol{x}_\perp, -L) = \Big(\frac{\omega}{2i\pi L c_0}\Big)^{\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} \breve{\alpha}_{\rm inc}(\omega, \boldsymbol{y}_\perp) \exp\Big(i\frac{\omega|\boldsymbol{y}_\perp - \boldsymbol{x}_\perp|^2}{2Lc_0}\Big) {\rm d}\boldsymbol{y}_\perp. \tag{2.8}$$

The configuration is illustrated in Figure 2.1. To summarize, the source is located at the surface in the plane $x_d = 0$ and we are interested in the transmitted wave evaluated at $x_d = -L$. We consider a scaling regime in which diffractive effects are of order one. Diffractive effects can be measured by the Rayleigh length L_R, defined as the distance from wave beam waist where the beam area is doubled by diffraction. In the homogeneous medium it follows from (2.8) that the Rayleigh length for a beam with initial beam width r_0 and carrier wavelength λ_0 is of the order of r_0^2/λ_0. In particular, in the case with Gaussian initial data

$$\check{\alpha}_{\rm inc}(\omega, \boldsymbol{x}_\perp) = \check{f}(\omega) \exp\Big(-\frac{|\boldsymbol{x}_\perp|^2}{2r_0^2}\Big), \tag{2.9}$$

we have

$$\check{\alpha}(\omega, \boldsymbol{x}_\perp, -L) = \check{f}(\omega)\Big(\frac{r_0}{r_L(\omega)}\Big)^{d-1} \exp\Big(-\frac{|\boldsymbol{x}_\perp|^2}{2r_L^2(\omega)}\Big), \qquad r_L(\omega) = r_0 \left(1 + i\frac{Lc_0}{r_0^2\omega}\right)^{1/2},$$

which shows that the Rayleigh length L_R (for $d = 3$ and with $\check{f}$ centered at ω_0) is

$$L_R = 2\pi \frac{r_0^2}{\lambda_0}.$$

Therefore we consider in this paper a high-frequency regime in which

$$\lambda_0 \ll r_0 \ll L, \qquad \lambda_0 L \sim r_0^2, \tag{2.10}$$

so that the slowly-varying envelope approximation will be valid and the diffractive effects will be of order one.

2.2. The White-Noise Approximation. We consider now the situation when the medium is not homogeneous, but rather fluctuating on a fine scale and modeled in terms of a random field. Such modeling is for instance motivated by wave propagation scenarios in the earth's crust, the atmosphere or the ocean. Then the medium is typically heterogeneous and its pointwise parameter values cannot be identified precisely. However, one can often characterize the statistics of the heterogeneities and then examine how the statistics of the wave field is affected by the statistics of the medium fluctuations.

Let the local index of refraction in (2.1) be modeled by

$$n^2(\boldsymbol{x}_\perp, x_d) = 1 + m(\boldsymbol{x}_\perp, x_d),$$

for m being the centered random medium fluctuations. We assume that m is a stationary zero-mean random process that is mixing in x_d and with integrable correlations. We denote by σ_c the typical amplitude of the fluctuations of the index of refraction (the standard deviation of m), and by l_c the characteristic length scale of the medium fluctuations (the correlation radius of m). We consider the situation in which the correlation length is of the same order as the beam width to capture the most delicate interaction in between the lateral fluctuations of the medium and the beam:

$$l_c \sim r_0 \ll L. \tag{2.11}$$

In this case (2.6) reads

$$\partial_{x_d} \check{\alpha} = -\frac{ic_0}{2\omega} \Delta_{\boldsymbol{x}_\perp} \check{\alpha} - \frac{i\omega}{2c_0} m(\boldsymbol{x}_\perp, x_d)\check{\alpha}, \tag{2.12}$$

the Schrödinger equation with a random potential, also referred to as the paraxial or forward-scattering wave equation [37]. In the case with very small or weak potentials the method of smooth potentials [35] for instance or a Rytov type method [43] can be used. However, the use of such perturbation methods do not describe a range of physical phenomena in situations with relatively strong medium interaction or "saturated fluctuations". This is the regime we consider in this paper corresponding to the scaling regime outlined above with a medium correlation length that is small relative to the propagation distance and large relative to the wavelength. Specifically, we shall assume that the potential has the white noise scaling when viewed as a stochastic process in the propagation coordinate. For m a rapidly decorrelating centered stochastic process as in (2.11) we indeed have for $x_d \in (-L, 0)$ that

$$B(\boldsymbol{x}_\perp, x_d) = \int_0^{x_d} m(\boldsymbol{x}_\perp, z) \mathrm{d}z \tag{2.13}$$

behaves weakly (or in distribution) as a non-standard Brownian field, that is a Gaussian process with mean zero and covariance

$$\mathbb{E}[B(\boldsymbol{x}_\perp, x_d) B(\boldsymbol{x}'_\perp, x'_d)] = \min\{|x_d|, |x'_d|\} \gamma(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp), \tag{2.14}$$

where the covariance $\gamma(\boldsymbol{x}_\perp)$ is given by

$$\gamma(\boldsymbol{x}_\perp) = \int_{-\infty}^{\infty} \mathbb{E}\big[m(\mathbf{0}, 0) m(\boldsymbol{x}_\perp, x_d)\big] \mathrm{d}x_d . \tag{2.15}$$

The main challenge is now to show that a "white-noise approximation", corresponding to replacing the contribution from the potential term m in (2.12) by an integral with respect to the Brownian field B is valid when starting from the Helmholtz equation. That is, we need to show that the simultaneous limits of the paraxial and white-noise approximations are valid. This was first accomplished in the case of stratified weakly fluctuating media in [2] and for three-dimensional random media in the case of scalar waves in [26], while here we prove this limit result in the case of rather general hyperbolic systems. The mathematical approach we use is an invariant imbedding technique that allows us to convert the boundary value problem associated with the Helmholtz equation to a nonlinear initial value problem which subsequently can be handled via averaging and diffusion approximation theorems to show the convergence of the wave field *in distribution* to the solution of an Itô-Schrödinger equation. Corresponding to the situation in (2.13) the convergence can only be obtained in a weak or distributional and not pathwise sense. We have that $\check{\alpha}$ converges to the solution of the Itô-Schrödinger equation as analyzed in [12] and extensively used to describe physical wave propagation [41], it reads:

$$\mathrm{d}\check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) = -\frac{ic_0}{2\omega} \Delta_{\boldsymbol{x}_\perp} \check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) \mathrm{d}x_d - \frac{i\omega}{2c_0} \check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) \circ \mathrm{d}B(\boldsymbol{x}_\perp, x_d), \tag{2.16}$$

where x_d runs backward from $x_d = 0$ to $x_d = -L$. The symbol $\circ$ stands for the Stratonovich stochastic integral in x_d and $B(\boldsymbol{x}_\perp, x_d)$ is the non-standard Brownian field or Gaussian process with the covariance (2.14). In the Itô form we have

$$\begin{aligned}\mathrm{d}\check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) = &-\frac{ic_0}{2\omega} \Delta_{\boldsymbol{x}_\perp} \check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) \mathrm{d}x_d + \frac{\omega^2 \gamma(\mathbf{0})}{8c_0^2} \check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) \mathrm{d}x_d \\ &- \frac{i\omega}{2c_0} \check{\alpha}(\omega, \boldsymbol{x}_\perp, x_d) \mathrm{d}B(\boldsymbol{x}_\perp, x_d) .\end{aligned} \tag{2.17}$$

The Itô correction has here a plus sign because the equation runs backward in x_d. In this paper we want to address the regime in which the random fluctuations of the medium are responsible for an effect of order one. From (2.17) this means that we should have $\gamma(\mathbf{0})L\omega_0^2/c_0^2$ of order one, or equivalently (using (2.15)),

$$\sigma_c^2 l_c L \sim \lambda_0^2. \tag{2.18}$$

Since $l_c \sim r_0$ and $L \sim r_0^2/\lambda_0$ (by (2.10)) this means that the typical amplitude σ_c of the medium fluctuations should be small, of the order of $(\lambda_0/r_0)^{3/2}$, in order to address the regime in which diffractive and random effects are of the same order. We now proceed by commenting on some consequences of the Itô-Schrödinger description.

2.3. Statistics of the Wave in the White-Noise Paraxial Approximation. First, we describe the mean or coherent field that we denote by $M_{1,0} = \mathbb{E}[\check{\alpha}]$. It follows from (2.17) that this solves

$$\frac{\partial M_{1,0}}{\partial x_d}(\omega, \boldsymbol{x}_\perp, x_d) = -\frac{ic_0}{2\omega}\Delta_{\boldsymbol{x}_\perp} M_{1,0}(\omega, \boldsymbol{x}_\perp, x_d) + \frac{\omega^2\gamma(\mathbf{0})}{8c_0^2} M_{1,0}(\omega, \boldsymbol{x}_\perp, x_d), \tag{2.19}$$

so that

$$M_{1,0}(\omega, \boldsymbol{x}_\perp, -L) = \check{\alpha}_{\text{homo}}(\omega, \boldsymbol{x}_\perp, -L) \exp\Big(-\frac{\omega^2\gamma(\mathbf{0})L}{8c_0^2}\Big),$$

where $\check{\alpha}_{\text{homo}}$ is the wave solution in the homogeneous medium. The exponential damping reflects the fact that the wave field becomes partly incoherent as it propagates into the medium. The characteristic depth of penetration L_p of the coherent field is

$$L_p = \frac{2}{\pi^2}\frac{\lambda_0^2}{\gamma(\mathbf{0})},$$

which is called scattering mean free path in the physical literature. Moreover we have the following expression for the mean transmitted pulse

$$\mathbb{E}\big[\alpha(t, \boldsymbol{x}_\perp, -L)\big] = \left[\mathcal{N}_{\sigma_L}(\cdot) * \alpha_{\text{inc}}(\cdot, \boldsymbol{x}_\perp, -L)\right](t) \tag{2.20}$$

where $*$ stands for a convolution product in time and

$$\mathcal{N}_{\sigma_L}(t) = \frac{1}{\sqrt{2\pi\sigma_L^2}} \exp\Big(-\frac{t^2}{2\sigma_L^2}\Big), \qquad \sigma_L^2 = \frac{\gamma(\mathbf{0})L}{4c_0^2}.$$

The result (2.20) follows directly from the fact that the damping term in (2.17) is independent of the offset $\boldsymbol{x}_\perp$ in our beam geometry. Therefore, for the mean, the random fluctuations in the medium have the effect of smearing the signal in time through convolution with a Gaussian kernel of width σ_L. It results in the loss of coherence of the wave field even though the total wave energy is conserved, the amount of smearing scales diffusively as $\sqrt{L}$. Note however, that the pulse in a particular realization of the medium may be very different from the mean pulse that is obtained by averaging over pulses in different media realizations.

In order to obtain more insight about the statistical structure of the wave and how the pulse in a particular realization typically differs from the mean it is useful to compute its higher-order moments, in particular the second-order moments that allow us to characterize a decoherence frequency and length and also the spreading of the pulse [18, 26]. The fourth-order moment can be used to describe intensity fluctuations and also the scintillation index, moreover, it is used for characterization of statistical stability of imaging functionals and in time-reversal problems [5, 39]. For the general moment we define the $(n,m)-$order moment of $\check{\alpha}$ in the context of the Itô-Schrödinger equation (2.17) by

$$M_{n,m}\big(\omega,\boldsymbol{x}_\perp^{(1)},\dots,\boldsymbol{x}_\perp^{(n)},\boldsymbol{y}_\perp^{(1)},\dots,\boldsymbol{y}_\perp^{(m)},x_d\big)=\mathbb{E}\Big[\prod_{j=1}^n\check{\alpha}(\omega,\boldsymbol{x}_\perp^{(j)},x_d)\prod_{k=1}^m\overline{\check{\alpha}(\omega,\boldsymbol{y}_\perp^{(k)},x_d)}\Big],\tag{2.21}$$

and it then follows from (2.17) and Itô's formula that $M_{n,m}$ solves the closed-form equation

$$\frac{\partial M_{n,m}}{\partial x_d}=\frac{ic_0}{2\omega}\Big(-\sum_{j=1}^n\Delta_{\boldsymbol{x}_\perp^{(j)}}+\sum_{k=1}^m\Delta_{\boldsymbol{y}_\perp^{(k)}}\Big)M_{n,m}-\frac{\omega^2}{8c_0^2}V_{n,m}M_{n,m},\tag{2.22}$$

running backward from $x_d=0$ to $x_d=-L$, where we have introduced the potential

$$V_{n,m}=2\sum_{1\le j\le n,1\le k\le m}\gamma(\boldsymbol{x}_\perp^{(j)}-\boldsymbol{y}_\perp^{(k)})-\sum_{1\le j,k\le n}\gamma(\boldsymbol{x}_\perp^{(j)}-\boldsymbol{x}_\perp^{(k)})-\sum_{1\le j,k\le m}\gamma(\boldsymbol{y}_\perp^{(j)}-\boldsymbol{y}_\perp^{(k)}).$$

This is a Schrödinger equation with a deterministic potential in $(n+m)(d-1)$ dimensions and it is not exactly solvable in general. However, the equations for the first- and second-order moments can readily be solved [34, Chap. 20]. We discussed the first-order moment above. The second-order moment solves

$$\frac{\partial M_{1,1}}{\partial x_d}=\frac{ic_0}{2\omega}\big(-\Delta_{\boldsymbol{x}_\perp}+\Delta_{\boldsymbol{y}_\perp}\big)M_{1,1}+\frac{\omega^2}{4c_0^2}\big(\gamma(\boldsymbol{0})-\gamma(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)\big)M_{1,1}.$$

We remark that the second-order moments can also be expressed in terms of the Wigner transform [26]

$$W(\omega,\boldsymbol{x}_\perp,\boldsymbol{\kappa}_\perp,x_d)=\int_{\mathbb{R}^{d-1}}\exp(-i\boldsymbol{\kappa}_\perp\cdot\boldsymbol{y}_\perp)\mathbb{E}\left[\check{\alpha}\big(\omega,\boldsymbol{x}_\perp+\frac{\boldsymbol{y}_\perp}{2},x_d\big)\overline{\check{\alpha}}\big(\omega,\boldsymbol{x}_\perp-\frac{\boldsymbol{y}_\perp}{2},x_d\big)\right]\mathrm{d}\boldsymbol{y}_\perp.$$

This representation is a convenient to deduce the explicit expression for the second-order moment of the field with the initial condition (2.9):

$$\begin{aligned}M_{1,1}\Big(\omega,\boldsymbol{x}_\perp+\frac{\boldsymbol{y}_\perp}{2},\boldsymbol{x}_\perp-\frac{\boldsymbol{y}_\perp}{2},-L\Big)&=\big|\check{f}(\omega)\big|^2\Big(\frac{r_0^2}{4\pi}\Big)^{\frac{d-1}{2}}\int_{\mathbb{R}^{d-1}}\mathrm{d}\boldsymbol{\kappa}_\perp\exp\big(i\boldsymbol{\kappa}_\perp\cdot\boldsymbol{x}_\perp\big)\\&\times\exp\Big(-\frac{1}{4r_0^2}\big|\boldsymbol{y}_\perp-\frac{\boldsymbol{\kappa}_\perp Lc_0}{\omega}\big|^2-\frac{r_0^2|\boldsymbol{\kappa}_\perp|^2}{4}+\frac{\omega^2}{4c_0^2}\int_{-L}^0\gamma\big(\boldsymbol{y}_\perp+\boldsymbol{\kappa}_\perp\frac{zc_0}{\omega}\big)-\gamma(\boldsymbol{0})\mathrm{d}z\Big).\end{aligned}$$

The second-order moment exhibits a very interesting multiscale behavior and we refer to [30] for a detailed discussion. This second-order moment along with the fourth-order moment play an important role in applications to time reversal and imaging.

The Itô-Schrödinger equation presented in this section in (2.17) corresponds to the one derived in the application to acoustic waves presented below in Section 8.1. There, the Itô-Schrödinger equation is presented for the transmission operator rather than for the field itself. The general Itô-Schrödinger result for general symmetric hyperbolic systems is presented in Section 5. A main point is that via the Itô-Schrödinger equation we have replaced a boundary value problem associated with the Helmholtz equation with radiation conditions by an initial value problem, which is important both from computational and analytic viewpoints. It is a fundamental modeling equation in particular for wave propagation in the atmosphere and the heterogeneous earth. It is therefore important to obtain a mathematical derivation which makes explicit the conditions under which it is valid.

3. Hyperbolic System in a Random Medium. We consider a general hyperbolic system for waves propagating in a d-dimensional medium with heterogeneous and random fluctuations:[1]

$$\mathbf{C}^{\varepsilon}(\boldsymbol{x})\frac{\partial \boldsymbol{u}^{\varepsilon}}{\partial t} + \sum_{k=1}^{d} \mathbf{D}_k \frac{\partial \boldsymbol{u}^{\varepsilon}}{\partial x_k} = \boldsymbol{f}^{\varepsilon}(t,\boldsymbol{x}), \tag{3.1}$$

where the vector wave field $\boldsymbol{u}^{\varepsilon}(t,\boldsymbol{x})$ takes values in $\mathbb{R}^n$ and $\boldsymbol{f}^{\varepsilon}(t,\boldsymbol{x})$ is the source term. This is a general model that can describe acoustic, elastic and electromagnetic wave propagation [11], as we discuss in Section 8. This model was used in a different scaling regime than the one we introduce below in order to derive transport equations for the energy density of waves in a random medium in [40]. In our paper we aim at deriving Itô-Schrödinger equations from the model (3.1).

We consider the situation in which a random section occupies the region $x_d \in (-L, 0)$ and is sandwiched in between two homogeneous half-spaces. The medium parameters are:

$$\mathbf{C}^{\varepsilon}(\boldsymbol{x}) = \begin{cases} \mathbf{C}_0 & \text{if } x_d \le -L \text{ or } x_d \ge 0, \\ \mathbf{C}_0 + \varepsilon^3 \mathbf{C}_1\left(\frac{\boldsymbol{x}}{\varepsilon^2}\right) & \text{if } x_d \in (-L,0), \end{cases} \tag{3.2}$$

with ε a small parameter.

The constant real matrix $\mathbf{C}_0$ is symmetric and positive definite. The real matrices $\mathbf{D}_k$ are symmetric and do not depend on $(t,\boldsymbol{x})$. They model the unperturbed hyperbolic system.

The random matrix-valued process $\mathbf{C}_1(\boldsymbol{x})$ describes the medium fluctuations. We assume that it is stationary and zero-mean and that it satisfies strong mixing conditions in x_d. The parameter ε^2 characterizes the ratio of the correlation length of the random medium fluctuations to the thickness of the random section.

We consider a source term with a transverse spatial extent of the order of ε^2 so that the medium fluctuations and the beam will interact strongly. Moreover, in order to deal with a situation in which diffractive effects are of order one we consider a high-frequency regime in which the typical wavelength is of order ε^4 (in order to satisfy (2.10), which means that the Rayleigh length is of the order of the propagation distance). Therefore we write the source term in the form

$$\boldsymbol{f}^{\varepsilon}(t,\boldsymbol{x}) = \boldsymbol{f}\Big(\frac{t}{\varepsilon^4}, \frac{\boldsymbol{x}_{\perp}}{\varepsilon^2}\Big)\delta(x_d), \tag{3.3}$$

where $\boldsymbol{x} = (\boldsymbol{x}_{\perp}, x_d)$, $\boldsymbol{x}_{\perp} = (x_1, \dots, x_{d-1})$. This is a model for a source acting on a thin region in the $x_d = 0$ plane. It will give jump conditions on the wave field across the interface. The source generates a wave beam that propagates mainly along the x_d-axis as we explain in detail in Section 5. We assume that the Fourier transform $\check{\boldsymbol{f}}(\omega, \boldsymbol{x}_{\perp})$ defined by

$$\check{\boldsymbol{f}}(\omega, \boldsymbol{x}_{\perp}) = \int_{-\infty}^{\infty} \boldsymbol{f}(t, \boldsymbol{x}_{\perp}) \exp\big(i\omega t\big) \mathrm{d}t$$

is compactly supported in ω in a domain away from zero.

[1] Throughout the paper, symbols of scalar quantities are printed in italic type, symbols of vectors are printed in bold italic type, and symbols of matrices are printed in bold type.

Finally, as seen in (3.2), we also assume that the amplitude of the random fluctuations is of order ε^3. This corresponds to the scaling (2.18) and this is the interesting regime in which the medium fluctuations give rise to effective terms of order one when ε goes to zero, as we will show in this paper.

