Timing the Smile

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Abstract

Within the general framework of stochastic volatility, the authors propose a method, which is consistent with no-arbitrage, to price complicated path-dependent derivatives using only the information contained in the implied volatility skew. This method exploits the time scale content of volatility to bridge the gap between skews and derivatives prices. Here they present their pricing formulas in terms of Greeks free from the details of the underlying models and mathematical techniques.

1 Underlying or Smile?

Our goal is to address the following fundamental question in pricing and hedging derivatives. How traded call options, quoted in terms of implied volatilities, can be used to price and hedge more complicated contracts. One

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can approach this difficult problem in two different ways: modeling the evolution of the underlying or modeling the evolution of the implied volatility surface. In both cases one requires that the model is free of arbitrage.

Modeling the underlying usually involves the specification of a multi-factor Markovian model under the risk-neutral pricing measure (see [7], for instance). The calibration to the observed implied volatilities of the parameters of that model, including the market prices of risk, is a challenging task because of the complex relation between call option prices and model parameters (through a pricing partial differential equation for instance). A main problem with this approach is to find the "right model" which will produce a stable parameter estimation. We like to think of this problem as the "(t, T, K)" problem: for a given present time t and a fixed maturity T, it is usually easy with low dimensional models to fit the skew with respect to strikes K. Getting a good fit of the term structure of implied volatility, that is when a range of observed maturities are taken into account, is a much harder problem which can be handled with a sufficient number of parameters and eventually including jumps in the model (see [7, 4] for instance). The main problem remains: the stability with respect to t of these calibrated parameters. However this is an highly desirable quality if one wants to use the model to compute no-arbitrage prices of more complex path-dependent derivatives, since in this case the distribution over time of the underlying is crucial.

Modeling directly the evolution of the implied volatility surface is a promising approach but involves some complicated issues. One has to make sure that the model is free of arbitrage or, in other words, that the surface is produced by some underlying under a risk-neutral measure. This is not an obvious task (see [6] and references therein). The choice of a model and its calibration is also an important issue in this approach. But most importantly, in order to use this modeling to price other path-dependent contracts, one has to identify a corresponding underlying which typically does not lead to a low dimensional Markovian evolution.

Wouldn't it be nice to have a direct and simple connection between the observed implied volatilities and prices of more complex path-dependent contracts! Our objective is to provide such a bridge. This is done by using a combination of singular and regular perturbations techniques corresponding respectively to fast and slow time scales in volatility. We obtain a parametrization of the implied volatility surface in terms of Greeks, which involves four parameters at the first order of approximation. This procedure leads to parameters which are exactly those needed to price other contracts at this level of approximation. In our previous work presented in [9] we used only the fast volatility time scale combined with a statistical estimation of an effective constant volatility from historical data. The introduction of the slow volatility time scale enables us to capture more accurately the behavior of the term structure of implied volatility at long maturities. Moreover in the framework presented here, statistics of historical data are not needed. Thus, in summary, we directly link the implied volatilities to prices of path-dependent contracts by exploiting volatility time scales. We refer to [11] for a detailed presentation of volatility time scales in the S&P 500 index. The mathematical derivation of the combined regular and singular perturbations can be found in [13].

2 Volatility Time Scales

Stochastic volatility models can be seen as continuous time versions of ARCH-type models which have been introduced by R. Engle. The importance of volatility modeling is reflected by the fact that R. Engle has just been awarded the 2003 Nobel Prize for Economics, shared with C. Granger whose work also deals with time scale modeling. Our modeling point of view is that volatility is driven by several stochastic factors running on different time scales. The presence of these volatility factors is well documented in the literature using underlying returns data (see for instance [1, 2, 5, 8, 11, 17, 18, 21, 22]). In fact these factors play a central role in derivatives pricing and generate in a complex way the term structure of implied volatility. Our perturbative approach vastly simplifies this complex relation and leads to simple formulas which reflect the main features of the implied volatilities that follow from the effects of these various volatility time scales.

Before going into formulas, we describe in simple words what these time scales represent and their effects on derivatives pricing.

A stochastic volatility factor running on a **slow scale** means that it takes a long time (compared with typical maturities) for this factor to change appreciably and decorrelate. In the slow scale limit this would then become a constant volatility factor frozen at the present level. In this limit, derivatives prices would be obtained by the usual Black-Scholes pricing theory at this constant volatility level. Our regular perturbation analysis gives corrections to this limit which affect long dated options and therefore are reflected in the behavior of the skew at large maturities. Slow scales, or small perturbations, have been considered in [15, 19, 23].