4. Wave Propagation in Random Media.

4.1. Wave Mode Decomposition. We rescale the transverse spatial variables and the time variable so as to observe the wave at the scale of the source and we take a scaled Fourier transform in time:

$$\check{\boldsymbol{u}}^\varepsilon(\omega, \boldsymbol{x}) = \int_{-\infty}^{\infty} \boldsymbol{u}^\varepsilon(\varepsilon^4 t, \varepsilon^2 \boldsymbol{x}_\perp, x_d) \exp\big(i\omega t\big) \mathrm{d}t.$$

The vector field $\check{\boldsymbol{u}}^\varepsilon(\omega, \boldsymbol{x})$ satisfies the time-harmonic form of (3.1):

$$-i\frac{\omega}{\varepsilon^4}\mathbf{C}^\varepsilon(\varepsilon^2\boldsymbol{x}_\perp, x_d)\check{\boldsymbol{u}}^\varepsilon + \frac{1}{\varepsilon^2}\sum_{k=1}^{d-1}\mathbf{D}_k\frac{\partial \check{\boldsymbol{u}}^\varepsilon}{\partial x_k} + \mathbf{D}_d\frac{\partial \check{\boldsymbol{u}}^\varepsilon}{\partial x_d} = \check{\boldsymbol{f}}(\omega, \boldsymbol{x}_\perp)\delta(x_d). \tag{4.1}$$

When the medium is homogeneous $\mathbf{C}_1 = 0$, then there exist plane wave solutions that depend only on x_d. They satisfy

$$\mathbf{C}_0^{-1}\mathbf{D}_d\frac{\partial \check{\boldsymbol{u}}^\varepsilon}{\partial x_d} = i\frac{\omega}{\varepsilon^4}\check{\boldsymbol{u}}^\varepsilon \tag{4.2}$$

away from $x_d = 0$. It is convenient to introduce the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_0 = \boldsymbol{u}^T\mathbf{C}_0\boldsymbol{v}, \tag{4.3}$$

where T stands for the transpose. This inner product is the natural one for the system (3.1). The matrix $\mathbf{C}_0^{-1}\mathbf{D}_d$ is self-adjoint with respect to the inner product (4.3). We denote the real eigenvalues of $\mathbf{C}_0^{-1}\mathbf{D}_d$ by $(c_j)_{j=1,\dots,n}$ and sort them in increasing order. There may be multiple eigenvalues. The corresponding real eigenvectors, denoted by $(\boldsymbol{u}_j)_{j=1,\dots,n}$, are chosen to be orthonormal with respect to the inner product (4.3). They satisfy:

$$\mathbf{C}_0^{-1}\mathbf{D}_d\boldsymbol{u}_j = c_j\boldsymbol{u}_j, \qquad j = 1,\dots,n. \tag{4.4}$$

Note that

$$\boldsymbol{u}_j^T\mathbf{D}_d\boldsymbol{u}_{j'} = \boldsymbol{u}_j^T\mathbf{C}_0\mathbf{C}_0^{-1}\mathbf{D}_d\boldsymbol{u}_{j'} = c_{j'}\boldsymbol{u}_j^T\mathbf{C}_0\boldsymbol{u}_{j'} = c_{j'}\delta_{jj'}, \qquad j, j' = 1,\dots,n. \tag{4.5}$$

We partition the index set $\{1,\dots,n\}$ of the pairs of eigenvalues/eigenvectors into three subsets:

$$\Lambda_d = \{j, c_j < 0\}, \qquad \Lambda_0 = \{j, c_j = 0\}, \qquad \Lambda_u = \{j, c_j > 0\}.$$

- The modes $(\boldsymbol{u}_j)_{j\in\Lambda_d}$ represent the downward-going wave modes, i.e. modes propagating towards the negative x_d, of the form:

$$\check{\boldsymbol{u}}^\varepsilon(\omega, \boldsymbol{x}) = \check{\alpha}_j(\omega)\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big)\boldsymbol{u}_j, \tag{4.6}$$

where $\check{\alpha}_j(\omega) \in \mathbb{C}$.
- The modes $(\boldsymbol{u}_j)_{j\in\Lambda_u}$ represent the upward-going wave modes of the form (4.6).
- The modes $(\boldsymbol{u}_{j_0})_{j_0\in\Lambda_0}$ represent the nonpropagating wave modes.

We assume that $\Lambda_d \neq \emptyset$ and denote $n_d = \mathrm{card}(\Lambda_d)$. We now introduce Hypothesis H1 that is fundamental to prove the forward-scattering approximation (which means that the wave goes downward and no energy is reflected in the asymptotic regime $\varepsilon \to 0$) and to derive a system of paraxial equations for the wave mode amplitudes:

Hypothesis H1: *For all $j, l \in \{1, \ldots, n\}^2$ such that $c_j = c_l$, we have $\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_l = 0$ for all $k \in \{1, \ldots, d-1\}$.*

Although it is not necessary for the forthcoming analysis, we can explain heuristically why Hypothesis H1 is a necessary condition for the validity of the paraxial approximation. Note first that a general plane wave in our scaling takes the form

$$\exp\Big(i\boldsymbol{\kappa} \cdot \boldsymbol{x} - i\omega \frac{t}{\varepsilon^4}\Big)\boldsymbol{u}\,,$$

with $\boldsymbol{\kappa}$ the wave vector that gives the spatial direction of the plane wave and $\boldsymbol{u}$ the vector profile. The paraxial approximation aims at describing beam waves that propagate along the x_d-axis. In (4.6) we have identified n_d plane wave modes propagating downward along the x_d-axis. The jth plane wave mode has wavevector $(0, \ldots, 0, \frac{\omega}{c_j \varepsilon^4})$, vector profile $\boldsymbol{u}_j$, and speed $|c_j|$. A paraxial perturbation (in the scales of the paraxial regime) of the jth mode has a wavevector of the form $\boldsymbol{\kappa}^\varepsilon = (\frac{\omega s_1}{\varepsilon^2}, \ldots, \frac{\omega s_{d-1}}{\varepsilon^2}, \frac{\omega}{c_j \varepsilon^4} + \frac{\omega s_d}{\varepsilon^2})$. It can exist provided there is a vector profile $\boldsymbol{u}^\varepsilon$ such that $\boldsymbol{u}^\varepsilon \exp(i\boldsymbol{\kappa}^\varepsilon \cdot \boldsymbol{x})$ is solution of the homogeneous time-harmonic wave equation. This vector profile can be written in the $(\boldsymbol{u}_l)_{l=1,\ldots,n}$-basis as $\boldsymbol{u}^\varepsilon = \boldsymbol{u}_j + \sum_{l=1}^n \varepsilon^2 d_l \boldsymbol{u}_l$. This perturbation should satisfy the dispersion relation $\frac{\omega}{\varepsilon^4}\mathbf{C}_0 \boldsymbol{u}^\varepsilon - \sum_{k=1}^d \mathbf{D}_k \boldsymbol{u}^\varepsilon \kappa_k^\varepsilon = 0$. Projecting this relation onto the vectors $\boldsymbol{u}_l$ and collecting the leading-order terms, we get that the coefficients d_l should satisfy $d_l(-1 + c_l/c_j) + \sum_{k=1}^{d-1} s_k \boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_l = 0$ for all $l \neq j$. A necessary condition for these equations to have a solution if that Hypothesis H1 is satisfied.

In fact, the situation $\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_l \neq 0$, for some k, represents a diffractive coupling between the modes j and l and in Hypothesis H1 excludes certain forms of such couplings. We remark that cross diffractive coupling between modes of different wave speeds is allowed since the separation in wave speeds then serves to control the coupling effect, see Eq. (4.10) below. We will show that a paraxial approximation can be defined with Hypothesis H1 and an additional technical hypothesis H2. We will also show that an additional hypothesis H3 introduced in Proposition 5.1 can serve to simplify the structure by making it diagonal in the case of a homogeneous medium. In the case of a random medium we shall introduce an additional hypothesis H4 in Proposition 5.2 which serves to retain the diagonal structure also in the random case, but Hypotheses H3-H4 are not necessary to write a Schrödinger-type system (we give such an example in Subsection 8.4).

4.2. Evolutions of the Wave Mode Amplitudes. When the medium is heterogeneous we expand the solution of the system (4.1) in the basis $(\boldsymbol{u}_j)_{j=1,\ldots,n}$:

$$\check{\boldsymbol{u}}^\varepsilon(\omega, \boldsymbol{x}) = \sum_{j \in \Lambda_d \cup \Lambda_u} \check{\alpha}_j^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d) \exp\Big(i\frac{\omega x_d}{c_j \varepsilon^4}\Big)\boldsymbol{u}_j + \sum_{j_0 \in \Lambda_0} \check{\alpha}_{j_0}^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d)\boldsymbol{u}_{j_0}. \qquad (4.7)$$

The evolution equations for $x_d \in (-L, 0)$ are obtained by substituting the form (4.7) into (4.1) and multiplying by $\mathbf{C}_0^{-1}$:

$$
\begin{aligned}
&-i\frac{\omega}{\varepsilon^4}\sum_{j_0\in\Lambda_0}\check{\alpha}^\varepsilon_{j_0}\boldsymbol{u}_{j_0} + \frac{1}{\varepsilon^2}\sum_{j\in\Lambda_d\cup\Lambda_u}\sum_{k=1}^{d-1}\mathbf{C}_0^{-1}\mathbf{D}_k\boldsymbol{u}_j\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_k}\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big)\\
&+\frac{1}{\varepsilon^2}\sum_{j_0\in\Lambda_0}\sum_{k=1}^{d-1}\mathbf{C}_0^{-1}\mathbf{D}_k\boldsymbol{u}_{j_0}\frac{\partial\check{\alpha}^\varepsilon_{j_0}}{\partial x_k} + \sum_{j\in\Lambda_d\cup\Lambda_u}c_j\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_d}\boldsymbol{u}_j\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big)\\
&=\sum_{j\in\Lambda_d\cup\Lambda_u}\frac{i\omega}{\varepsilon}\mathbf{C}_0^{-1}\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_j\check{\alpha}^\varepsilon_j\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big)\\
&\quad+\sum_{j_0\in\Lambda_0}\frac{i\omega}{\varepsilon}\mathbf{C}_0^{-1}\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_{j_0}\check{\alpha}^\varepsilon_{j_0}.
\end{aligned}
\tag{4.8}
$$

By projecting (4.8) onto $\mathbf{C}_0\boldsymbol{u}_{j_0}$, $j_0 \in \Lambda_0$, we find by using the orthonormality property of the eigenvectors that

$$
\begin{aligned}
&-i\frac{\omega}{\varepsilon^4}\check{\alpha}^\varepsilon_{j_0} + \frac{1}{\varepsilon^2}\sum_{j\in\Lambda_d\cup\Lambda_u}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_{j_0}^T\mathbf{D}_k\boldsymbol{u}_j\big)\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_k}\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big) + \frac{1}{\varepsilon^2}\sum_{l_0\in\Lambda_0}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_{j_0}^T\mathbf{D}_k\boldsymbol{u}_{l_0}\big)\frac{\partial\check{\alpha}^\varepsilon_{l_0}}{\partial x_k}\\
&=\sum_{j\in\Lambda_d\cup\Lambda_u}\frac{i\omega}{\varepsilon}\big(\boldsymbol{u}_{j_0}^T\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_j\big)\check{\alpha}^\varepsilon_j\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big) + \sum_{l_0\in\Lambda_0}\frac{i\omega}{\varepsilon}\big(\boldsymbol{u}_{j_0}^T\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_{l_0}\big)\check{\alpha}^\varepsilon_{l_0}.
\end{aligned}
$$

Using Hypothesis H1 for $j_0, l_0 \in \Lambda_0$ (which are such that $c_{j_0} = c_{l_0} = 0$) gives

$$
\check{\alpha}^\varepsilon_{j_0}(\omega,\boldsymbol{x}_\perp,x_d) = -i\frac{\varepsilon^2}{\omega}\sum_{j\in\Lambda_d\cup\Lambda_u}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_{j_0}^T\mathbf{D}_k\boldsymbol{u}_j\big)\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_k}(\omega,\boldsymbol{x}_\perp,x_d)\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big) + O(\varepsilon^3).
\tag{4.9}
$$

This shows that to leading order the complex amplitudes of the nonpropagating wave modes can be expressed in terms of the complex amplitudes of the propagating modes.

By projecting (4.8) onto $\mathbf{C}_0\boldsymbol{u}_j$, $j \in \Lambda_d \cup \Lambda_u$, we find

$$
\begin{aligned}
&c_j\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_d}\exp\Big(i\frac{\omega x_d}{c_j\varepsilon^4}\Big)\\
&= -\frac{1}{\varepsilon^2}\sum_{l\in\Lambda_d\cup\Lambda_u}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_l\big)\frac{\partial\check{\alpha}^\varepsilon_l}{\partial x_k}\exp\Big(i\frac{\omega x_d}{c_l\varepsilon^4}\Big) - \frac{1}{\varepsilon^2}\sum_{j_0\in\Lambda_0}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_{j_0}\big)\frac{\partial\check{\alpha}^\varepsilon_{j_0}}{\partial x_k}\\
&\quad+\sum_{l\in\Lambda_d\cup\Lambda_u}\frac{i\omega}{\varepsilon}\big(\boldsymbol{u}_j^T\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_l\big)\check{\alpha}^\varepsilon_l\exp\Big(i\frac{\omega x_d}{c_l\varepsilon^4}\Big) + \sum_{j_0\in\Lambda_0}\frac{i\omega}{\varepsilon}\big(\boldsymbol{u}_j^T\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_{j_0}\big)\check{\alpha}^\varepsilon_{j_0}.
\end{aligned}
$$

Substituting (4.9) into this equation gives

$$
\begin{aligned}
\frac{\partial\check{\alpha}^\varepsilon_j}{\partial x_d} &= -\frac{1}{\varepsilon^2 c_j}\sum_{l\in\Lambda_d\cup\Lambda_u}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_l\big)\frac{\partial\check{\alpha}^\varepsilon_l}{\partial x_k}\exp\Big(i\frac{\omega x_d}{\varepsilon^4}\big(\frac{1}{c_l}-\frac{1}{c_j}\big)\Big)\\
&\quad+\frac{i}{\omega c_j}\sum_{l\in\Lambda_d\cup\Lambda_u}\sum_{k,k'=1}^{d-1}\sum_{j_0\in\Lambda_0}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_{j_0}\big)\big(\boldsymbol{u}_{j_0}^T\mathbf{D}_{k'}\boldsymbol{u}_l\big)\frac{\partial^2\check{\alpha}^\varepsilon_l}{\partial x_{k'}\partial x_k}\exp\Big(i\frac{\omega x_d}{\varepsilon^4}\big(\frac{1}{c_l}-\frac{1}{c_j}\big)\Big)\\
&\quad+\frac{i\omega}{\varepsilon c_j}\sum_{l\in\Lambda_d\cup\Lambda_u}\big(\boldsymbol{u}_j^T\mathbf{C}_1\big(\boldsymbol{x}_\perp,\frac{x_d}{\varepsilon^2}\big)\boldsymbol{u}_l\big)\check{\alpha}^\varepsilon_l\exp\Big(i\frac{\omega x_d}{\varepsilon^4}\big(\frac{1}{c_l}-\frac{1}{c_j}\big)\Big) + O(\varepsilon).
\end{aligned}
\tag{4.10}
$$

It can be seen from Eq. (4.10) why Hypothesis H1 is necessary to ensure the validity of the paraxial approximation: the first term of the right-hand side of (4.10) has amplitude ε^{-2}. This term will blow up if, for some pair (j,l), we have $c_j = c_l$ (so that the rapid phase is zero and it cannot average out the expression) and the amplitude $\big(\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_l\big)$ is not zero for some $k = 1, \dots, d-1$. Hypothesis H1 ensures that this situation does not occur for any pair (j,l).

4.3. Boundary Conditions. The boundary conditions at $x_d = -L$ correspond to a radiation condition which means that there are no upward-going waves coming from $-\infty$:

$$\check{\alpha}_j^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d = -L) = 0, \qquad j \in \Lambda_u. \tag{4.11}$$

The boundary conditions at $x_d = 0^+$ correspond to a radiation condition which means that there are no downward-going waves coming from $+\infty$:

$$\check{\alpha}_j^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d = 0^+) = 0, \qquad j \in \Lambda_d. \tag{4.12}$$

The source imposes jump conditions at the interface $x_d = 0$. From (4.1) these jump conditions read:

$$\mathbf{D}_d \big[\check{\boldsymbol{u}}^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d)\big]_{0^-}^{0^+} = \check{\boldsymbol{f}}(\omega, \boldsymbol{x}_\perp). \tag{4.13}$$

Projecting onto $\boldsymbol{u}_j$ and using (4.5) and (4.12) give boundary conditions at $x_d = 0^-$:

$$\check{\alpha}_j^\varepsilon(\omega, \boldsymbol{x}_\perp, 0^-) = \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}_\perp), \qquad j \in \Lambda_d, \tag{4.14}$$

with

$$\check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}_\perp) = -\frac{1}{c_j} \boldsymbol{u}_j^T \check{\boldsymbol{f}}(\omega, \boldsymbol{x}_\perp). \tag{4.15}$$

4.4. Energy Conservation. The energy density and the energy flux in the kth direction are respectively

$$\mathcal{E}^\varepsilon(t, \boldsymbol{x}) = \frac{1}{2} \boldsymbol{u}^\varepsilon(t, \boldsymbol{x})^T \mathbf{C}^\varepsilon(\boldsymbol{x}) \boldsymbol{u}^\varepsilon(t, \boldsymbol{x}), \qquad \mathcal{F}_k^\varepsilon(t, \boldsymbol{x}) = \frac{1}{2} \boldsymbol{u}^\varepsilon(t, \boldsymbol{x})^T \mathbf{D}_k \boldsymbol{u}^\varepsilon(t, \boldsymbol{x}).$$

From the boundary conditions (4.11) and (4.14) the total energy flux incoming in the section $x_d \in (-L, 0)$ is

$$\mathcal{F}_{\mathrm{inc}}^\varepsilon = \frac{1}{2} \iint \boldsymbol{u}_{\mathrm{inc}}^\varepsilon(t, \boldsymbol{x}_\perp)^T \mathbf{D}_d \boldsymbol{u}_{\mathrm{inc}}^\varepsilon(t, \boldsymbol{x}_\perp) \mathrm{d}\boldsymbol{x}_\perp \, \mathrm{d}t,$$

where the incoming wave field is

$$\boldsymbol{u}_{\mathrm{inc}}^\varepsilon(t, \boldsymbol{x}_\perp) = \sum_{j \in \Lambda_d} v_{j,\mathrm{inc}}\Big(\frac{t}{\varepsilon^4}, \frac{\boldsymbol{x}_\perp}{\varepsilon^2}\Big) \boldsymbol{u}_j, \tag{4.16}$$

$$v_{j,\mathrm{inc}}(t, \boldsymbol{x}_\perp) = \frac{1}{2\pi} \int \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}_\perp) \exp(-i\omega t) \mathrm{d}\omega. \tag{4.17}$$

Using (4.5) we find

$$\mathcal{F}_{\mathrm{inc}}^\varepsilon = \frac{\varepsilon^{4+2(d-1)}}{4\pi} \sum_{j \in \Lambda_d} c_j \iint \big|\check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}_\perp)\big|^2 \mathrm{d}\boldsymbol{x}_\perp \, \mathrm{d}\omega \tag{4.18}$$

(the flux is signed and here $\mathcal{F}^\varepsilon_{\rm inc} < 0$ since $c_j < 0$ for $j \in \Lambda_d$). Similarly the total transmitted flux is

$$\mathcal{F}^\varepsilon_{\rm tr} = \frac{\varepsilon^{4+2(d-1)}}{4\pi} \sum_{j\in\Lambda_d} c_j \iint \big|\check{\alpha}^\varepsilon_j(\omega, \boldsymbol{x}_\perp, x_d = -L)\big|^2 {\rm d}\boldsymbol{x}_\perp {\rm d}\omega, \tag{4.19}$$

and the total reflected flux is

$$\mathcal{F}^\varepsilon_{\rm ref} = \frac{\varepsilon^{4+2(d-1)}}{4\pi} \sum_{j\in\Lambda_u} c_j \iint \big|\check{\alpha}^\varepsilon_j(\omega, \boldsymbol{x}_\perp, x_d = 0^-)\big|^2 {\rm d}\boldsymbol{x}_\perp {\rm d}\omega. \tag{4.20}$$

Energy conservation imposes that

$$\mathcal{F}^\varepsilon_{\rm inc} + \mathcal{F}^\varepsilon_{\rm ref} = \mathcal{F}^\varepsilon_{\rm tr}. \tag{4.21}$$

4.5. The Two-point Boundary Value Problem. Let us next introduce the column vector

$$\check{\boldsymbol{X}}^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d) = \big(\check{\alpha}^\varepsilon_j(\omega, \boldsymbol{x}_\perp, x_d)\big)_{j\in\Lambda_d\cup\Lambda_u}. \tag{4.22}$$

From (4.10) the vector $\check{\boldsymbol{X}}^\varepsilon$ satisfies in the section $x_d \in (-L, 0)$ the linear system

$$\frac{{\rm d}\check{\boldsymbol{X}}^\varepsilon}{{\rm d}x_d} = \check{\mathbf{A}}^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d)\check{\boldsymbol{X}}^\varepsilon\,, \tag{4.23}$$

where the (j,l)-entry of the square matrix $\check{\mathbf{A}}^\varepsilon = (\check{A}_{jl})_{j,l\in\Lambda_d\cup\Lambda_u}$ is given by

$$\check{A}^\varepsilon_{jl}(\omega, \boldsymbol{x}_\perp, x_d) = \Big[\frac{1}{\varepsilon^2}\check{A}^{(2)}_{jl}(\omega) + \frac{1}{\varepsilon}\check{A}^{(1)}_{jl}\big(\omega, \boldsymbol{x}_\perp, \frac{x_d}{\varepsilon^2}\big) + \check{A}^{(0)}_{jl}(\omega)\Big] \exp\Big(i\frac{\omega x_d}{\varepsilon^4}\big(\frac{1}{c_l} - \frac{1}{c_j}\big)\Big),$$

$$\check{A}^{(0)}_{jl}(\omega) = \frac{i}{\omega c_j} \sum_{k,k'=1}^{d-1} \sum_{j_0\in\Lambda_0} \big(\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_{j_0}\big)\big(\boldsymbol{u}_{j_0}^T \mathbf{D}_{k'} \boldsymbol{u}_l\big) \frac{\partial^2}{\partial x_{k'}\partial x_k}, \tag{4.24}$$

$$\check{A}^{(1)}_{jl}\big(\omega, \boldsymbol{x}_\perp, x_d\big) = \frac{i\omega}{c_j}\big(\boldsymbol{u}_j^T \mathbf{C}_1(\boldsymbol{x}_\perp, x_d)\boldsymbol{u}_l\big), \tag{4.25}$$

$$\check{A}^{(2)}_{jl}(\omega) = -\frac{1}{c_j} \sum_{k=1}^{d-1} \big(\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_l\big) \frac{\partial}{\partial x_k}. \tag{4.26}$$

From (4.11-4.14) the vector $\check{\boldsymbol{X}}^\varepsilon$ satisfies the following two-point boundary conditions at $x_d = -L$ and $x_d = 0$:

$$\mathbf{H}^{-L}\check{\boldsymbol{X}}^\varepsilon(\omega, \boldsymbol{x}_\perp, -L) + \mathbf{H}^0\check{\boldsymbol{X}}^\varepsilon(\omega, \boldsymbol{x}_\perp, 0) = \check{\boldsymbol{V}}(\omega, \boldsymbol{x}_\perp)\,, \tag{4.27}$$

where the vector $\check{\boldsymbol{V}}(\omega, \boldsymbol{x}_\perp)$ and the diagonal matrices $\mathbf{H}^{-L}$ and $\mathbf{H}^0$ are defined by

$$\check{V}_j(\omega, \boldsymbol{x}_\perp) = \begin{cases} \check{v}_{j,{\rm inc}}(\omega, \boldsymbol{x}_\perp) & \text{if } j \in \Lambda_d, \\ 0 & \text{if } j \in \Lambda_u, \end{cases} \tag{4.28}$$

$$H^{-L}_{jl} = \begin{cases} 1 & \text{if } j = l \in \Lambda_u, \\ 0 & \text{otherwise,} \end{cases} \qquad H^0_{jl} = \begin{cases} 1 & \text{if } j = l \in \Lambda_d, \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

In the expression (4.24-4.26) of $\check{\mathbf{A}}^\varepsilon$ we have neglected terms of order ε and smaller, and kept only the terms of order ε^{-2}, ε^{-1}, and 1.

If the source is $\boldsymbol{x}_\perp$-independent and if the medium is homogeneous ($\mathbf{C}_1 = \mathbf{0}$), then the solution of (4.23)-(4.27) is a constant vector which corresponds to a collection of downward-going modes:

$$\check{X}_j^\varepsilon(\omega, \boldsymbol{x}_\perp, x_d) = \begin{cases} \check{v}_{j,\mathrm{inc}}(\omega) & \text{if } j \in \Lambda_d \\ 0 & \text{if } j \in \Lambda_u \end{cases} \qquad \forall (\boldsymbol{x}_\perp, x_d) \in \mathbb{R}^{d-1} \times [-L, 0].$$

If the source is $\boldsymbol{x}_\perp$-dependent and/or the medium is heterogeneous, then transverse spatial effects and/or random effects have to be taken into account:
1) The terms associated with the matrices $\check{\mathbf{A}}^{(0)}$ and $\check{\mathbf{A}}^{(2)}$ in (4.26) correspond to deterministic transverse spatial effects.
The terms associated with the matrix $\check{\mathbf{A}}^{(0)}$ have amplitudes of order one and rapid phases. In the limit $\varepsilon \to 0$ only the terms with no rapid phase will survive, that is, the entries (j, l) for which we have $c_j = c_l$.
The terms associated with the matrix $\check{\mathbf{A}}^{(2)}$ have large amplitudes, of order ε^{-2}, but they also have rapid phases that vary at the scale ε^4. Note that Hypothesis H1 imposes that the entries (j, l) for which there is no rapid phase $c_j = c_l$ satisfy $\check{A}^{(2)}_{jl} = 0$. We will see that these large-amplitude and rapidly fluctuating terms give rise to effective terms of order one in the limit $\varepsilon \to 0$ through the application of an averaging theorem for highly oscillatory differential equations.
The deterministic effects associated with the matrices $\check{\mathbf{A}}^{(0)}$ and $\check{\mathbf{A}}^{(2)}$ consist in coupling between modes with different velocities, including coupling between upward- and downward-going modes, and it is crucial to take them into account in the determination of the paraxial wave equation. As we will see, the reflected upward-going mode amplitudes are vanishing in the limit $\varepsilon \to 0$, but the coupling terms between upward- and downward-going modes give important contributions to the dynamics of the transmitted waves.
2) The terms associated with the matrix $\check{\mathbf{A}}^{(1)}$ in (4.26) correspond to downward and upward scattering and coupling between modes due to the random heterogeneities of the medium. These terms have large amplitudes, of order ε^{-1}, but they vary rapidly at the scale ε^2, and the driving processes have mean zero and mixing properties. They will also give rise to effective terms of order one in the limit $\varepsilon \to 0$ through the application of a diffusion approximation theorem. The limit of $\check{\boldsymbol{X}}^\varepsilon$ will be characterized by a system of stochastic partial differential equations driven by correlated Brownian fields.

5. The Paraxial Wave Equations. In this section we state the main results of the paper. Proposition 5.1 describes the paraxial equations for waves when the medium is homogeneous and it is proved in Section 6. Proposition 5.2 describes the white-noise paraxial equations for waves when the medium is randomly heterogeneous and it is proved in Section 7. The proofs are based on the wave mode decomposition set forth in Section 4.

We assume that there are $\tilde{n}$ distinct nonzero eigenvalues:

$$\{\tilde{c}_1, \ldots, \tilde{c}_{\tilde{n}}\} = \big\{c_j,\, j \in \Lambda_d \cup \Lambda_u\big\}, \text{ with } \tilde{c}_1 < \cdots < \tilde{c}_{\tilde{n}}. \tag{5.1}$$

We denote by $\tilde{n}_d$ the number of distinct negative eigenvalues. For $p = 1, \ldots, \tilde{n}_d$ (which is such that $\tilde{c}_p < 0$) we denote by

$$\Lambda_d^{(p)} = \big\{j = 1, \ldots, n \text{ such that } c_j = \tilde{c}_p\big\}, \tag{5.2}$$

the set of modes with the velocity $\tilde{c}_p$. The multiplicity $\tilde{m}_p$ of the eigenvalue $\tilde{c}_p$ is the cardinal of the set $\Lambda_d^{(p)}$. We now state an additional hypothesis that prevents anomalous coupling between pairs of modes:

Hypothesis H2: *The positive numbers* $\left|\tilde{c}_p^{-1} - \tilde{c}_q^{-1}\right|$ *for* $1 \leq p < q \leq \tilde{n}$ *are distinct.*

PROPOSITION 5.1. *When the medium is homogeneous* $\mathbf{C}_1 \equiv 0$*, under Hypotheses H1-H2, in the limit* $\varepsilon \to 0$*, the transmitted wave at* $x_d = -L$ *consists of the succession of* $\tilde{n}_d$ *wave fields that emerge around times* $-L/\tilde{c}_p$*,* $p = 1, \dots, \tilde{n}_d$*:*

$$\boldsymbol{u}^\varepsilon\Big(t = -\frac{L}{\tilde{c}_p} + \varepsilon^4 s, \varepsilon^2 \boldsymbol{x}_\perp, x_d = -L\Big) \stackrel{\varepsilon \to 0}{\longrightarrow} \boldsymbol{u}_{\rm tr}^{(p)}(s, \boldsymbol{x}_\perp). \tag{5.3}$$

The field $\boldsymbol{u}_{\rm tr}^{(p)}$ *is a superposition of wave modes with the velocity* $\tilde{c}_p$*:*

$$\boldsymbol{u}_{\rm tr}^{(p)}(s, \boldsymbol{x}_\perp) = \sum_{j \in \Lambda_d^{(p)}} \alpha_j(s, \boldsymbol{x}_\perp) \boldsymbol{u}_j. \tag{5.4}$$

The Fourier transforms $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$*,* $j \in \Lambda_d^{(p)}$*, of the wave amplitudes* $\alpha_j(s, \boldsymbol{x}_\perp)$ *are given by*

$$\big(\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)\big)_{j \in \Lambda_d^{(p)}} = \int_{\mathbb{R}^{d-1}} \check{\mathbf{T}}^{(p)}(\omega, \boldsymbol{x}_\perp - \boldsymbol{x}'_\perp, x_d = 0)\big(\check{v}_{j,{\rm inc}}(\omega, \boldsymbol{x}'_\perp)\big)_{j \in \Lambda_d^{(p)}} {\rm d}\boldsymbol{x}'_\perp,$$

with $\check{v}_{j,{\rm inc}}$ *given in (4.15). The* $\tilde{m}_p \times \tilde{m}_p$*-matrix* $\check{\mathbf{T}}^{(p)}(\omega, \boldsymbol{x}_\perp, x_d)$ *is solution of the coupled system of paraxial wave equations:*

$$\frac{\partial \check{\mathbf{T}}^{(p)}}{\partial x_d} = \frac{i}{\omega} \sum_{k,k'=1}^{d-1} \frac{\partial^2 \check{\mathbf{T}}^{(p)}}{\partial x_k \partial x_{k'}} \boldsymbol{\Gamma}^{(p,k,k')}, \tag{5.5}$$

starting from $\check{\mathbf{T}}^{(p)}(\omega, \boldsymbol{x}_\perp, x_d = -L) = \delta(\boldsymbol{x}_\perp)\mathbf{I}^{(p)}$*, where* $\mathbf{I}^{(p)}$ *is the* $\tilde{m}_p \times \tilde{m}_p$ *identity matrix and the* $\tilde{m}_p \times \tilde{m}_p$ *symmetric matrix* $\boldsymbol{\Gamma}^{(p,k,k')}$ *is defined by*

$$\begin{aligned}\Gamma_{j,l}^{(p,k,k')} = \frac{1}{2} \sum_{j', c_{j'} \neq \tilde{c}_p} \frac{1}{c_{j'} - \tilde{c}_p} \Big[&\big(\boldsymbol{u}_{\tilde{M}_p + j}^T \mathbf{D}_k \boldsymbol{u}_{j'}\big)\big(\boldsymbol{u}_{\tilde{M}_p + l}^T \mathbf{D}_{k'} \boldsymbol{u}_{j'}\big) \\ &+ \big(\boldsymbol{u}_{\tilde{M}_p + j}^T \mathbf{D}_{k'} \boldsymbol{u}_{j'}\big)\big(\boldsymbol{u}_{\tilde{M}_p + l}^T \mathbf{D}_k \boldsymbol{u}_{j'}\big)\Big], \qquad j, l = 1, \dots, \tilde{m}_p,\end{aligned} \tag{5.6}$$

where $\tilde{M}_1 = 0$ *and* $\tilde{M}_p = \sum_{q=1}^{p-1} \tilde{m}_q$*.*

Let

Hypothesis H3: *The matrices* $\boldsymbol{\Gamma}^{(p,k,k')}$ *are diagonal for all* $p = 1, \dots, \tilde{n}_d$ *and* $k, k' = 1, \dots, d-1$*.*

Under Hypotheses H1-H3, for any $j \in \Lambda_d$ *the Fourier transform* $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$ *of the wave mode amplitude* $\alpha_j(s, \boldsymbol{x}_\perp)$ *is given by*

$$\check{\alpha}_j(\omega, \boldsymbol{x}_\perp) = \int_{\mathbb{R}^{d-1}} \check{T}_j(\omega, \boldsymbol{x}_\perp - \boldsymbol{x}'_\perp, x_d = 0) \check{v}_{j,{\rm inc}}(\omega, \boldsymbol{x}'_\perp) {\rm d}\boldsymbol{x}'_\perp,$$

where the scalar-valued process $\check{T}_j(\omega, \boldsymbol{x}_\perp, x_d)$ *is solution of the paraxial wave equation:*

$$\frac{\partial \check{T}_j}{\partial x_d} = \frac{i}{\omega} \sum_{k,k'=1}^{d-1} G^{(j,k,k')} \frac{\partial^2 \check{T}_j}{\partial x_k \partial x_{k'}}, \tag{5.7}$$

starting from $\check{T}_j(\omega, \boldsymbol{x}_\perp, x_d = -L) = \delta(\boldsymbol{x}_\perp)$, where the real-valued coefficient $G^{(j,k,k')}$ is defined by

$$G^{(j,k,k')} = \sum_{j', c_{j'} \neq c_j} \frac{1}{c_{j'} - c_j} \big(\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_{j'}\big)\big(\boldsymbol{u}_j^T \mathbf{D}_{k'} \boldsymbol{u}_{j'}\big). \tag{5.8}$$

Note that:
- Equation (5.5) can be solved explicitly in the transverse spatial Fourier domain

$$\widehat{\mathbf{T}}^{(p)}(\omega, \boldsymbol{\kappa}_\perp, x_d) = \exp\Big(- \frac{i(x_d - L)}{\omega} \sum_{k,k'=1}^{d-1} \mathbf{\Gamma}^{(p,k,k')} \kappa_k \kappa_{k'} \Big),$$

which shows that (5.5) can also be written as

$$\frac{\partial \check{\mathbf{T}}^{(p)}}{\partial x_d} = \frac{i}{\omega} \sum_{k,k'=1}^{d-1} \mathbf{\Gamma}^{(p,k,k')} \frac{\partial^2 \check{\mathbf{T}}^{(p)}}{\partial x_k \partial x_{k'}}. \tag{5.9}$$

- Hypothesis H3 is readily satisfied if the negative eigenvalues are simple, since then $\tilde{m}_p = 1$ and the matrices $\mathbf{\Gamma}^{(p,k,k')}$ are of size 1×1.

Proposition 5.1 describes how the downward-going wave modes propagate into the homogeneous medium. The form of this Schrödinger system is simple but its derivation is not obvious. One has to show that the forward-scattering approximation is valid. One has to take into account the coupling of downward-going waves with upward-going wave modes and with non-propagative wave modes. As we will see, the brutal suppression of the upward-going (i.e. reflected) and non-propagative modes from the equations leads to wrong results. The reflected modes are vanishing in the limit $\varepsilon \to 0$, which means that there is no transfer of energy from the downward-going modes to the upward-going modes, but the averaging of the coupling terms between upward- and downward-going modes give effective terms that contribute to the expressions of the diffraction operators.

We now give the result when the medium has random fluctuations.

Proposition 5.2. *When the medium is randomly heterogeneous $\mathbf{C}_1 \not\equiv 0$, under Hypotheses H1-H2, in the limit $\varepsilon \to 0$, the transmitted wave consists of the succession of $\tilde{n}_d$ wave fields that emerge at $x_d = -L$ around times $t = -L/\tilde{c}_p$, $p = 1, \ldots, \tilde{n}_d$, as described by (5.3). They have the form (5.4) and the wave mode amplitudes are described below.*

1) Let us write the random matrix $\mathbf{C}_1(\boldsymbol{x})$ in the form

$$\mathbf{C}_1(\boldsymbol{x}_\perp, x_d) = \sum_{r=1}^{n_r} g^{(r)}(\boldsymbol{x}_\perp, x_d) \mathbf{h}^{(r)}, \tag{5.10}$$

where the real-valued fields $g^{(r)}(\boldsymbol{x}_\perp, x_d)$ are stationary, zero-mean, and mixing in x_d and the matrices $\mathbf{h}^{(r)}$ are constant.

For $p = 1, \ldots, \tilde{n}_p$, the time-harmonic fields $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$, $j \in \Lambda_d^{(p)}$, are given by

$$\big(\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)\big)_{j \in \Lambda_d^{(p)}} = \int_{\mathbb{R}^{d-1}} \check{\boldsymbol{\mathcal{T}}}^{(p)}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d = 0) \big(\check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}'_\perp)\big)_{j \in \Lambda_d^{(p)}} \mathrm{d}\boldsymbol{x}'_\perp.$$

The $\tilde{m}_p \times \tilde{m}_p$*-matrix kernel* $\check{\boldsymbol{\mathcal{T}}}^{(p)}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d)$ *is solution of the coupled system of paraxial wave equations:*

$$\mathrm{d}\check{\boldsymbol{\mathcal{T}}}^{(p)} = \frac{i}{\omega} \sum_{k,k'=1}^{d-1} \frac{\partial^2 \check{\boldsymbol{\mathcal{T}}}^{(p)}}{\partial x'_k \partial x'_{k'}} \boldsymbol{\Gamma}^{(p,k,k')} \mathrm{d}x_d + \frac{i\omega}{\tilde{c}_p} \sum_{r=1}^{n_{\mathrm{r}}} \check{\boldsymbol{\mathcal{T}}}^{(p)} \mathbf{N}^{(r,p)} \circ \mathrm{d}W^{(r)}(\boldsymbol{x}'_\perp, x_d)$$
$$- \frac{\omega^2}{2\tilde{c}_p^2} \sum_{1 \le r < s \le n_{\mathrm{r}}} \gamma^A_{r,s}(\mathbf{0}) \check{\boldsymbol{\mathcal{T}}}^{(p)} \big[\mathbf{N}^{(s,p)}\mathbf{N}^{(r,p)} - \mathbf{N}^{(r,p)}\mathbf{N}^{(s,p)}\big] \mathrm{d}x_d, \qquad (5.11)$$

starting from $\check{\boldsymbol{\mathcal{T}}}^{(p)}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d = -L) = \delta(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)\mathbf{I}^{(p)}$*. Here the symbol* $\circ$ *stands for the Stratonovich stochastic integral, the* $\tilde{m}_p \times \tilde{m}_p$*-matrix* $\boldsymbol{\Gamma}^{(p,k,k')}$ *is defined by (5.6), the* $W^{(r)}(\boldsymbol{x}_\perp, x_d)$*,* $r = 1, \dots, n_{\mathrm{r}}$ *are correlated Brownian fields with the covariance*

$$\mathbb{E}[W^{(r)}(\boldsymbol{x}_\perp, x_d) W^{(s)}(\boldsymbol{x}'_\perp, x'_d)] = \min\{(L + x_d), (L + x'_d)\} \gamma^S_{rs}(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp), \quad (5.12)$$

the matrices $\boldsymbol{\gamma}^S(\boldsymbol{x}_\perp)$ *and* $\boldsymbol{\gamma}^A(\boldsymbol{x}_\perp)$ *are given by*

$$\gamma^S_{rs}(\boldsymbol{x}_\perp) = \int_{-\infty}^{\infty} \mathbb{E}\big[g^{(r)}(\mathbf{0}, 0) g^{(s)}(\boldsymbol{x}_\perp, x_d)\big] \mathrm{d}x_d,$$
$$\gamma^A_{rs}(\boldsymbol{x}_\perp) = \int_0^{\infty} \mathbb{E}\big[g^{(r)}(\mathbf{0}, 0) g^{(s)}(\boldsymbol{x}_\perp, x_d)\big] \mathrm{d}x_d - \int_{-\infty}^{0} \mathbb{E}\big[g^{(r)}(\mathbf{0}, 0) g^{(s)}(\boldsymbol{x}_\perp, x_d)\big] \mathrm{d}x_d,$$

and, for $r = 1, \dots, n_{\mathrm{r}}$*,* $p = 1, \dots, \tilde{n}_d$*, the* $\tilde{m}_p \times \tilde{m}_p$ *matrix* $\mathbf{N}^{(r,p)}$ *is defined by*

$$N^{(r,p)}_{jl} = \boldsymbol{u}^T_{\tilde{M}_p + j} \mathbf{h}^{(r)} \boldsymbol{u}_{\tilde{M}_p + l} \qquad j, l = 1, \dots, \tilde{m}_p, \qquad (5.13)$$

where $\tilde{M}_1 = 0$ *and* $\tilde{M}_p = \sum_{q=1}^{p-1} \tilde{m}_q$*.*

2) Let

Hypothesis H4: *For all* $r = 1, \dots, n_{\mathrm{r}}$*,* $p = 1, \dots, \tilde{n}_d$*, the matrix* $\mathbf{N}^{(r,p)}$ *is diagonal.*

Under Hypotheses H1-H4, the time-harmonic fields $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$*,* $j \in \Lambda_d$ *are given by*

$$\check{\alpha}_j(\omega, \boldsymbol{x}_\perp) = \int_{\mathbb{R}^{d-1}} \check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d = 0) \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}'_\perp) \mathrm{d}\boldsymbol{x}'_\perp,$$

with $\check{v}_{j,\mathrm{inc}}$ *given in (4.15). The operators* $\check{\mathcal{T}}_j$ *are the solutions of the following Itô-Schrödinger diffusion models for* $x_d \in (-L, 0)$*:*

$$\mathrm{d}\check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d) = \frac{i}{\omega} \sum_{k,k'=1}^{d-1} G^{(j,k,k')} \frac{\partial^2}{\partial x'_k \partial x'_{k'}} \check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d) \mathrm{d}x_d$$
$$+ \frac{i\omega}{c_j} \check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d) \circ \mathrm{d}B_j(\boldsymbol{x}'_\perp, x_d), \qquad (5.14)$$

with the initial conditions

$$\check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d = -L) = \delta(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp). \qquad (5.15)$$

Here the $B_j(\boldsymbol{x}_\perp, x_d)$*,* $j \in \Lambda_d$*, are correlated Brownian fields with the covariance*

$$\mathbb{E}[B_j(\boldsymbol{x}_\perp, x_d) B_l(\boldsymbol{x}'_\perp, x'_d)] = \min\{(L + x_d), (L + x'_d)\} \gamma_{jl}(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp), \qquad (5.16)$$

where the covariance matrix $\boldsymbol{\gamma}(\boldsymbol{x}_\perp)$ *is given by*

$$\gamma_{jl}(\boldsymbol{x}_\perp) = \int_{-\infty}^{\infty} \mathbb{E}\big[\big(\boldsymbol{u}_j^T \mathbf{C}_1(\mathbf{0},0)\boldsymbol{u}_j\big)\big(\boldsymbol{u}_l^T \mathbf{C}_1(\boldsymbol{x}_\perp, x_d)\boldsymbol{u}_l\big)\big]\mathrm{d}x_d. \tag{5.17}$$

Note that:
- The stochastic partial differential equations (5.11) and (5.14) are written using the Statonovich stochastic integral. In Itô's form they read respectively

$$\begin{aligned}\mathrm{d}\check{\boldsymbol{\mathcal{T}}}^{(p)} = {} & \frac{i}{\omega}\sum_{k,k'=1}^{d-1} \frac{\partial^2 \check{\boldsymbol{\mathcal{T}}}^{(p)}}{\partial x'_k \partial x'_{k'}} \boldsymbol{\Gamma}^{(p,k,k')}\mathrm{d}x_d + \frac{i\omega}{\tilde{c}_p}\sum_{r=1}^{n_\mathrm{r}} \check{\boldsymbol{\mathcal{T}}}^{(p)}\mathbf{N}^{(r,p)}\mathrm{d}W^{(r)}(\boldsymbol{x}'_\perp, x_d) \\ & -\frac{\omega^2}{2\tilde{c}_p^2}\sum_{1\le r,s\le n_\mathrm{r}} \gamma^S_{r,s}(\mathbf{0})\check{\boldsymbol{\mathcal{T}}}^{(p)}\mathbf{N}^{(s,p)}\mathbf{N}^{(r,p)}\mathrm{d}x_d \\ & -\frac{\omega^2}{2\tilde{c}_p^2}\sum_{1\le r<s\le n_\mathrm{r}} \gamma^A_{r,s}(\mathbf{0})\check{\boldsymbol{\mathcal{T}}}^{(p)}\big[\mathbf{N}^{(s,p)}\mathbf{N}^{(r,p)} - \mathbf{N}^{(r,p)}\mathbf{N}^{(s,p)}\big]\mathrm{d}x_d, \end{aligned} \tag{5.18}$$

and

$$\mathrm{d}\check{\mathcal{T}}_j = \frac{i}{\omega}\sum_{k,k'=1}^{d-1} G^{(j,k,k')}\frac{\partial^2}{\partial x'_k \partial x'_{k'}}\check{\mathcal{T}}_j \mathrm{d}x_d + \frac{i\omega}{c_j}\check{\mathcal{T}}_j \mathrm{d}B_j(\boldsymbol{x}'_\perp, x_d) - \frac{\omega^2}{2c_j^2}\gamma_{jj}(\mathbf{0})\check{\mathcal{T}}_j\mathrm{d}x_d\,. \tag{5.19}$$

The Itô corrections have minus signs because the equations run forward in x_d.
- When the random process $\mathbf{C}_1(\boldsymbol{x})$ is of the form (5.10), the matrix $\boldsymbol{\gamma}(\boldsymbol{x}_\perp)$ defined by (5.17) is also given by

$$\gamma_{jl}(\boldsymbol{x}_\perp) = \sum_{r,s=1}^{n_\mathrm{r}} \big(\boldsymbol{u}_j^T\mathbf{h}^{(r)}\boldsymbol{u}_j\big)\gamma^S_{rs}(\boldsymbol{x}_\perp)\big(\boldsymbol{u}_l^T\mathbf{h}^{(s)}\boldsymbol{u}_l\big), \quad j,l\in\Lambda_d.$$

- Hypotheses H3 and H4 are satisfied in particular when the negative eigenvalues $(c_j)_{j=1,\dots,n_d}$ are simple.

The Itô-Schrödinger equations (5.11) or (5.14) describe how the wave modes propagate into the random medium. The important qualitative results are that the forward scattering approximation is valid in the scaled regime considered in the paper, and that the scattering effects of the random medium can be modeled by correlated Brownian fields that only depend on the two-point statistics of the random fluctuations of the parameters of the medium.

In order to give a complete characterization of the transmitted field, we add that there is no other transmitted wave in the sense that for any time $t_0 \notin \cup_{j\in\Lambda_d}\{-L/c_j\}$,

$$\boldsymbol{u}^\varepsilon\big(t = t_0 + \varepsilon^4 s, \varepsilon^2\boldsymbol{x}_\perp, x_d = -L\big) \xrightarrow{\varepsilon\to 0} \mathbf{0}.$$

There is no reflected wave in the sense that for any time $t_0 > 0$,

$$\boldsymbol{u}^\varepsilon\big(t = t_0 + \varepsilon^4 s, \varepsilon^2\boldsymbol{x}_\perp, x_d = 0\big) \xrightarrow{\varepsilon\to 0} \mathbf{0}.$$

In [12] the existence and uniqueness have been established for the random process

$$V_j(\omega, \boldsymbol{x}_\perp, x_d) = \int_{\mathbb{R}^{d-1}} \check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_d)\phi_j(\boldsymbol{x}'_\perp)\mathrm{d}\boldsymbol{x}'_\perp,$$

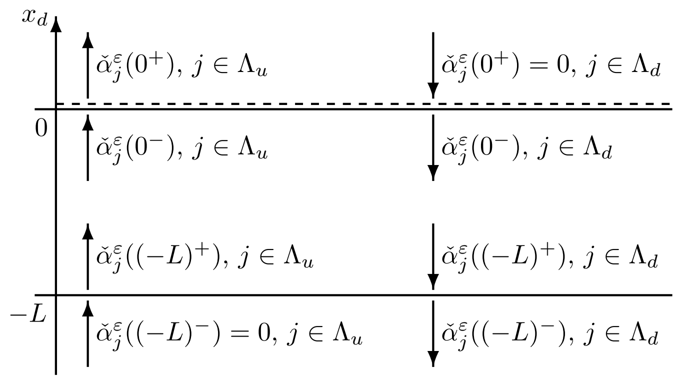

FIG. 5.1. *Boundary conditions for the wave modes with a source at $x_d = 0$ and radiation conditions at $\pm\infty$.*

for any test functions ϕ_j with unit $L^2(\mathbb{R}^{d-1}, \mathbb{C})$-norm and $j \in \Lambda_d$. Furthermore, it is shown that the process $V_j(\omega, \boldsymbol{x}_\perp, x_d)$ is a continuous Markov diffusion process on the unit ball of $L^2(\mathbb{R}^{d-1}, \mathbb{C})$. The moment equations moreover satisfy a closed system at each order [20]. It is straightforward to generalize these results for the joint distribution of the joint process $(V_j)_{j \in \Lambda_d}$. The interested reader can find explicit calculations for the mean intensity, the autocorrelation function, and other relevant quantities of the transmitted wave in [26].

The fact that the solutions of Itô-Schrödinger equations have constant L^2-norms shows that the sum of the energies of the transmitted mode waves (in homogeneous or random media) is equal to the energy of the incoming wave. Indeed the total energy flux generated by the source and entering the random section at $x_d = 0$ is given by (4.18) (in the limit of ε small). Moreover, the corresponding total energy flux of the transmitted wave modes (5.3) exiting at $x_d = -L$ is

$$\mathcal{F}^\varepsilon_{\mathrm{tr},j} = \frac{\varepsilon^{4+2(d-1)}}{4\pi} c_j \iiint \Big| \int \check{\mathcal{T}}_j(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, 0) \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}'_\perp) \mathrm{d}\boldsymbol{x}'_\perp \Big|^2 \mathrm{d}\omega \mathrm{d}\boldsymbol{x}_\perp .$$

This shows that we have $\mathcal{F}^\varepsilon_{\mathrm{inc}} = \sum_{j \in \Lambda_d} \mathcal{F}^\varepsilon_{\mathrm{tr},j}$, which illustrates the fact that no energy is reflected and all the energy is transmitted in the form of the $\tilde{n}_d$ wave fields described in the propositions.

6. The Derivation of the Paraxial Wave Equations in the Homogeneous Case. In this section we assume that there are no random fluctuations $\mathbf{C}_1 = 0$ and we prove Proposition 5.1. We transform the two-point boundary value problem (4.23)-(4.27) into an initial value problem. This is done by an invariant imbedding step in which we introduce transmission and reflection matrices. The principle of the invariant imbedding approach is explained in detail in [19, Chapter 4]. First, we define the lateral Fourier modes for $j \in \Lambda_u \cup \Lambda_d$:

$$\widehat{\alpha}^\varepsilon_j(\omega, \boldsymbol{\kappa}_\perp, x_d) = \int \check{\alpha}^\varepsilon_j(\omega, \boldsymbol{x}_\perp, x_d) \exp\big(- i \boldsymbol{\kappa}_\perp \cdot \boldsymbol{x}_\perp \big) \mathrm{d}\boldsymbol{x}_\perp , \tag{6.1}$$

where we have denoted $\boldsymbol{\kappa}_\perp = (\kappa_1, \dots, \kappa_{d-1})$. The reflected wave is characterized by $\widehat{\alpha}^\varepsilon_j(\omega, \boldsymbol{\kappa}_\perp, x_d = 0)$ for $j \in \Lambda_u$ and the transmitted wave by $\widehat{\alpha}^\varepsilon_j(\omega, \boldsymbol{\kappa}_\perp, x_d = -L)$ for $j \in \Lambda_d$.

The parameters ω and $\boldsymbol{\kappa}_\perp$ are frozen in the problem, so we shall not write explicitly the $(\omega, \boldsymbol{\kappa}_\perp)$-dependence of the vectors and matrices. The column vector $\widehat{\boldsymbol{X}}^\varepsilon(x_d)$ defined by

$$\widehat{\boldsymbol{X}}^\varepsilon(x_d) = \big(\widehat{\alpha}_j^\varepsilon(\omega, \boldsymbol{\kappa}_\perp, x_d)\big)_{j\in\Lambda_d\cup\Lambda_u} \tag{6.2}$$

is the solution of the two-point boundary value problem:

$$\frac{\mathrm{d}\widehat{\boldsymbol{X}}^\varepsilon}{\mathrm{d}x_d} = \widehat{\mathbf{A}}^\varepsilon(x_d)\widehat{\boldsymbol{X}}^\varepsilon\,, \qquad \mathbf{H}^{-L}\widehat{\boldsymbol{X}}^\varepsilon(-L) + \mathbf{H}^0\widehat{\boldsymbol{X}}^\varepsilon(0) = \widehat{\boldsymbol{V}}\,,$$

where the matrix $\widehat{\mathbf{A}}^\varepsilon(x_d)$ is defined by

$$\widehat{A}_{jl}^\varepsilon(x_d) = \Big[\frac{1}{\varepsilon^2}\widehat{P}_{jl} + \widehat{M}_{jl}\Big]\exp\Big(i\frac{\omega x_d}{\varepsilon^4}\big(\frac{1}{c_l} - \frac{1}{c_j}\big)\Big), \tag{6.3}$$

$$\widehat{P}_{jl} = -\frac{i}{c_j}\sum_{k=1}^{d-1}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_l\big)\kappa_k, \tag{6.4}$$

$$\widehat{M}_{jl} = -\frac{i}{\omega c_j}\sum_{k,k'=1}^{d-1}\sum_{j_0\in\Lambda_0}\big(\boldsymbol{u}_j^T\mathbf{D}_k\boldsymbol{u}_{j_0}\big)\big(\boldsymbol{u}_{j_0}^T\mathbf{D}_{k'}\boldsymbol{u}_l\big)\kappa_k\kappa_{k'}, \tag{6.5}$$

$\mathbf{H}^{-L}$ and $\mathbf{H}^0$ are given by (4.29), and

$$\widehat{V}_j = \begin{cases} \widehat{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{\kappa}_\perp) & \text{if } j\in\Lambda_d,\\ 0 & \text{if } j\in\Lambda_u,\end{cases} \tag{6.6}$$

$$\widehat{v}_{\mathrm{inc},j}(\omega, \boldsymbol{\kappa}_\perp) = \int \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{x}_\perp)\exp(-i\boldsymbol{\kappa}_\perp\cdot\boldsymbol{x}_\perp)\mathrm{d}\boldsymbol{x}_\perp\,, \qquad j\in\Lambda_d. \tag{6.7}$$

By applying Proposition A.1 of Appendix A , we get that the reflected and transmitted modes are given by

$$\widehat{\alpha}_j^\varepsilon(x_d = 0) = [\widehat{\mathbf{R}}^\varepsilon(x_d = 0)\widehat{\boldsymbol{V}}]_j\,, \qquad j\in\Lambda_u, \tag{6.8}$$

$$\widehat{\alpha}_j^\varepsilon(x_d = -L) = [\widehat{\mathbf{T}}^\varepsilon(x_d = 0)\widehat{\boldsymbol{V}}]_j\,, \qquad j\in\Lambda_d, \tag{6.9}$$

where the reflection and transmission matrices $\widehat{\mathbf{R}}^\varepsilon$ and $\widehat{\mathbf{T}}^\varepsilon$ are solution of the initial value problem

$$\frac{\mathrm{d}\widehat{\mathbf{R}}^\varepsilon}{\mathrm{d}x_d} = (\mathbf{I} - \widehat{\mathbf{R}}^\varepsilon\mathbf{H}^0)\widehat{\mathbf{A}}^\varepsilon(x_d)\widehat{\mathbf{R}}^\varepsilon, \qquad \widehat{\mathbf{R}}^\varepsilon(x_d = -L) = \mathbf{I}, \tag{6.10}$$

$$\frac{\mathrm{d}\widehat{\mathbf{T}}^\varepsilon}{\mathrm{d}x_d} = -\widehat{\mathbf{T}}^\varepsilon\mathbf{H}^0\widehat{\mathbf{A}}^\varepsilon(x_d)\widehat{\mathbf{R}}^\varepsilon, \qquad \widehat{\mathbf{T}}^\varepsilon(x_d = -L) = \mathbf{I}, \tag{6.11}$$

and $\mathbf{I}$ is the identity matrix. Note that the linear boundary value problem has been transformed into a nonlinear initial value problem, that has the form of a matrix Riccati equation.

Next we wish to apply Proposition B.1, and write the system in a form which is convenient for applying this result. We assume that there are $\tilde{n}$ distinct nonzero eigenvalues $\tilde{c}_1 < \cdots < \tilde{c}_{\tilde{n}}$ defined by (5.1) and we denote by $\tilde{m}_p$ the multiplicity of the eigenvalue $\tilde{c}_p$ for $p = 1, \ldots, \tilde{n}$.

Remember that $\widehat{P}_{jl} = 0$ if $c_j = c_l$ by Hypothesis H1. We introduce the matrices $\widehat{\mathbf{M}}^{(0)}$, $\widehat{\mathbf{M}}^{(p,q)}$, and $\widehat{\mathbf{P}}^{(p,q)}$ for $p \neq q \in \{1, \dots, \tilde{n}\}$:

$$\widehat{P}_{jl}^{(p,q)} = \begin{cases} \widehat{P}_{jl} & \text{if } c_j = \tilde{c}_p \text{ and } c_l = \tilde{c}_q, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widehat{M}_{jl}^{(0)} = \begin{cases} \widehat{M}_{jl} & \text{if } c_j = c_l, \\ 0 & \text{otherwise,} \end{cases} \qquad \widehat{M}_{jl}^{(p,q)} = \begin{cases} \widehat{M}_{jl} & \text{if } c_j = \tilde{c}_p \text{ and } c_l = \tilde{c}_q, \\ 0 & \text{otherwise,} \end{cases}$$

which allows us to expand the matrix $\widehat{\mathbf{A}}^\varepsilon(x_d)$ into matrices which have the same rapid phases:

$$\begin{aligned} \widehat{\mathbf{A}}^\varepsilon(x_d) = \frac{1}{\varepsilon^2} \sum_{1 \le p < q \le \tilde{n}} &\Big\{ \widehat{\mathbf{P}}^{(p,q)} \exp\Big(i \frac{\omega x_d}{\varepsilon^4} \big(\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p} \big) \Big) + \widehat{\mathbf{P}}^{(q,p)} \exp\Big(i \frac{\omega x_d}{\varepsilon^4} \big(\frac{1}{\tilde{c}_p} - \frac{1}{\tilde{c}_q} \big) \Big) \Big\} \\ + \widehat{\mathbf{M}}^{(0)} + \sum_{1 \le p < q \le \tilde{n}} &\Big\{ \widehat{\mathbf{M}}^{(p,q)} \exp\Big(i \frac{\omega x_d}{\varepsilon^4} \big(\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p} \big) \Big) + \widehat{\mathbf{M}}^{(q,p)} \exp\Big(i \frac{\omega x_d}{\varepsilon^4} \big(\frac{1}{\tilde{c}_p} - \frac{1}{\tilde{c}_q} \big) \Big) \Big\}. \end{aligned}$$

Using Hypothesis H2 we can apply Proposition B.1 which gives the asymptotic behavior of the reflection and transmission matrices. The limits $(\widehat{\mathbf{R}}, \widehat{\mathbf{T}})$ of the reflection/transmission matrices $(\widehat{\mathbf{R}}^\varepsilon, \widehat{\mathbf{T}}^\varepsilon)$ as $\varepsilon \to 0$ satisfy

$$\begin{aligned} \frac{\mathrm{d}\widehat{\mathbf{R}}}{\mathrm{d}x_d} &= (\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{M}}^{(0)}\widehat{\mathbf{R}} + \frac{i}{\omega} \sum_{1 \le p < q \le \tilde{n}} \frac{1}{\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p}} \mathbf{L}^{(\mathbf{R},p,q)}, \\ \frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}x_d} &= -\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{M}}^{(0)}\widehat{\mathbf{R}} + \frac{i}{\omega} \sum_{1 \le p < q \le \tilde{n}} \frac{1}{\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p}} \mathbf{L}^{(\mathbf{T},p,q)}, \end{aligned}$$

where the brackets stand for Lie brackets as defined in Proposition B.1:

$$\begin{aligned} \mathbf{L}^{(\mathbf{R},p,q)} &= \sum_{j,k} \frac{\partial \big[(\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big]}{\partial R_{jk}} \big((\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big)_{jk} \\ &\quad - \sum_{j,k} \frac{\partial \big[(\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big]}{\partial R_{jk}} \big((\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big)_{jk}, \\ \mathbf{L}^{(\mathbf{T},p,q)} &= \sum_{j,k} \frac{\partial \big[\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big]}{\partial T_{jk}} \big(\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big)_{jk} \\ &\quad - \sum_{j,k} \frac{\partial \big[\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big]}{\partial R_{jk}} \big((\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big)_{jk} \\ &\quad - \sum_{j,k} \frac{\partial \big[\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big]}{\partial T_{jk}} \big(\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big)_{jk} \\ &\quad + \sum_{j,k} \frac{\partial \big[\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{R}} \big]}{\partial R_{jk}} \big((\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{R}} \big)_{jk}. \end{aligned}$$

It turns out that these equations can be dramatically simplified. Indeed, the Lie brackets can be computed

$$\begin{aligned} \mathbf{L}^{(\mathbf{R},p,q)} &= (\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)(\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{P}}^{(q,p)} - \widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{P}}^{(p,q)})\widehat{\mathbf{R}}, \\ \mathbf{L}^{(\mathbf{T},p,q)} &= -\widehat{\mathbf{T}}\mathbf{H}^0(\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{P}}^{(q,p)} - \widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{P}}^{(p,q)})\widehat{\mathbf{R}}, \end{aligned}$$

which gives the following equations for the reflection and transmission matrices:

$$\frac{\mathrm{d}\widehat{\mathbf{R}}}{\mathrm{d}x_d} = (\mathbf{I} - \widehat{\mathbf{R}}\mathbf{H}^0)\widehat{\mathbf{D}}\widehat{\mathbf{R}}\,, \qquad \widehat{\mathbf{R}}(x_d = -L) = \mathbf{I}, \tag{6.12}$$

$$\frac{\mathrm{d}\widehat{\mathbf{T}}}{\mathrm{d}x_d} = -\widehat{\mathbf{T}}\mathbf{H}^0\widehat{\mathbf{D}}\widehat{\mathbf{R}}\,, \qquad \widehat{\mathbf{T}}(x_d = -L) = \mathbf{I}, \tag{6.13}$$

where

$$\begin{aligned}\widehat{\mathbf{D}} &= \widehat{\mathbf{M}}^{(0)} + \frac{i}{\omega}\sum_{1\le p<q\le\tilde{n}} \frac{1}{\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p}}\big(\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{P}}^{(q,p)} - \widehat{\mathbf{P}}^{(q,p)}\widehat{\mathbf{P}}^{(p,q)}\big) \\ &= \widehat{\mathbf{M}}^{(0)} + \frac{i}{\omega}\sum_{p\neq q=1}^{\tilde{n}} \frac{1}{\frac{1}{\tilde{c}_q} - \frac{1}{\tilde{c}_p}}\widehat{\mathbf{P}}^{(p,q)}\widehat{\mathbf{P}}^{(q,p)}.\end{aligned} \tag{6.14}$$

The matrix $\widehat{\mathbf{D}}$ is block-diagonal. There are $\tilde{n}$ blocks and the pth block $\widehat{\mathbf{D}}^{(p)}$ has size $\tilde{m}_p \times \tilde{m}_p$ and it corresponds to the pairs of indices (j,l) such that $c_j = c_l = \tilde{c}_p$:

$$\widehat{\mathbf{D}} = \oplus_{p=1}^{\tilde{n}}\widehat{\mathbf{D}}^{(p)} = \begin{pmatrix} \widehat{\mathbf{D}}^{(1)} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{D}}^{(2)} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \widehat{\mathbf{D}}^{(\tilde{n})} \end{pmatrix},$$

where the $\tilde{m}_p \times \tilde{m}_p$ matrix $\widehat{\mathbf{D}}^{(p)}$ is given by:

$$\widehat{D}^{(p)}_{j,l} = \frac{i}{\omega}\sum_{j', c_{j'}\neq\tilde{c}_p} \frac{1}{c_{j'} - \tilde{c}_p}\Big[\sum_{k=1}^{d-1}\big(\boldsymbol{u}^T_{\tilde{M}_p+j}\mathbf{D}_k\boldsymbol{u}_{j'}\big)\kappa_k\Big]\Big[\sum_{k=1}^{d-1}\big(\boldsymbol{u}^T_{\tilde{M}_p+l}\mathbf{D}_k\boldsymbol{u}_{j'}\big)\kappa_k\Big],$$

for $j,l = 1,\dots,\tilde{m}_p$, where $\tilde{M}_1 = 0$ and $\tilde{M}_p = \sum_{q=1}^{p-1}\tilde{m}_q$. From the block-diagonal structure of $\widehat{\mathbf{D}}$ and the diagonal structure of $\mathbf{H}^0$ we conclude that the system (6.12-6.13) only couples modes with the same eigenvalue. Therefore $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{T}}$ are block-diagonal:

$$\widehat{\mathbf{R}}(x_d) = \oplus_{p=1}^{\tilde{n}}\widehat{\mathbf{R}}^{(p)}(x_d), \qquad \widehat{\mathbf{T}}(x_d) = \oplus_{p=1}^{\tilde{n}}\widehat{\mathbf{T}}^{(p)}(x_d),$$

and we describe the forms of the matrices $\widehat{\mathbf{R}}^{(p)}$ and $\widehat{\mathbf{T}}^{(p)}$ below:

- Let p be such that $\tilde{c}_p > 0$. Since $H^0_{jl} = 0$ for all j,l such that $c_j = c_l = \tilde{c}_p$, we find that the $\tilde{m}_p \times \tilde{m}_p$ matrices $\widehat{\mathbf{R}}^{(p)}$ and $\widehat{\mathbf{T}}^{(p)}$ satisfy:

$$\frac{\mathrm{d}\widehat{\mathbf{R}}^{(p)}}{\mathrm{d}x_d} = \widehat{\mathbf{D}}^{(p)}\widehat{\mathbf{R}}^{(p)}\,, \qquad \widehat{\mathbf{R}}^{(p)}(x_d = -L) = \mathbf{I}^{(p)}\,, \tag{6.15}$$

$$\frac{\mathrm{d}\widehat{\mathbf{T}}^{(p)}}{\mathrm{d}x_d} = \mathbf{0}^{(p)}\,, \qquad \widehat{\mathbf{T}}^{(p)}(x_d = -L) = \mathbf{I}^{(p)}\,. \tag{6.16}$$

- Let p be such that $\tilde{c}_p < 0$. Since the matrix $\mathbf{H}^0$ restricted to the indices (j,l) such that $c_j = c_l = \tilde{c}_p$ is the identity, we find that the $\tilde{m}_p \times \tilde{m}_p$ matrices $\widehat{\mathbf{R}}^{(p)}$ and $\widehat{\mathbf{T}}^{(p)}$ satisfy:

$$\frac{\mathrm{d}\widehat{\mathbf{R}}^{(p)}}{\mathrm{d}x_d} = (\mathbf{I}^{(p)} - \widehat{\mathbf{R}}^{(p)})\widehat{\mathbf{D}}^{(p)}\widehat{\mathbf{R}}^{(p)}\,, \qquad \widehat{\mathbf{R}}^{(p)}(x_d = -L) = \mathbf{I}^{(p)}\,, \quad (6.17)$$

$$\frac{\mathrm{d}\widehat{\mathbf{T}}^{(p)}}{\mathrm{d}x_d} = -\widehat{\mathbf{T}}^{(p)}\widehat{\mathbf{D}}^{(p)}\widehat{\mathbf{R}}^{(p)}\,, \qquad \widehat{\mathbf{T}}^{(p)}(x_d = -L) = \mathbf{I}^{(p)}\,. \quad (6.18)$$

  In particular we get that $\widehat{\mathbf{R}}^{(p)}(x_d) = \mathbf{I}^{(p)}$ for any $x_d \in (-L, 0)$ and p such that $\tilde{c}_p < 0$ and $\widehat{\mathbf{T}}^{(p)}$ satisfies

$$\frac{\mathrm{d}\widehat{\mathbf{T}}^{(p)}}{\mathrm{d}x_d} = -\widehat{\mathbf{T}}^{(p)}\widehat{\mathbf{D}}^{(p)}\,, \qquad \widehat{\mathbf{T}}^{(p)}(x_d = -L) = \mathbf{I}^{(p)}\,.$$

The first consequence of the block-diagonal form of the reflection matrix and the form (6.6) of $\widehat{\boldsymbol{V}}$ is that, for $j \in \Lambda_u$, we have

$$\lim_{\varepsilon \to 0} \widehat{\alpha}_j^{\varepsilon}(x_d = 0) = [\widehat{\mathbf{R}}(x_d = 0)\widehat{\boldsymbol{V}}]_j = 0.$$

This shows that the paraxial (or forward-scattering) approximation is valid in the sense that the reflected modes $\widehat{\alpha}_j^{\varepsilon}(x_d = 0)$ for $j \in \Lambda_u$ are vanishing in the limit $\varepsilon \to 0$.

The second consequence of the previous asymptotic results and the application of an inverse Fourier transform in $\boldsymbol{\kappa}_\perp$ is that the transmitted wave modes $\widehat{\alpha}_j^{\varepsilon}(x_d = -L)$, $j \in \Lambda_d$, given by (6.9) satisfy the paraxial wave system (5.5) of Proposition 5.1 in the limit $\varepsilon \to 0$.

With the hypothesis H2 the matrix $\widehat{\mathbf{D}}$ is diagonal:

$$\widehat{D}_{jl} = \begin{cases} \dfrac{i}{\omega}\hat{d}_j(\boldsymbol{\kappa}_\perp) & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\hat{d}_j(\boldsymbol{\kappa}_\perp) = \sum_{j', c_{j'} \neq c_j} \frac{1}{c_{j'} - c_j}\Big[\sum_{k=1}^{d-1} (\boldsymbol{u}_j^T \mathbf{D}_k \boldsymbol{u}_{j'})\kappa_k\Big]^2.$$

The solution of (6.13) is also a diagonal matrix with

$$\widehat{T}_{jj} = \begin{cases} 1 & \text{if } j \in \Lambda_u, \\ \widehat{t}_j & \text{if } j \in \Lambda_d, \end{cases}$$

where

$$\frac{\mathrm{d}\widehat{t}_j}{\mathrm{d}x_d}(x_d) = -\frac{i\hat{d}_j(\boldsymbol{\kappa}_\perp)}{\omega}\widehat{t}_j(x_d)\,, \qquad \widehat{t}_j(x_d = -L) = 1.$$

By using (6.9) and by taking an inverse Fourier transform in $\boldsymbol{\kappa}_\perp$, we obtain the expression of the transmitted wave field and the result (5.7) of Proposition 5.1. Finally, the solutions of the Schrödinger equations in a homogeneous medium can be computed explicitly and we obtain for any $j \in \Lambda_d$

$$\alpha_j(s, \boldsymbol{x}_\perp) = \frac{1}{(2\pi)^d} \iint \check{v}_{j,\mathrm{inc}}(\omega, \boldsymbol{\kappa}_\perp) \exp\Big(-i\frac{\hat{d}_j(\boldsymbol{\kappa}_\perp)}{\omega}L + i\boldsymbol{\kappa}_\perp \cdot \boldsymbol{x}_\perp\Big)\mathrm{d}\boldsymbol{\kappa}_\perp \exp(-i\omega s)\mathrm{d}\omega.$$

7. The Derivation of the Paraxial Wave Equations in the Random Case. In this section we prove Proposition 5.2. We transform the two-point boundary value problem (4.23)-(4.27) into an initial value problem. This is very important in the random case so that we can deal exclusively with quantities that are adapted to the filtration of the driving process $\mathbf{C}_1$ (which is necessary for the application of a diffusion approximation theorem). This transformation is done by an invariant imbedding step in which we introduce transmission and reflection operators. The algebra is more complicated than in the homogeneous case since the random medium fluctuations involve coupling not only between the n modes (as in the homogeneous case) but also between different $\boldsymbol{\kappa}_\perp$-modes. This is why we need to introduce matrix operators. The vector $\widehat{\boldsymbol{X}}^\varepsilon$ defined as in (4.22) is solution of the two-point boundary value problem:

$$\frac{\mathrm{d}\widehat{\boldsymbol{X}}^\varepsilon}{\mathrm{d}x_d} = \widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(x_d)\widehat{\boldsymbol{X}}^\varepsilon , \qquad \mathbf{H}^{-L}\widehat{\boldsymbol{X}}^\varepsilon(-L) + \mathbf{H}^0\widehat{\boldsymbol{X}}^\varepsilon(0) = \widehat{\boldsymbol{V}} .$$

Here $\widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(x_d)$ is the matrix operator acting on vector fields $\widehat{\boldsymbol{Y}}(\boldsymbol{\kappa}_\perp) = (\widehat{Y}_j(\boldsymbol{\kappa}_\perp))_{j\in\Lambda_d\cup\Lambda_u}$ as

$$[\widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(x_d)\widehat{\boldsymbol{Y}}](\boldsymbol{\kappa}_\perp) = \int \widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(\boldsymbol{\kappa}_\perp, \boldsymbol{\kappa}'_\perp, x_d)\widehat{\boldsymbol{Y}}(\boldsymbol{\kappa}'_\perp)\mathrm{d}\boldsymbol{\kappa}'_\perp ,$$

with the matrix kernel given by

$$\widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(\boldsymbol{\kappa}_\perp, \boldsymbol{\kappa}'_\perp, x_d) = \delta(\boldsymbol{\kappa}_\perp - \boldsymbol{\kappa}'_\perp)\widehat{\mathbf{A}}^\varepsilon(\omega, \boldsymbol{\kappa}_\perp, x_d) + \widehat{\boldsymbol{\mathcal{N}}}^\varepsilon(\omega, \boldsymbol{\kappa}_\perp - \boldsymbol{\kappa}'_\perp, x_d) , \tag{7.1}$$

$$\widehat{\mathcal{N}}^\varepsilon_{jl}(\omega, \boldsymbol{\kappa}_\perp, x_d) = \frac{1}{\varepsilon}\widehat{N}_{jl}(\omega, \boldsymbol{\kappa}_\perp, \frac{x_d}{\varepsilon^2}) \exp\Big(i\frac{\omega x_d}{\varepsilon^4}(\frac{1}{c_l} - \frac{1}{c_j})\Big), \quad j, l \in \Lambda_d \cup \Lambda_u . \tag{7.2}$$

The matrix $\widehat{\mathbf{A}}^\varepsilon$ is defined by (6.3) and the matrix $\widehat{\mathbf{N}}$ is defined by

$$\widehat{N}_{jl}(\omega, \boldsymbol{\kappa}_\perp, x_d) = \frac{i\omega}{c_j(2\pi)^{d-1}}\big(\boldsymbol{u}_j^T\widehat{\mathbf{C}}_1(\boldsymbol{\kappa}_\perp, x_d)\boldsymbol{u}_l\big) . \tag{7.3}$$

The matrix operator $\widehat{\boldsymbol{\mathcal{N}}}^\varepsilon$ can be expanded into a matrix operator without rapid phase and a matrix operator with rapid phase:

$$\widehat{\mathcal{N}}^\varepsilon_{jl}(\omega, \boldsymbol{\kappa}_\perp, x_d) = \frac{1}{\varepsilon}\widehat{N}^{(\mathrm{a})}_{jl}(\omega, \boldsymbol{\kappa}_\perp, \frac{x_d}{\varepsilon^2}) + \frac{1}{\varepsilon}\widehat{N}^{(\mathrm{b})}_{jl}(\omega, \boldsymbol{\kappa}_\perp, \frac{x_d}{\varepsilon^2}) \exp\Big(i\frac{\omega x_d}{\varepsilon^4}(\frac{1}{c_j} - \frac{1}{c_l})\Big),$$

where we have introduced the matrices $\widehat{\mathbf{N}}^{(\mathrm{a})}$ and $\widehat{\mathbf{N}}^{(\mathrm{b})}$:

$$\widehat{N}^{(\mathrm{a})}_{jl} = \begin{cases} \widehat{N}_{jl} & \text{if } c_j = c_l, \\ 0 & \text{otherwise,} \end{cases} \qquad \widehat{N}^{(\mathrm{b})}_{jl} = \begin{cases} \widehat{N}_{jl} & \text{if } c_j \neq c_l, \\ 0 & \text{otherwise.} \end{cases}$$

Using (5.10) we can expand

$$\widehat{\mathbf{N}}^{(\mathrm{a})}(\omega, \boldsymbol{\kappa}_\perp, x_d) = i\omega\sum_{r=1}^{n_{\mathrm{r}}}\widehat{g}^{(r)}(\boldsymbol{\kappa}_\perp, x_d)\mathbf{N}^{(r)},$$

where the constant matrices $\mathbf{N}^{(r)}$ are given by

$$N^{(r)}_{jl} = \frac{1}{c_j}\big(\boldsymbol{u}^T_{\tilde{M}_p+j}\mathbf{h}^{(r)}\boldsymbol{u}_{\tilde{M}_p+l}\big) \text{ for } j, l = 1, \dots, \tilde{m}_p,$$

where $\tilde{M}_1 = 0$ and $\tilde{M}_p = \sum_{q=1}^{p-1} \tilde{m}_q$. The matrices $\mathbf{N}^{(r)}$ are block-diagonal:

$$\mathbf{N}^{(r)} = \oplus_{p=1}^{\tilde{n}} \frac{1}{\tilde{c}_p} \mathbf{N}^{(r,p)},$$

where the $\tilde{m}_p \times \tilde{m}_p$ matrix $\mathbf{N}^{(r,p)}$ is given by

$$N_{jl}^{(r,p)} = \boldsymbol{u}_j^T \mathbf{h}^{(r)} \boldsymbol{u}_l \text{ for } j,l \text{ such that } c_j = c_l = \tilde{c}_p. \tag{7.4}$$

By applying Proposition A.1, we get that the reflected and transmitted modes are given by

$$\widehat{\alpha}_j^\varepsilon(x_d = 0) = [\widehat{\boldsymbol{\mathcal{R}}}^\varepsilon(x_d = 0)\widehat{\boldsymbol{V}}]_j\,, \qquad j \in \Lambda_u, \tag{7.5}$$

$$\widehat{\alpha}_j^\varepsilon(x_d = -L) = [\widehat{\boldsymbol{\mathcal{T}}}^\varepsilon(x_d = 0)\widehat{\boldsymbol{V}}]_j\,, \qquad j \in \Lambda_d, \tag{7.6}$$

where the reflection and transmission matrix-operator kernels are solution of the initial value problem

$$\frac{\mathrm{d}\widehat{\boldsymbol{\mathcal{R}}}^\varepsilon}{\mathrm{d}x_d} = (\widehat{\boldsymbol{\mathcal{I}}} - \widehat{\boldsymbol{\mathcal{R}}}^\varepsilon \widehat{\boldsymbol{\mathcal{H}}}^0)\widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(x_d)\widehat{\boldsymbol{\mathcal{R}}}^\varepsilon, \qquad \widehat{\boldsymbol{\mathcal{R}}}^\varepsilon(x_d = -L) = \widehat{\boldsymbol{\mathcal{I}}}, \tag{7.7}$$

$$\frac{\mathrm{d}\widehat{\boldsymbol{\mathcal{T}}}^\varepsilon}{\mathrm{d}x_d} = -\widehat{\boldsymbol{\mathcal{T}}}^\varepsilon \widehat{\boldsymbol{\mathcal{H}}}^0 \widehat{\boldsymbol{\mathcal{A}}}^\varepsilon(x_d)\widehat{\boldsymbol{\mathcal{R}}}^\varepsilon, \qquad \widehat{\boldsymbol{\mathcal{T}}}^\varepsilon(x_d = -L) = \widehat{\boldsymbol{\mathcal{I}}}, \tag{7.8}$$

with

$$\widehat{\boldsymbol{\mathcal{I}}}(\boldsymbol{\kappa}_\perp, \boldsymbol{\kappa}_\perp') = \delta(\boldsymbol{\kappa}_\perp - \boldsymbol{\kappa}_\perp')\mathbf{I}\,, \qquad \widehat{\boldsymbol{\mathcal{H}}}^0(\boldsymbol{\kappa}_\perp, \boldsymbol{\kappa}_\perp') = \delta(\boldsymbol{\kappa}_\perp - \boldsymbol{\kappa}_\perp')\mathbf{H}^0\,.$$

Explicitly, the transmitted wave components are, for $j \in \Lambda_d$:

$$\boldsymbol{u}_j^T \mathbf{C}_0 \boldsymbol{u}^\varepsilon(t, \varepsilon^2 \boldsymbol{x}_\perp, x_d = -L) = \frac{1}{(2\pi)^d} \iiint \sum_{k \in \Lambda_d} \widehat{\mathcal{T}}_{j,k}^\varepsilon(\omega, \boldsymbol{\kappa}_\perp, \boldsymbol{\kappa}_\perp', x_d = 0)\widehat{v}_{\mathrm{inc},k}(\omega, \boldsymbol{\kappa}_\perp')$$
$$\times \exp(i\boldsymbol{\kappa}_\perp \cdot \boldsymbol{x}_\perp)\mathrm{d}\boldsymbol{\kappa}_\perp' \mathrm{d}\boldsymbol{\kappa}_\perp \exp\Big(-i\frac{\omega}{c_j \varepsilon^4}(L + c_j t)\Big)\mathrm{d}\omega\,.$$

We first note that the rapid phase in ω will give a localization in time of the transmitted waves (provided we show that $\widehat{\boldsymbol{\mathcal{T}}}^\varepsilon$ has a limit). Therefore we focus our attention on

$$\alpha_j^\varepsilon(s, \boldsymbol{x}_\perp) = \boldsymbol{u}_j^T \mathbf{C}_0 \boldsymbol{u}^\varepsilon\Big(t = -\frac{L}{c_j} + \varepsilon^4 s, \varepsilon^2 \boldsymbol{x}_\perp, x_d = -L\Big), \qquad j \in \Lambda_d.$$

The proof of the convergence follows closely the strategy adopted in [26], where the paraxial wave equation is obtained from the acoustic wave equations in the same distinguished limit. The main step of the proof consists in showing the convergence of the general moments of the transmitted wave components to the ones given by the limit system of stochastic partial differential equations (5.11). For $N \in \mathbb{N}$, $j_r \in \Lambda_d$, $m_r \in \mathbb{N}$, $s_r \in \mathbb{R}$ and $\boldsymbol{x}_{\perp r} \in \mathbb{R}^{d-1}$, $r = 1, \ldots, N$, the general moment of transmitted wave components

$$I^\varepsilon = \mathbb{E}\Big[\prod_{r=1}^N \alpha_{j_r}^\varepsilon(s_r, \boldsymbol{x}_{\perp r})^{m_r}\Big]$$

can be expressed as a sum of $|\Lambda_d|^M$ multiple integrals, for $M = \sum_{r=1}^N m_r$:

$$I^\varepsilon = \sum_{\boldsymbol{k}=(k_{r,h})_{h=1,\dots,m_r,r=1,\dots,N}\in\Lambda_d^M} I^\varepsilon_{\boldsymbol{k}}\,,$$

$$I^\varepsilon_{\boldsymbol{k}} = \frac{1}{(2\pi)^{dM}}\int\cdots\int\prod_{r=1}^N\prod_{h=1}^{m_r}\mathrm{d}\boldsymbol{\kappa}'_{\perp r,h}\mathrm{d}\boldsymbol{\kappa}_{\perp r,h}\mathrm{d}\omega_{r,h}$$
$$\times\prod_{r,h}\big(\widehat{v}_{\mathrm{inc},k_{r,h}}(\omega_{r,h},\boldsymbol{\kappa}'_{\perp r,h})\exp\big(i(\boldsymbol{\kappa}_{\perp r,h}\cdot\boldsymbol{x}_{\perp r}-\omega_{r,h}s_r)\big)\big)$$
$$\times\mathbb{E}\Big[\prod_{r,h}\widehat{\mathcal{T}}^\varepsilon_{j_r,k_{r,h}}(\omega_{r,h},\boldsymbol{\kappa}_{\perp r,h},\boldsymbol{\kappa}'_{\perp r,h},0)\Big]\,.$$

Therefore, the convergence of the general moment of the transmitted wave components in the white-noise limit will follow from the convergence of the moments $\mathbb{E}[J^\varepsilon(0)]$ of the transmission matrix-operator kernels, where

$$J^\varepsilon(x_d) = \prod_{r=1}^M\widehat{\mathcal{T}}^\varepsilon_{j_r,k_r}(\omega_r,\boldsymbol{\kappa}_{\perp r},\boldsymbol{\kappa}'_{\perp r},x_d)\,. \tag{7.9}$$

We use diffusion approximation theorems in the same way as in [26] combined with the application of the averaging theorem in Appendix B as in the homogeneous case to obtain the limit of the expectation of $J^\varepsilon(0)$ as $\varepsilon\to 0$. We find that this limit can be expressed as the expectation of the corresponding product of components of an "effective" transmission kernel in the following way:

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[J^\varepsilon(0)\big] = \mathbb{E}\Big[\prod_{r=1}^M\widehat{\mathcal{T}}_{j_r,k_r}(\omega_r,\boldsymbol{\kappa}_{\perp r},\boldsymbol{\kappa}'_{\perp r},0)\Big]. \tag{7.10}$$

Under Hypotheses 1-2 the expectation on the right-hand side of (7.10) is taken with respect to the following Itô-Schrödinger model for the transmission operator $\widehat{\boldsymbol{\mathcal{T}}} = (\widehat{\mathcal{T}}_{j,l})_{j,l\in\Lambda_d\cup\Lambda_u}$:

$$\widehat{\boldsymbol{\mathcal{T}}}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d) = \oplus_{p=1}^{\tilde{n}}\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d),$$

where, for $p=\tilde{n}_d+1,\dots,\tilde{n}$

$$\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d) = \delta(\boldsymbol{\kappa}_\perp-\boldsymbol{\kappa}'_\perp)\mathbf{I}^{(p)},$$

and for $p=1,\dots,\tilde{n}_d$, the $\tilde{m}_p\times\tilde{m}_p$ matrix kernel $\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)$ is solution of

$$\mathrm{d}\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d) = -\frac{i}{\omega}\sum_{k,k'=1}^{d-1}\kappa'_k\kappa'_{k'}\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)\boldsymbol{\Gamma}^{(p,k,k')}\mathrm{d}x_d$$
$$-\frac{\omega^2}{2\tilde{c}_p^2}\sum_{1\le r,s\le n_{\mathrm{r}}}\gamma^S_{rs}(\mathbf{0})\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)\mathbf{N}^{(s,p)}\mathbf{N}^{(r,p)}\mathrm{d}x_d$$
$$+\frac{i\omega}{(2\pi)^{d-1}\tilde{c}_p}\sum_{r=1}^{n_{\mathrm{r}}}\int\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}''_\perp,x_d)\mathbf{N}^{(r,p)}\mathrm{d}\widehat{W}^{(r)}(\boldsymbol{\kappa}''_\perp-\boldsymbol{\kappa}'_\perp,x_d)\mathrm{d}\boldsymbol{\kappa}''_\perp$$
$$-\frac{\omega^2}{2\tilde{c}_p^2}\sum_{1\le r<s\le n_{\mathrm{r}}}\gamma^A_{rs}(\mathbf{0})\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}''_\perp,x_d)\big[\mathbf{N}^{(s,p)}\mathbf{N}^{(r,p)}-\mathbf{N}^{(r,p)}\mathbf{N}^{(s,p)}\big]\mathrm{d}x_d,$$

starting from $\widehat{\boldsymbol{\mathcal{T}}}^{(p)}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d=-L)=\delta(\boldsymbol{\kappa}_\perp-\boldsymbol{\kappa}'_\perp)\mathbf{I}^{(p)}$. Here we use the standard Itô stochastic integral and the Brownian field $\widehat{\boldsymbol{W}}=(\widehat{W}^{(r)})_{r=1,\dots,n_{\rm r}}$ is the partial Fourier transform of the field $\boldsymbol{W}=(W^{(r)})_{r=1,\dots,n_{\rm r}}$ whose covariance is (5.12). It has the following operator-valued spatial covariance:

$$\mathbb{E}\big[\widehat{\boldsymbol{W}}(\boldsymbol{\kappa}_\perp,x_d)\widehat{\boldsymbol{W}}^T(\boldsymbol{\kappa}'_\perp,x'_d)\big]=\min\{(L+x_d),(L+x'_d)\}(2\pi)^{d-1}\widehat{\boldsymbol{\gamma}}^S(\boldsymbol{\kappa}_\perp)\delta(\boldsymbol{\kappa}_\perp+\boldsymbol{\kappa}'_\perp)\,, \tag{7.11}$$

where

$$\widehat{\boldsymbol{\gamma}}^S(\boldsymbol{\kappa}_\perp)=\int_{\mathbb{R}^{d-1}}\boldsymbol{\gamma}^S(\boldsymbol{x}_\perp)\exp(-i\boldsymbol{\kappa}_\perp\cdot\boldsymbol{x}_\perp)\mathrm{d}\boldsymbol{x}_\perp\,. \tag{7.12}$$

By considering the transmission kernels in the original spatial variables:

$$\check{\mathcal{T}}_j(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_d)=\frac{1}{(2\pi)^{d-1}}\iint\exp\big(i(\boldsymbol{\kappa}_\perp\cdot\boldsymbol{x}_\perp-\boldsymbol{\kappa}'_\perp\cdot\boldsymbol{x}'_\perp)\big)\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)\mathrm{d}\boldsymbol{\kappa}_\perp\mathrm{d}\boldsymbol{\kappa}'_\perp,$$

we obtain that they satisfy the system (5.11) where we have used the Stratonovich integral instead of Itô integral.

Under Hypotheses 1-4 the expectation on the right-hand side of (7.10) is taken with respect to the following Itô-Schrödinger model for the transmission operator $\widehat{\boldsymbol{\mathcal{T}}}=(\widehat{\mathcal{T}}_{j,l})_{j,l\in\Lambda_d\cup\Lambda_u}$:

$$\widehat{\mathcal{T}}_{j,l}(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)=\left\{\begin{array}{ll}0 & \text{if } j\neq l,\\ \widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d) & \text{if } j=l\in\Lambda_d,\\ \delta(\boldsymbol{\kappa}_\perp-\boldsymbol{\kappa}'_\perp) & \text{if } j=l\in\Lambda_u,\end{array}\right.$$

where the kernel $\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)$ is solution of

$$\begin{aligned}\mathrm{d}\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)=&-\frac{i\hat{d}_j(\boldsymbol{\kappa}'_\perp)}{\omega}\check{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)\mathrm{d}x_d-\frac{\omega^2\gamma_{jj}(\mathbf{0})}{2c_j^2}\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d)\mathrm{d}x_d\\&+\frac{i\omega}{c_j(2\pi)^{d-1}}\int\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}''_\perp,x_d)\mathrm{d}\widehat{B}_j(\boldsymbol{\kappa}''_\perp-\boldsymbol{\kappa}'_\perp,x_d)\mathrm{d}\boldsymbol{\kappa}''_\perp,\end{aligned}$$

starting from $\widehat{\mathcal{T}}_j(\omega,\boldsymbol{\kappa}_\perp,\boldsymbol{\kappa}'_\perp,x_d=-L)=\delta(\boldsymbol{\kappa}_\perp-\boldsymbol{\kappa}'_\perp)$. Here the Brownian field $\widehat{\boldsymbol{B}}=(\widehat{B}_j)_{j\in\Lambda_d}$ is the partial Fourier transform of the field $\boldsymbol{B}=(B_j)_{j\in\Lambda_d}$ whose covariance is (5.16). By considering the transmission kernels in the original spatial variables we obtain that they satisfy the system (5.14).

8. Applications. In this section we apply the general results to physical systems in the three-dimensional case $d=3$.

8.1. Acoustic Waves in Random Media. We consider linear acoustic waves propagating in a three-dimensional random medium. The governing equations are the conservation equations

$$\rho(\boldsymbol{x})\frac{\partial\boldsymbol{u}}{\partial t}+\nabla p=\boldsymbol{f}_{\boldsymbol{u}}\big(\frac{t}{\varepsilon^4},\frac{\boldsymbol{x}_\perp}{\varepsilon^2}\big)\delta(x_3),\qquad\frac{1}{K(\boldsymbol{x})}\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=f_p\big(\frac{t}{\varepsilon^4},\frac{\boldsymbol{x}_\perp}{\varepsilon^2}\big)\delta(x_3), \tag{8.1}$$

where p is the pressure field, $\boldsymbol{u}$ is the velocity field, ρ is the density of the medium, K is the bulk modulus of the medium. They have the form:

$$\frac{1}{K(\boldsymbol{x})}=\begin{cases}K_0^{-1} & \text{if } x_3\leq -L \text{ or } x_3\geq 0,\\ K_0^{-1}\big[1+\varepsilon^3 m_K(\frac{\boldsymbol{x}}{\varepsilon^2})\big] & \text{if } x_3\in(-L,0),\end{cases}$$

$$\rho(\boldsymbol{x})=\begin{cases}\rho_0 & \text{if } x_3\leq -L \text{ or } x_3\geq 0,\\ \rho_0\big[1+\varepsilon^3 m_\rho(\frac{\boldsymbol{x}}{\varepsilon^2})\big] & \text{if } x_3\in(-L,0),\end{cases}$$

where the random processes $m_K(\boldsymbol{x})$ and $m_\rho(\boldsymbol{x})$ model the medium fluctuations. We can write the problem in the hyperbolic form (3.1). The four-dimensional acoustic field is characterized by the three-dimensional velocity field $\boldsymbol{u}(t,\boldsymbol{x})$ and the scalar pressure field $p(t,\boldsymbol{x})$:

$$\mathbf{C}^\varepsilon(\boldsymbol{x})\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix}+\sum_{k=1}^3 \mathbf{D}_k\frac{\partial}{\partial x_k}\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix}=\begin{pmatrix}\boldsymbol{f_u}\\ f_p\end{pmatrix}\big(\frac{t}{\varepsilon^4},\frac{\boldsymbol{x}_\perp}{\varepsilon^2}\big)\delta(x_3),$$

with $\mathbf{C}^\varepsilon(\boldsymbol{x})=\mathbf{C}_0+\varepsilon^3\mathbf{C}_1(x/\varepsilon^2)$, $\mathbf{C}_1(\boldsymbol{x})=m_\rho(\boldsymbol{x})\mathbf{h}^\rho+m_K(\boldsymbol{x})\mathbf{h}^K$,

$$\mathbf{C}_0=\begin{pmatrix}\rho_0&0&0&0\\0&\rho_0&0&0\\0&0&\rho_0&0\\0&0&0&K_0^{-1}\end{pmatrix},\quad \mathbf{h}^\rho=\begin{pmatrix}\rho_0&0&0&0\\0&\rho_0&0&0\\0&0&\rho_0&0\\0&0&0&0\end{pmatrix},\quad \mathbf{h}^K=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&K_0^{-1}\end{pmatrix},$$

$$\mathbf{D}_1=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\quad \mathbf{D}_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad \mathbf{D}_3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

There are four eigenvalues of $\mathbf{C}_0^{-1}\mathbf{D}_3$ (that are wave velocities):

$$c_1=-c_0,\quad c_2=c_3=0,\quad c_4=c_0,$$

where $c_0=\sqrt{K_0/\rho_0}$ is the acoustic speed of propagation. They are associated with the four eigenvectors (that is, wave modes):

$$\begin{pmatrix}\boldsymbol{u}_1\\ p_1\end{pmatrix}=\frac{1}{\sqrt{2\rho_0}}\begin{pmatrix}0\\0\\1\\-\zeta_0\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{u}_2\\ p_2\end{pmatrix}=\frac{1}{\sqrt{\rho_0}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

$$\begin{pmatrix}\boldsymbol{u}_3\\ p_3\end{pmatrix}=\frac{1}{\sqrt{\rho_0}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{u}_4\\ p_4\end{pmatrix}=\frac{1}{\sqrt{2\rho_0}}\begin{pmatrix}0\\0\\1\\\zeta_0\end{pmatrix},$$

where $\zeta_0=\sqrt{K_0\rho_0}$ is the acoustic impedance. It is easy to check that Hypotheses H1-H4 are satisfied. By applying Proposition 5.2 we find that the transmitted wave at $x_3=-L$ emerges around time L/c_0:

$$\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix}\Big(t=\frac{L}{c_0}+\varepsilon^4 s,\varepsilon^2\boldsymbol{x}_\perp,x_3=-L\Big)\overset{\varepsilon\to 0}{\longrightarrow}\alpha_{\rm tr}(s,\boldsymbol{x}_\perp)\begin{pmatrix}\boldsymbol{u}_1\\ p_1\end{pmatrix}.$$

The Fourier transform $\check{\alpha}_{\rm tr}(\omega,\boldsymbol{x}_\perp)$ of the wave amplitude $\alpha_{\rm tr}(s,\boldsymbol{x}_\perp)$ is given by

$$\check{\alpha}_{\rm tr}(\omega,\boldsymbol{x}_\perp)=\frac{1}{c_0}\int\check{\mathcal{T}}(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_3=0)[\boldsymbol{u}_1^T\check{\boldsymbol{f}}_{\boldsymbol{u}}+p_1\check{f}_p](\omega,\boldsymbol{x}'_\perp)\mathrm{d}\boldsymbol{x}'_\perp,$$

where the kernel of the transmission operator $\check{\mathcal{T}}(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_3)$ is solution of the paraxial wave equation:

$$\mathrm{d}\check{\mathcal{T}}(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_3)=\frac{ic_0}{2\omega}\Delta_{\boldsymbol{x}'_\perp}\check{\mathcal{T}}(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_3)\mathrm{d}x_3+\frac{i\omega}{2c_0}\check{\mathcal{T}}(\omega,\boldsymbol{x}_\perp,\boldsymbol{x}'_\perp,x_3)\circ\mathrm{d}B(\boldsymbol{x}'_\perp,x_3),$$

starting from $\check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = -L) = \delta(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)$. Here $\Delta_{\boldsymbol{x}'_\perp} = \frac{\partial^2}{\partial {x'_1}^2} + \frac{\partial^2}{\partial {x'_2}^2}$ and B is a Brownian field with the covariance function:

$$
\begin{aligned}
\mathbb{E}[B(\boldsymbol{x}_\perp, x_3)B(\boldsymbol{x}'_\perp, x'_3)] &= \min\{(L + x_3), (L + x'_3)\}\gamma(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp), \\
\gamma(\boldsymbol{x}_\perp) &= \int_{-\infty}^{\infty} \mathbb{E}[(m_\rho + m_K)(\boldsymbol{0}, 0)(m_\rho + m_K)(\boldsymbol{x}_\perp, x_3)]\mathrm{d}x_3.
\end{aligned}
$$

This example shows that the Itô-Schrödinger model is the basic model for wave propagation in the paraxial white-noise regime. This result was obtained in [26].

8.2. Elastic Waves in Isotropic Random Media. We consider linear elastic waves propagating in a three-dimensional medium with heterogeneous and random fluctuations. The linearized elastic wave equations in an isotropic solid have the form [13]

$$
\rho \frac{\partial \xi_k}{\partial t} - \sum_{l=1}^{3} \frac{\partial \tau_{kl}}{\partial x_l} = f_k, \qquad k = 1, 2, 3, \tag{8.2}
$$

$$
\frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j}\right) - \sum_{k,l=1}^{3} S_{ijkl} \frac{\partial \tau_{kl}}{\partial t} = h_{ij}, \qquad i, j = 1, 2, 3, \tag{8.3}
$$

where $(\tau_{kl}(t, \boldsymbol{x}))_{k,l=1,2,3}$ is the (symmetric) dynamic stress, $(\xi_k(t, \boldsymbol{x}))_{k=1,2,3}$ is the velocity, $(f_k(t, \boldsymbol{x}))_{k=1,2,3}$ is the volume source density of force, $h_{ij}(t, \boldsymbol{x})$ is the (symmetric) volume source density of deformation rate,

$$
S_{ijkl}(\boldsymbol{x}) = \frac{1}{4\mu(\boldsymbol{x})}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda(\boldsymbol{x})}{(3\lambda(\boldsymbol{x}) + 2\mu(\boldsymbol{x}))2\mu(\boldsymbol{x})}\delta_{ij}\delta_{kl}, \qquad i, j, k, l = 1, 2, 3
$$

is the compliance, $\rho(\boldsymbol{x})$ is the density of the medium, $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ are the Lamé coefficients of the medium. We also use the Kronecker symbol $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Equations (8.2-8.3) correspond respectively to the equation of motion and the deformation equation for the small amplitude variations around the equilibrium distributions.

If we introduce the pressure p and part of the stress tensor $(\epsilon_{kl})_{k,l=1,2,3}$ that are given respectively by

$$
p(t, \boldsymbol{x}) = \lambda(\boldsymbol{x}) \left[\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}(t, \boldsymbol{x})\right], \tag{8.4}
$$

$$
\epsilon_{kl}(t, \boldsymbol{x}) = \mu(\boldsymbol{x}) \left[\frac{\partial u_l}{\partial x_k}(t, \boldsymbol{x}) + \frac{\partial u_k}{\partial x_l}(t, \boldsymbol{x})\right], \quad k, l = 1, 2, 3, \tag{8.5}
$$

where $(u_j)_{j=1,2,3}$ is the displacement vector, then the stress tensor $(\tau_{kl})_{k,l=1,2,3}$ and the velocity vector $(\xi_k)_{k=1,2,3}$ have the following form in an isotropic solid [13]:

$$
\tau_{kl}(t, \boldsymbol{x}) = p(t, \boldsymbol{x})\delta_{kl} + \epsilon_{kl}(t, \boldsymbol{x}), \qquad k, l = 1, 2, 3, \tag{8.6}
$$

$$
\xi_k(t, \boldsymbol{x}) = \frac{\partial u_k}{\partial t}(t, \boldsymbol{x}), \qquad k = 1, 2, 3. \tag{8.7}
$$

We consider the case in which the medium parameters are randomly perturbed in the region $x_3 \in (-L, 0)$:

$$\frac{1}{\mu(\boldsymbol{x})} = \begin{cases} \mu_0^{-1} & \text{if } x_3 \le -L \text{ or } x_3 \ge 0, \\ \mu_0^{-1}\left[1+\varepsilon^3 m_\mu\left(\frac{\boldsymbol{x}}{\varepsilon^2}\right)\right] & \text{if } x_3 \in (-L,0), \end{cases} \tag{8.8}$$

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if } x_3 \le -L \text{ or } x_3 \ge 0, \\ \rho_0\left[1+\varepsilon^3 m_\rho\left(\frac{\boldsymbol{x}}{\varepsilon^2}\right)\right] & \text{if } x_3 \in (-L,0), \end{cases} \tag{8.9}$$

$$\frac{1}{\lambda(\boldsymbol{x})} = \begin{cases} \lambda_0^{-1} & \text{if } x_3 \le -L \text{ or } x_3 \ge 0, \\ \lambda_0^{-1}\left[1+\varepsilon^3 m_\lambda\left(\frac{\boldsymbol{x}}{\varepsilon^2}\right)\right] & \text{if } x_3 \in (-L,0), \end{cases} \tag{8.10}$$

where m_μ, m_ρ, and m_λ are zero-mean stationary random processes.

We can write the elastic problem in the hyperbolic form (3.1). The ten-dimensional elastic field is characterized by the three-dimensional velocity field $\boldsymbol{\xi}(t,\boldsymbol{x})$, the six-dimensional stress field $\boldsymbol{\epsilon}(t,\boldsymbol{x}) = (\epsilon_{11},\epsilon_{22},\epsilon_{33},\epsilon_{23},\epsilon_{13},\epsilon_{12})^T$ and the scalar pressure field $p(t,\boldsymbol{x})$:

$$\mathbf{C}_0 \frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\epsilon}\\ p\end{pmatrix} + \sum_{k=1}^3 \mathbf{D}_k \frac{\partial}{\partial x_k}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\epsilon}\\ p\end{pmatrix} = \begin{pmatrix}\boldsymbol{f}_{\boldsymbol{\xi}}\\ \boldsymbol{f}_{\boldsymbol{\epsilon}}\\ 0\end{pmatrix}\left(\frac{t}{\varepsilon^4},\frac{\boldsymbol{x}_\perp}{\varepsilon^2}\right)\delta(x_3),$$

with $\boldsymbol{f}_{\boldsymbol{\xi}} = \boldsymbol{f}$,

$$f_{\boldsymbol{\epsilon} jk}(t,\boldsymbol{x}_\perp) = -2\mu_0 h_{jk}(t,\boldsymbol{x}) - \lambda_0\Big[\sum_{l=1}^3 h_{ll}(t,\boldsymbol{x})\Big]\delta_{jk},$$

$\mathbf{C}^\varepsilon(\boldsymbol{x}) = \mathbf{C}_0 + \varepsilon^3\mathbf{C}_1(x/\varepsilon^2)$, $\mathbf{C}_1(\boldsymbol{x}) = m_\rho(\boldsymbol{x})\mathbf{h}^\rho + m_\mu(\boldsymbol{x})\mathbf{h}^\mu + m_\lambda(\boldsymbol{x})\mathbf{h}^\lambda$,

$$\mathbf{C}_0 = \begin{pmatrix} \rho_0\mathbf{I} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \frac{1}{2\mu_0}\mathbf{I} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \frac{1}{\mu_0}\mathbf{I} & \boldsymbol{e}_0^T \\ \boldsymbol{e}_0 & \boldsymbol{e}_0 & \boldsymbol{e}_0 & \frac{1}{\lambda_0} \end{pmatrix}, \qquad \mathbf{h}^\rho = \begin{pmatrix} \rho_0\mathbf{I} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \boldsymbol{e}_0 & \boldsymbol{e}_0 & \boldsymbol{e}_0 & 0 \end{pmatrix},$$

$$\mathbf{h}^\mu = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \frac{1}{2\mu_0}\mathbf{I} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \frac{1}{\mu_0}\mathbf{I} & \boldsymbol{e}_0^T \\ \boldsymbol{e}_0 & \boldsymbol{e}_0 & \boldsymbol{e}_0 & 0 \end{pmatrix}, \qquad \mathbf{h}^\lambda = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \boldsymbol{e}_0 & \boldsymbol{e}_0 & \boldsymbol{e}_0 & \frac{1}{\lambda_0} \end{pmatrix},$$

$$\mathbf{D}_k = \begin{pmatrix} \mathbf{0} & \mathbf{U}_k & \mathbf{V}_k & \boldsymbol{e}_k^T \\ \mathbf{U}_k & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \mathbf{V}_k & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_0^T \\ \boldsymbol{e}_k & \boldsymbol{e}_0 & \boldsymbol{e}_0 & 0 \end{pmatrix}, \quad k = 1,2,3,$$

where $\boldsymbol{e}_0 = (0,0,0)^T$, $\boldsymbol{e}_1 = (1,0,0)^T$, $\boldsymbol{e}_2 = (0,1,0)^T$, and $\boldsymbol{e}_3 = (0,0,1)^T$, $\mathbf{0}$ is the 3×3 null matrix, $\mathbf{I}$ is the 3×3 identity matrix,

$$\mathbf{U}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{V}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are ten eigenvalues of $\mathbf{C}_0^{-1}\mathbf{D}_3$:

$$c_1 = -c_P, \quad c_2 = c_3 = -c_S, \quad c_4 = c_5 = c_6 = c_7 = 0, \quad c_8 = c_9 = c_S, \quad c_{10} = c_P,$$

with c_S and c_P the shear-wave and pressure-wave speeds of propagation:

$$c_S = \frac{\sqrt{\mu_0}}{\sqrt{\rho_0}}, \qquad c_P = \frac{\sqrt{2\mu_0 + \lambda_0}}{\sqrt{\rho_0}}. \tag{8.11}$$

The associated eigenvectors $\boldsymbol{u}_j$, $j = 1, \ldots, 10$, are

$$\begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\epsilon}_1 \\ p_1 \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} -\boldsymbol{e}_3 \\ \zeta_P \boldsymbol{e}_3 \\ \boldsymbol{e}_0 \\ \frac{\lambda_0}{2\mu_0}\zeta_P \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_2 \\ \boldsymbol{\epsilon}_2 \\ p_2 \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} -\boldsymbol{e}_1 \\ \boldsymbol{e}_0 \\ \zeta_S \boldsymbol{e}_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_3 \\ \boldsymbol{\epsilon}_3 \\ p_3 \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} -\boldsymbol{e}_2 \\ \boldsymbol{e}_0 \\ \zeta_S \boldsymbol{e}_1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\xi}_4 \\ \boldsymbol{\epsilon}_4 \\ p_4 \end{pmatrix} = \sqrt{2\mu_0} \begin{pmatrix} \boldsymbol{e}_0 \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_5 \\ \boldsymbol{\epsilon}_5 \\ p_5 \end{pmatrix} = \sqrt{2\mu_0} \begin{pmatrix} \boldsymbol{e}_0 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\xi}_6 \\ \boldsymbol{\epsilon}_6 \\ p_6 \end{pmatrix} = \sqrt{\mu_0} \begin{pmatrix} \boldsymbol{e}_0 \\ \boldsymbol{e}_0 \\ \boldsymbol{e}_3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_7 \\ \boldsymbol{\epsilon}_7 \\ p_7 \end{pmatrix} = \frac{\sqrt{2\lambda_0\mu_0}}{\sqrt{\lambda_0 + 2\mu_0}} \begin{pmatrix} \boldsymbol{e}_0 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_0 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\xi}_8 \\ \boldsymbol{\epsilon}_8 \\ p_8 \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_0 \\ \zeta_S \boldsymbol{e}_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_9 \\ \boldsymbol{\epsilon}_9 \\ p_9 \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_0 \\ \zeta_S \boldsymbol{e}_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\xi}_{10} \\ \boldsymbol{\epsilon}_{10} \\ p_{10} \end{pmatrix} = \frac{1}{\sqrt{2\rho_0}} \begin{pmatrix} \boldsymbol{e}_3 \\ \zeta_P \boldsymbol{e}_3 \\ \boldsymbol{e}_0 \\ \frac{\lambda_0}{2\mu_0}\zeta_P \end{pmatrix},$$

where ζ_S and ζ_P are the shear-wave and pressure-wave impedances:

$$\zeta_S = \sqrt{\mu_0 \rho_0}, \qquad \zeta_P = \frac{2\mu_0\sqrt{\rho_0}}{\sqrt{2\mu_0 + \lambda_0}}. \tag{8.12}$$

The first mode (and the tenth mode) is a pressure wave mode. The second and third modes (and the eighth and ninth modes) are shear wave modes.

Hypotheses H1-H4 are satisfied. Note that $\boldsymbol{u}_6^T \mathbf{D}_2 \boldsymbol{u}_2 \neq 0$ and $\boldsymbol{u}_6^T \mathbf{D}_1 \boldsymbol{u}_3 \neq 0$; $\boldsymbol{u}_j^T \mathbf{D}_1 \boldsymbol{u}_2 \neq 0$ and $\boldsymbol{u}_j^T \mathbf{D}_2 \boldsymbol{u}_3 \neq 0$ for $j = 1, 7, 10$, but we have $\Gamma_{12}^{(2,k,k')} = 0$ which ensures that Hypothesis H3 is satisfied. By applying Proposition 5.2 we find that the transmitted wave at $x_3 = -L$ consists of a pressure wave and a shear wave that emerge around times L/c_P and L/c_S. On the one hand we have for the pressure wave mode

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\epsilon} \\ p \end{pmatrix} \Big(t = \frac{L}{c_P} + \varepsilon^4 s, \varepsilon^2 \boldsymbol{x}_\perp, x_3 = -L \Big) \xrightarrow{\varepsilon \to 0} \alpha_P(s, \boldsymbol{x}_\perp) \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\epsilon}_1 \\ p_1 \end{pmatrix}.$$

The Fourier transform $\check{\alpha}_P(\omega, \boldsymbol{x}_\perp)$ of the wave amplitude $\alpha_P(s, \boldsymbol{x}_\perp)$ is given by

$$\check{\alpha}_P(\omega, \boldsymbol{x}_\perp) = \frac{1}{c_P} \int \check{\mathcal{T}}_P(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = 0)[\boldsymbol{\xi}_1^T \check{\boldsymbol{f}}_{\boldsymbol{\xi}} + \boldsymbol{\epsilon}_1^T \check{\boldsymbol{f}}_{\boldsymbol{\epsilon}}](\omega, \boldsymbol{x}'_\perp) \mathrm{d}\boldsymbol{x}'_\perp,$$

On the other hand we have for the shear wave modes

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\epsilon} \\ p \end{pmatrix} \Big(t = \frac{L}{c_S} + \varepsilon^4 s, \varepsilon^2 \boldsymbol{x}_\perp, x_3 = -L \Big) \xrightarrow{\varepsilon \to 0} \alpha_{S,2}(s, \boldsymbol{x}_\perp) \begin{pmatrix} \boldsymbol{\xi}_2 \\ \boldsymbol{\epsilon}_2 \\ p_2 \end{pmatrix} + \alpha_{S,3}(s, \boldsymbol{x}_\perp) \begin{pmatrix} \boldsymbol{\xi}_3 \\ \boldsymbol{\epsilon}_3 \\ p_3 \end{pmatrix}.$$

The Fourier transform $\check{\alpha}_{S,j}(\omega, \boldsymbol{x}_\perp)$ of the wave amplitude $\alpha_{S,j}(s, \boldsymbol{x}_\perp)$ is given by

$$\check{\alpha}_{S,j}(\omega, \boldsymbol{x}_\perp) = \frac{1}{c_S} \int \check{\mathcal{T}}_S(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = 0)[\boldsymbol{\xi}_j^T \check{\boldsymbol{f}}_{\boldsymbol{\xi}} + \boldsymbol{\epsilon}_j^T \check{\boldsymbol{f}}_{\boldsymbol{\epsilon}}](\omega, \boldsymbol{x}'_\perp) \mathrm{d}\boldsymbol{x}'_\perp, \qquad j = 2, 3.$$

The kernels of the transmission operators satisfy, for $q \in \{P, S\}$

$$\begin{aligned} \mathrm{d}\check{\mathcal{T}}_q(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) = {} & \frac{i c_q}{2\omega} \Delta_{\boldsymbol{x}'_\perp} \check{\mathcal{T}}_q(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) \mathrm{d}x_3 \\ & + \frac{i\omega}{2c_q} \check{\mathcal{T}}_q(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) \circ \mathrm{d}B_q(\boldsymbol{x}'_\perp, x_3), \end{aligned}$$

with the initial conditions $\check{\mathcal{T}}_q(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = -L) = \delta(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)$, $q \in \{P, S\}$. Here $B_P(\boldsymbol{x}_\perp, x_3)$ and $B_S(\boldsymbol{x}_\perp, x_3)$ are two correlated Brownian fields with the covariance

$$\begin{aligned} \mathbb{E}[B_q(\boldsymbol{x}_\perp, x_3) B_r(\boldsymbol{x}'_\perp, x'_3)] &= \min\{(L + x_3), (L + x'_3)\} \boldsymbol{\gamma}_{qr}(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp), \\ \gamma_{qr}(\boldsymbol{x}_\perp) &= \int_{-\infty}^{\infty} \mathbb{E}[m_q(\boldsymbol{0}, 0) m_r(\boldsymbol{x}_\perp, x_3)] \mathrm{d}x_3, \qquad q, r \in \{P, S\}, \end{aligned}$$

with

$$m_S(\boldsymbol{x}) = m_\mu(\boldsymbol{x}) + m_\rho(\boldsymbol{x}), \qquad m_P(\boldsymbol{x}) = \frac{2\mu_0}{2\mu_0 + \lambda_0} m_\mu(\boldsymbol{x}) + m_\rho(\boldsymbol{x}) + \frac{\lambda_0}{2\mu_0 + \lambda_0} m_\lambda(\boldsymbol{x}).$$

This example shows that in the paraxial white-noise regime and in the vector case wave propagation is governed by a set of Itô-Schrödinger equations that are driven by a set of correlated Brownian fields. The Itô-Schrödinger equations are dynamically uncoupled, but coupled statistically. Note in particular that the Brownian motions driving the two shear modes are perfectly correlated, and that they are partially correlated with the one driving the pressure mode. A preliminary version of this result was obtained in [27].

8.3. Electromagnetic Waves in Isotropic Random Media. We consider electromagnetic waves propagating in a three-dimensional randomly heterogeneous medium. Maxwell's equations for the electric field $\boldsymbol{E}(t, \boldsymbol{x})$ and the magnetic field $\boldsymbol{H}(t, \boldsymbol{x})$ are

$$\nabla \times \boldsymbol{E} = -\partial_t(\mu(\boldsymbol{x})\boldsymbol{H}), \qquad \nabla \cdot (\epsilon_\chi(\boldsymbol{x})\boldsymbol{E}) = \rho, \tag{8.13}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{(s)}\Big(\frac{t}{\varepsilon^4}, \frac{\boldsymbol{x}_\perp}{\varepsilon^2}\Big)\delta(x_3) + \partial_t(\epsilon_\chi(\boldsymbol{x})\boldsymbol{E}), \qquad \nabla \cdot (\mu(\boldsymbol{x})\boldsymbol{H}) = 0. \tag{8.14}$$

The term $\boldsymbol{J}^{(s)}$ is a current source term, ρ is the charge density (the equation of continuity of charge $\partial_t \rho + \nabla \cdot \boldsymbol{J}^{(s)} = 0$ is automatically satisfied), $\mu(\boldsymbol{x})$ is the magnetic permeability of the medium, and $\epsilon_\chi(\boldsymbol{x})$ is the dielectric permittivity of the medium. They have the form:

$$\mu(\boldsymbol{x}) = \begin{cases} \mu_0 & \text{if } x_3 \le -L \text{ or } x_3 \ge 0, \\ \mu_0\big[1+\varepsilon^3 m_\mu(\frac{\boldsymbol{x}}{\varepsilon^2})\big] & \text{if } x_3 \in (-L,0), \end{cases}$$
$$\epsilon_\chi(\boldsymbol{x}) = \begin{cases} \epsilon_0 & \text{if } x_3 \le -L \text{ or } x_3 \ge 0, \\ \epsilon_0\big[1+\varepsilon^3 m_\chi(\frac{\boldsymbol{x}}{\varepsilon^2})\big] & \text{if } x_3 \in (-L,0), \end{cases}$$

where the random processes $m_\mu(\boldsymbol{x})$ and $m_\chi(\boldsymbol{x})$ model the medium fluctuations. We can write the electromagnetic problem in the hyperbolic form (3.1). The six-dimensional electromagnetic field is characterized by the three-dimensional electric field $\boldsymbol{E}(t,\boldsymbol{x})$ and the three-dimensional magnetic field $\boldsymbol{H}(t,\boldsymbol{x})$. It satisfies the hyperbolic system:

$$\mathbf{C}^\varepsilon(\boldsymbol{x})\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} + \sum_{k=1}^3 \mathbf{D}_k\frac{\partial}{\partial x_k}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \begin{pmatrix}-\boldsymbol{J}^{(s)}\\ \boldsymbol{e}_0\end{pmatrix}\Big(\frac{t}{\varepsilon^4},\frac{\boldsymbol{x}_\perp}{\varepsilon^2}\Big)\delta(x_3),$$

with $\mathbf{C}^\varepsilon(\boldsymbol{x}) = \mathbf{C}_0 + \varepsilon^3\mathbf{C}_1(\boldsymbol{x}/\varepsilon^2)$, $\mathbf{C}_1(\boldsymbol{x}) = m_\chi(\boldsymbol{x})\mathbf{h}^\chi + m_\mu(\boldsymbol{x})\mathbf{h}^\mu$,

$$\mathbf{C}_0 = \begin{pmatrix}\epsilon_0\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mu_0\mathbf{I}\end{pmatrix}, \qquad \mathbf{D}_k = \begin{pmatrix}\mathbf{0} & \mathbf{W}_k\\ \mathbf{W}_k^T & \mathbf{0}\end{pmatrix}, \quad k=1,2,3,$$

$$\mathbf{W}_1 = \begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}, \quad \mathbf{W}_3 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},$$

$$\mathbf{h}^\chi = \begin{pmatrix}\epsilon_0\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \qquad \mathbf{h}^\mu = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mu_0\mathbf{I}\end{pmatrix},$$

$\mathbf{0}$ is the 3×3 null matrix, and $\mathbf{I}$ is the 3×3 identity matrix. The six eigenvalues of $\mathbf{C}_0^{-1}\mathbf{D}_3$ are:

$$c_1 = c_2 = -c_0, \quad c_3 = c_4 = 0, \quad c_5 = c_6 = c_0,$$

where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the electromagnetic speed of propagation. The associated eigenvectors are:

$$\begin{pmatrix}\boldsymbol{E}_1\\ \boldsymbol{H}_1\end{pmatrix} = \frac{1}{\sqrt{2\mu_0}}\begin{pmatrix}\zeta_0\boldsymbol{e}_1\\ -\boldsymbol{e}_2\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{E}_2\\ \boldsymbol{H}_2\end{pmatrix} = \frac{1}{\sqrt{2\mu_0}}\begin{pmatrix}\zeta_0\boldsymbol{e}_2\\ \boldsymbol{e}_1\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{E}_3\\ \boldsymbol{H}_3\end{pmatrix} = \frac{1}{\sqrt{\mu_0}}\begin{pmatrix}\zeta_0\boldsymbol{e}_3\\ \boldsymbol{e}_0\end{pmatrix},$$

$$\begin{pmatrix}\boldsymbol{E}_4\\ \boldsymbol{H}_4\end{pmatrix} = \frac{1}{\sqrt{\mu_0}}\begin{pmatrix}\boldsymbol{e}_0\\ \boldsymbol{e}_3\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{E}_5\\ \boldsymbol{H}_5\end{pmatrix} = \frac{1}{\sqrt{2\mu_0}}\begin{pmatrix}\zeta_0\boldsymbol{e}_2\\ -\boldsymbol{e}_1\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{E}_6\\ \boldsymbol{H}_6\end{pmatrix} = \frac{1}{\sqrt{2\mu_0}}\begin{pmatrix}\zeta_0\boldsymbol{e}_1\\ \boldsymbol{e}_2\end{pmatrix},$$

where $\boldsymbol{e}_0 = (0,0,0)^T$, $\boldsymbol{e}_1 = (1,0,0)^T$, $\boldsymbol{e}_2 = (0,1,0)^T$, $\boldsymbol{e}_3 = (0,0,1)^T$, and $\zeta_0 = \sqrt{\mu_0}/\sqrt{\epsilon_0}$ is the impedance.

Hypotheses H1-H4 are satisfied. By applying Proposition 5.2 we find that the transmitted wave at $x_3 = -L$ consists of the superposition of two correlated wave modes that emerge around time L/c_0:

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}\Big(t = \frac{L}{c_0} + \varepsilon^4 s, \varepsilon^2\boldsymbol{x}_\perp, x_3 = -L\Big) \overset{\varepsilon\to 0}{\longrightarrow} \alpha_1(s,\boldsymbol{x}_\perp)\begin{pmatrix}\boldsymbol{E}_1\\ \boldsymbol{H}_1\end{pmatrix} + \alpha_2(s,\boldsymbol{x}_\perp)\begin{pmatrix}\boldsymbol{E}_2\\ \boldsymbol{H}_2\end{pmatrix}. \quad (8.15)$$

For $j = 1, 2$, the Fourier transform $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$ of the wave amplitude $\alpha_j(s, \boldsymbol{x}_\perp)$ is given by

$$\check{\alpha}_j(\omega, \boldsymbol{x}_\perp) = -\frac{1}{c_0} \int \check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = 0) \boldsymbol{E}_j^T \check{\boldsymbol{J}}^{(s)}(\omega, \boldsymbol{x}'_\perp) \mathrm{d}\boldsymbol{x}'_\perp,$$

where the kernel $\check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3)$ satisfies

$$\mathrm{d}\check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) = \frac{ic_0}{2\omega} \Delta_{\boldsymbol{x}'_\perp} \check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) \mathrm{d}x_3 + \frac{i\omega}{2c_0} \check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3) \circ \mathrm{d}B(\boldsymbol{x}'_\perp, x_3),$$

starting from $\check{\mathcal{T}}(\omega, \boldsymbol{x}_\perp, \boldsymbol{x}'_\perp, x_3 = -L) = \delta(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)$. Here $\Delta_{\boldsymbol{x}'_\perp} = \frac{\partial^2}{\partial {x'_1}^2} + \frac{\partial^2}{\partial {x'_2}^2}$ and B is a Brownian field with the covariance function:

$$\mathbb{E}[B(\boldsymbol{x}_\perp, x_3) B(\boldsymbol{x}'_\perp, x'_3)] = \min\{(L + x_3), (L + x'_3)\} \gamma(\boldsymbol{x}'_\perp - \boldsymbol{x}_\perp),$$
$$\gamma(\boldsymbol{x}_\perp) = \int_{-\infty}^{\infty} \mathbb{E}[(m_\chi + m_\mu)(\boldsymbol{0}, 0)(m_\chi + m_\mu)(\boldsymbol{x}_\perp, x_3)] \mathrm{d}x_3.$$

Note in particular that there is a unique Brownian motion driving the two electromagnetic modes, as shown originally in [28].

8.4. Electromagnetic Waves in Anisotropic Homogeneous Media. We consider electromagnetic waves propagating in a three-dimensional anisotropic homogeneous medium. Maxwell's equations for the electric field $\boldsymbol{E}(t, \boldsymbol{x})$ and the magnetic field $\boldsymbol{H}(t, \boldsymbol{x})$ are

$$\nabla \times \boldsymbol{E} = -\partial_t(\mu_0 \boldsymbol{H}), \qquad \nabla \cdot (\boldsymbol{\epsilon}_\chi \boldsymbol{E}) = \rho, \tag{8.16}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{(s)}\big(\frac{t}{\varepsilon^4}, \frac{\boldsymbol{x}_\perp}{\varepsilon^2}\big)\delta(x_3) + \partial_t(\boldsymbol{\epsilon}_\chi \boldsymbol{E}), \qquad \nabla \cdot (\mu_0 \boldsymbol{H}) = 0. \tag{8.17}$$

Here $\boldsymbol{J}^{(s)}$ is the current source term, ρ is the charge density, μ_0 is the magnetic permeability of the medium, and $\boldsymbol{\epsilon}_\chi$ is the dielectric permittivity tensor of the medium. We consider a uniaxial crystal whose permittivity tensor is:

$$\boldsymbol{\epsilon}_\chi = \begin{pmatrix} \epsilon_0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix},$$

with $\epsilon_1 \neq \epsilon_0$. This is a special situation in which the optical axis of the crystal coincides with the propagation axis (x_d). The general case can be addressed but leads to additional phenomena such as angular walk-off that we will not address in this paper. We refer the interested reader to [22] for instance.

We can write the electromagnetic problem in the hyperbolic form (3.1). The six-dimensional electromagnetic field satisfies the hyperbolic system:

$$\mathbf{C}_0 \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} + \sum_{k=1}^{3} \mathbf{D}_k \frac{\partial}{\partial x_k} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{J}^{(s)} \\ \boldsymbol{e}_0 \end{pmatrix} \big(\frac{t}{\varepsilon^4}, \frac{\boldsymbol{x}_\perp}{\varepsilon^2}\big), \text{ with } \mathbf{C}_0 = \begin{pmatrix} \boldsymbol{\epsilon}_\chi & \boldsymbol{0} \\ \boldsymbol{0} & \mu_0 \mathbf{I} \end{pmatrix},$$

and the matrices $\mathbf{D}_k$ are defined as in the previous subsection. The eigenvalues are the same as in the isotropic case as well as the eigenvectors, except the third one which is now given by

$$\begin{pmatrix} \boldsymbol{E}_3 \\ \boldsymbol{H}_3 \end{pmatrix} = \frac{\sqrt{\epsilon_0}}{\sqrt{\mu_0}\sqrt{\epsilon_1}} \begin{pmatrix} \zeta_0 \boldsymbol{e}_3 \\ \boldsymbol{e}_0 \end{pmatrix}.$$

Hypotheses H1-H2 are fulfilled, but not H3 because of the new form of the third eigenvector. As a consequence, by applying Proposition 5.1 we find that the transmitted wave at $x_3 = -L$ consists of the superposition (8.15) of two wave modes that emerge around time L/c_0, but now the Fourier transforms $\check{\alpha}_j(\omega, \boldsymbol{x}_\perp)$ of the wave amplitude $\alpha_j(s, \boldsymbol{x}_\perp)$ are given by

$$\begin{pmatrix} \check{\alpha}_1(\omega, \boldsymbol{x}_\perp) \\ \check{\alpha}_2(\omega, \boldsymbol{x}_\perp) \end{pmatrix} = -\frac{1}{c_0} \int \check{\mathbf{T}}(\omega, \boldsymbol{x}_\perp - \boldsymbol{x}'_\perp, x_3 = 0) \begin{pmatrix} \boldsymbol{E}_1^T \check{\boldsymbol{J}}^{(s)}(\omega, \boldsymbol{x}'_\perp) \\ \boldsymbol{E}_2^T \check{\boldsymbol{J}}^{(s)}(\omega, \boldsymbol{x}'_\perp) \end{pmatrix} \mathrm{d}\boldsymbol{x}'_\perp ,$$

where the 2×2 matrix $\check{\mathbf{T}}(\omega, \boldsymbol{x}_\perp, x_3)$ satisfies

$$\frac{\partial \check{\mathbf{T}}}{\partial x_3} = \frac{i c_0}{2\omega} \Big[\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2 \check{\mathbf{T}}}{\partial x_1^2} + \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \frac{\partial^2 \check{\mathbf{T}}}{\partial x_2^2} + (\rho - 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial^2 \check{\mathbf{T}}}{\partial x_1 \partial x_2} \Big],$$

starting from $\check{\mathbf{T}}(\omega, \boldsymbol{x}_\perp, x_3 = -L) = \delta(\boldsymbol{x}_\perp)\mathbf{I}$. Here $\mathbf{I}$ is the 2×2 identity matrix and $\rho = \epsilon_0/\epsilon_1$. This result can also be stated as follows: the transverse electric field $(\check{E}_j(\omega, \boldsymbol{x}_\perp, x_3))_{j=1,2}$ satisfies the Schrödinger system

$$\begin{aligned}
\frac{\partial \check{E}_1}{\partial x_3} &= -\frac{i c_0}{2\omega} \Big[\frac{\partial^2 \check{E}_1}{\partial x_1^2} + \rho \frac{\partial^2 \check{E}_1}{\partial x_2^2} + (\rho - 1) \frac{\partial^2 \check{E}_2}{\partial x_1 \partial x_2} \Big], \\
\frac{\partial \check{E}_2}{\partial x_3} &= -\frac{i c_0}{2\omega} \Big[\rho \frac{\partial^2 \check{E}_2}{\partial x_1^2} + \frac{\partial^2 \check{E}_2}{\partial x_2^2} + (\rho - 1) \frac{\partial^2 \check{E}_1}{\partial x_1 \partial x_2} \Big],
\end{aligned}$$

for $x_3 \in (-L, 0)$, starting from $\check{E}_j(\omega, \boldsymbol{x}_\perp, x_3 = 0) = -(\sqrt{\zeta_0}/2)\check{J}_j^{(s)}(\omega, \boldsymbol{x}_\perp)$. We can clearly see in these equations that diffractive terms are more complicated than the standard transverse Laplacian. The Shrödinger equations for the two electromagnetic modes are dynamically coupled through the complex form of the diffraction operator. This example shows that the paraxial approximation can hold true in anisotropic media but then the diffraction can be anisotropic as well.

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9. Conclusion. In this paper we have used invariant imbedding and limit theorems to obtain the paraxial equations for general vector waves in random media in the paraxial white-noise regime. We have shown that the paraxial systems for the wave modes have the form of a system of Schrödinger equations that are driven by a set of correlated Brownian fields. We have identified the structure of the correlations between the Brownian fields and between the wave fields and we have clarified the form of the diffraction operators. These equations give a new and efficient way of modeling wave propagation in a consistent manner via averaging and diffusion-approximation results. The results presented here are for the transmitted field. However, the framework can be used to analyze also the wave field reflected by a strong interface or the relatively weak reflections generated by the medium heterogeneities. In the context of acoustic waves this is discussed in [26, 29, 30].

Appendix A. An Invariant Imbedding Theorem. Let us consider the two-point boundary value problem:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}z}(z) = \mathbf{A}(z)\boldsymbol{X}(z) , \qquad \boldsymbol{X}(z) \in \mathbb{R}^m , \tag{A.1}$$

with the boundary condition $\mathbf{H}^{-L}\boldsymbol{X}(-L)+\mathbf{H}^0\boldsymbol{X}(0)=\boldsymbol{V}^0$, where $\mathbf{A}(z)$, $\mathbf{H}^{-L}$ and $\mathbf{H}^0$ are $m\times m$-matrices and $\boldsymbol{V}^0$ is an m-dimensional vector. Assume that $\mathbf{H}^{-L}+\mathbf{H}^0$ is invertible. In this linear framework the invariant imbedding approach leads to the following proposition [6].

Proposition A.1. *Let us define the $m\times m$-matrix-valued functions $(\mathbf{R}(\zeta))_{-L\le\zeta\le 0}$ and $(\mathbf{Q}(z,\zeta))_{-L\le z\le\zeta\le 0}$ as the solutions of the initial value problems:*

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\zeta}(\zeta)=\mathbf{A}(\zeta)\mathbf{R}(\zeta)-\mathbf{R}(\zeta)\mathbf{H}^0\mathbf{A}(\zeta)\mathbf{R}(\zeta)\,,\qquad -L\le\zeta\le 0\,, \tag{A.2}$$

starting from $\zeta=-L$: $\mathbf{R}(\zeta=-L)=(\mathbf{H}^{-L}+\mathbf{H}^0)^{-1}$, and

$$\frac{\partial\mathbf{Q}}{\partial\zeta}(z,\zeta)=-\mathbf{Q}(z,\zeta)\mathbf{H}^0\mathbf{A}(\zeta)\mathbf{R}(\zeta)\,,\qquad -L\le z\le\zeta\le 0\,, \tag{A.3}$$

starting from $\zeta=z$: $\mathbf{Q}(z,\zeta=z)=\mathbf{R}(z)$. If the solution $\mathbf{R}(\zeta)$ of the nonlinear initial value problem (A.2) exists up to $\zeta=0$, then $\mathbf{P}(z)=\mathbf{Q}(z,0)$ is the solution of:

$$\begin{aligned}&\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}z}(z)=\mathbf{A}(z)\mathbf{P}(z)\,,\qquad -L\le z\le 0\,,\\ &\mathbf{H}^{-L}\mathbf{P}(-L)+\mathbf{H}^0\mathbf{P}(0)=\mathbf{I}\,,\end{aligned}$$

and consequently $\boldsymbol{X}(z)=\mathbf{P}(z)\boldsymbol{V}^0$ is the solution of (A.1) with the boundary condition $\mathbf{H}^{-L}\boldsymbol{X}(-L)+\mathbf{H}^0\boldsymbol{X}(0)=\boldsymbol{V}^0$.

Appendix B. An Averaging Theorem. The following proposition can be found in [36].

Proposition B.1. *Let us consider the solution $\mathbf{X}^\varepsilon=(X^\varepsilon_{jk})_{j,k=1,\dots,N}$ of the initial value problem*

$$\frac{\mathrm{d}\mathbf{X}^\varepsilon}{\mathrm{d}z}=\mathbf{F}^{(0)}(\mathbf{X}^\varepsilon)+\frac{1}{\varepsilon^2}\sum_{q=1}^{Q}\Big(\mathbf{F}^{(q)}(\mathbf{X}^\varepsilon)\exp\Big(i\frac{\alpha^{(q)}z}{\varepsilon^4}\Big)+\mathbf{G}^{(q)}(\mathbf{X}^\varepsilon)\exp\Big(-i\frac{\alpha^{(q)}z}{\varepsilon^4}\Big)\Big)\,,$$

starting from $\mathbf{X}^\varepsilon(z_0)=\mathbf{X}_{\rm ini}$. Here $\mathbf{F}^{(q)}:\mathbb{R}^{N^2}\to\mathbb{R}^{N^2}$ and $\mathbf{G}^{(q)}:\mathbb{R}^{N^2}\to\mathbb{R}^{N^2}$ are smooth functions with bounded derivatives and the $\alpha^{(q)}\in(0,\infty)$ are distinct. Then $\mathbf{X}^\varepsilon$ converges (uniformly for z over compact intervals) as $\varepsilon\to 0$ to $\mathbf{X}$ the solution of the effective equation

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}z}=\mathbf{F}^{(0)}(\mathbf{X})+\sum_{q=1}^{Q}\frac{i}{\alpha^{(q)}}\big[\mathbf{F}^{(q)}(\mathbf{X}),\mathbf{G}^{(q)}(\mathbf{X})\big]\,,\qquad \mathbf{X}(z_0)=\mathbf{X}_{\rm ini}\,,$$

where $[\cdot,\cdot]$ stands for the Lie brackets defined by

$$\big[\mathbf{F}(\mathbf{X}),\mathbf{G}(\mathbf{X})\big]=\sum_{j,k=1}^{N}\Big(\frac{\partial\mathbf{F}(\mathbf{X})}{\partial X_{jk}}G_{jk}(\mathbf{X})-\frac{\partial\mathbf{G}(\mathbf{X})}{\partial X_{jk}}F_{jk}(\mathbf{X})\Big)\,.$$

REFERENCES

[1] Andrews, L.C. , Phillips, R.L.: Laser Beam Propagation through Random Media. SPIE Press, Bellingham (2005)

[2] Bailly, F., Clouet, J.-F., Fouque, J.-P.: Parabolic and white noise approximation for waves in random media. SIAM J. Appl. Math. 56, 1445-1470 (1996)
[3] Bal, G.: On the self-averaging of wave energy in random media. SIAM Multiscale Model. Simul. 2, 398-420 (2004)
[4] Bal, G., Pinaud, O.: Dynamics of wave scintillation in random media, Comm. Part. Differ. Equat. 35, 1176-1235 (2010)
[5] Bal, G., Papanicolaou, G., Ryzhik, L.: Self-averaging in time reversal for the parabolic wave equation. Stochastics and Dynamics 2, 507-532 (2002)
[6] Bellman, R., Wing, G.M.: Introduction to Invariant Imbedding. Wiley, New York (1975)
[7] Blomgren, P., Papanicolaou, G., Zhao, H.: Super-resolution in time-reversal acoustics. J. Acoust. Soc. Am. 111, 230-248 (2002)
[8] Borcea, L., Garnier, J.: Paraxial coupling of propagating modes in three-dimensional waveguides with random boundaries. SIAM Multiscale Model. Simul. 12, 832-878 (2014)
[9] Carmona, R., Wang, L.: Monte Carlo Malliavin computation of the sensitivities of solutions of SPDEs. SIAM J. Appl. Math. 69, 1682-1711 (2009)
[10] Claerbout, J.F.: Imaging the Earth's Interior. Blackwell Scientific Publications, Palo Alto (1985)
[11] Courant, R., Hilbert, D.: Methods of Mathematical Physics, Volume 2. Wiley, New York (1962)
[12] Dawson, D., Papanicolaou, G.: A random wave process. Appl. Math. Optim. 12, 97-114 (1984)
[13] de Hoop, A.T.: Handbook of Radiation and Scattering of Waves, London, Academic Press, Chapter 10, 1995. Available via http://www.atdehoop.com
[14] de Hoop, M., Hörmann, G., Oberguggenberger, M.: Evolution systems for paraxial wave equations of Schrödinger-type with non-smooth coefficients. J. Diff. Eqs. 245, 1413-1432 (2008)
[15] de Hoop, M.V., Garnier, J., Sølna, K.: Retrieval of a Green's function with reflections from partly coherent waves generated by a wave packet using cross correlations. SIAM J. Appl. Math. 73, 493-522 (2013)
[16] Fannjiang, A.: Self-averaging scaling limits for random parabolic waves. Arch. Rational Mech. Anal. 175, 343-387 (2005)
[17] Fannjiang, A.: Introduction to propagation, time reversal and imaging in random media. In Multi-Scale Phenomena in Complex Fluids -Modeling, Analysis and Numerical Simulation, Hou, T.Y, Liu, C., and J.G. Liu, J.G., eds., World Scientific, 2009.
[18] Fannjiang, A., Sølna, K.: Propagation and time-reversal of wave beams in atmospheric turbulence. SIAM Multiscale Model. Simul. 3, 522-558 (2005)
[19] Fouque, J.-P., Garnier, J., Papanicolaou, G., Sølna, K.: Wave Propagation and Time Reversal in Randomly Layered Media. Springer, New York (2007)
[20] Fouque, J.-P., Papanicolaou, G., Samuelides, Y.: Forward and Markov approximation: the strong-intensity-fluctuations regime revisited. Waves in Random Media 8, 303-314 (1998)
[21] Frank, S.D., Odom, R.I., Collis, J.M.: Elastic parabolic equation solutions for underwater acoustic problems using seismic sources. J. Acoust. Soc. Am. 133, 1358-1367 (2013)
[22] Garnier, J.: High-frequency asymptotics for Maxwell's equations in anisotropic media. Part I: Linear geometric and diffractive optics. J. Math. Phys. 42, 1612-1635 (2001)
[23] Garnier, J., Papanicolaou, G.: Correlation based virtual source imaging in strongly scattering media. Inverse Problems 28, 075002 (2012)
[24] Garnier, J., Papanicolaou, G.: Role of scattering in virtual source array imaging. SIAM J. Imaging Sciences 7, 1210-1236 (2014)
[25] Garnier, J., Sølna, K.: Effective transport equations and enhanced backscattering in random waveguides. SIAM J. Appl. Math. 68, 1574-1599 (2008)
[26] Garnier, J., Sølna, K.: Coupled paraxial wave equations in random media in the white-noise regime. Ann. Appl. Probab. 19, 318-346 (2009)
[27] Garnier, J., Sølna, K.: Parabolic and white-noise approximations for elastic waves in random media. Wave Motion 46, 237-254 (2009)
[28] Garnier, J., Sølna, K.: Paraxial coupling of electromagnetic waves in random media. SIAM Multiscale Model. Simul. 7, 1928-1955 (2009)
[29] Garnier, J., Sølna, K.: Random backscattering in the parabolic scaling. J. Stat. Phys. 131, 445-486 (2008)
[30] Garnier, J., Sølna, K.: Scaling limits for wave pulse transmission and reflection operators. Wave Motion 2, 122-143 (2009)
[31] Garnier, J., Sølna, K.: Scintillation in the white-noise paraxial regime. Comm. Part. Differ. Equat. 39, 626-650 (2014)
[32] Huang, K., Papanicolaou, G., Sølna, K., Tsogka, C., Zhao, H.: Efficient numerical simulation for long range wave propagation. J. Comput. Phys. 215, 448-464 (2005)
[33] Huang, K., Sølna, K., Zhao, H.: Coupled parabolic equations for wave propagation. Methods

and Applications of Analysis 11, 399-412 (2004)
[34] Ishimaru, A.: Wave Propagation and Scattering in Random Media. IEEE Press, Piscataway (1997)
[35] Keller, J.B.: Stochastic equations and wave propagation in random media. Proc. Symp. appl. Math., 16, 145-170 (1964)
[36] Kurzweil, J., Jarník, J.: Limit processes in ordinary differential equations. Journal of Applied Mathematics and Physics 38, 241-256 (1987)
[37] Klyatskin, V.I., Tatarskii, V.I.: A new method of successive approximations in the problem of the propagation of waves in a random medium having random large-scale inhomogenieties. Radiophys. and Quantum Electronics (USSR), 14, 1110-1111 (1970)
[38] Levy, M.: Parabolic Equation Methods for Electromagnetic Wave Propagation. Institution of Electrical Engineers, London (2000)
[39] Papanicolaou, G., Ryzhik, L., Sølna, K.: Statistical stability in time reversal. SIAM J. Appl. Math. 64, 1133-1155 (2004)
[40] Ryzhik, L., Keller, J., Papanicolaou, G.: Transport equations for elastic and other waves in random media. Wave Motion 24, 327-370 (1996)
[41] Strohbehn, J.W. (ed.): Laser beam propagation in the atmosphere. Springer, Berlin (1978)
[42] Tappert, F.: The parabolic approximation method. In: Wave propagation and underwater acoustics, Keller, J.B., Papadakis, J.S. (eds.), pp. 224-287, Springer, Berlin (1977)
[43] Tatarskii, V.I.: The effects of the turbulent atmosphere on wave propagation. National Technical Service: Springfield, VA (1971)
[44] Uscinski, B.: The Elements of Wave Propagation in Random Media. McGraw Hill, New York (1977)