A stochastic volatility factor running on a **fast scale** means that it takes a short time (compared with typical maturities) for this factor to come back to its mean level and decorrelate. In the fast scale limit this would then also become a constant volatility factor at an effective level $\bar{\sigma}$ determined by the averaged square volatility

$$\bar{\sigma}^2 \approx \frac{1}{T-t} \int_t^T \sigma^2(s) ds, \tag{1}$$

the slow volatility factor being frozen, and where we assume that the fast volatility factor is mean-reverting with rapid mixing properties. Our singular perturbation analysis gives corrections to this Black-Scholes limit which affect options over various maturities and therefore are reflected in the behavior of the skew.

The formulas presented below are obtained by considering that volatility is driven by both slow and fast scale factors. Our analysis, which combines regular and singular perturbations, leads to a parametrization of the term structure of implied volatility which is valid over a wide range of maturities. In that sense, to the leading order, we solve the "(T, K) problem". In fact it turns out that the calibration of our parameters is stable in time and therefore, to the leading order, we provide a solution to the full (t, T, K) problem, and we demonstrate that modeling volatility with at least two factors (a slow and a fast) is consistent with the behavior of derivative markets.

3 Volatility Skew Formulas

3.1 Vanilla Prices

Our asymptotic analysis performed on European vanilla options leads to an explicit formula for the approximated price when the underlying model has a volatility driven by a slow and a fast factor. The leading order term, $P_{BS}(\sigma^\star)$, is the classical Black-Scholes price of the contract evaluated at the constant volatility $\sigma^\star$ which will be calibrated from the observed implied volatilities in Section 3.2. The correction is a combination of three terms expressed in terms of the Greeks of the Black-Scholes price at the volatility level $\sigma^\star$:

$$P \approx P_{BS}(\sigma^\star) + (T-t) \left\{ v_0 \mathcal{V} + v_1 S \Delta(\mathcal{V}) + v_3 S \Delta(S^2 \Gamma) \right\}, \tag{2}$$

where S denotes the present value at time t of the underlying, T denotes the maturity, and the Greeks are given by

$$\begin{aligned}
\mathcal{V} &= \frac{\partial P_{BS}}{\partial \sigma}(\sigma^\star) && \text{(Vega)} \\
S\Delta(\mathcal{V}) &= S\frac{\partial^2 P_{BS}}{\partial S \partial \sigma}(\sigma^\star) && (S\text{Delta(Vega))} \\
S\Delta(S^2\Gamma) &= S\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 P_{BS}}{\partial S^2}\right)(\sigma^\star) && (S\text{Delta}(S^2\text{Gamma})).
\end{aligned}$$

An extensive discussion of the role of the Greeks can be found in [16].

The small parameters (v_0, v_1, v_3) will also be calibrated from the observed implied volatilities as we will explain in Section 3.2. The terms involving v_0 and v_1 are price corrections that come from the effect of the slow factor. The term involving v_3 is caused by the fast factor in the volatility and its leverage effect. We remark that the **effective volatility** $\sigma^\star$ includes a correction that comes from the market price of *fast* volatility risk; this volatility level correction could alternatively have been incorporated as a price correction term proportional to S^2Gamma (the apparently missing v_2 term). In that sense $\sigma^\star$ is a corrected value of the average volatility $\bar{\sigma}$ introduced in (1). The main advantage of introducing $\sigma^\star$ is that it can be estimated from the smile as explained below in Section 3.2. In contrast, $\bar{\sigma}$ can only be estimated from long records of historical returns data.

Observe that for European vanilla options we have the explicit relation:

$$\mathcal{V} = (T-t)\sigma S^2\Gamma,$$

and therefore the price approximation can be written in the form

$$P \approx P_{BS}(\sigma^\star) + (T-t)v_0\mathcal{V} + \{(T-t)v_1 + (v_3/\sigma^\star)\}\, S\Delta(\mathcal{V}). \qquad (3)$$

It is crucial to observe that we can implement this level of price approximation knowing only the present value, S, and the four parameters $\sigma^\star, v_0, v_1$ and v_3. We next show that these parameters in fact can be estimated from the implied volatilities.

3.2 Calibrating the smile

The price approximation given above in the case with European call options leads to the following approximation of the implied volatility skew:

$$I(t, S; T, K) \approx b_0 + b_1(T-t) + \{m_0 + m_1(T-t)\}\,\text{LMMR}, \qquad (4)$$

where as in [9] the Log-Moneyness-to-Maturity Ratio is defined by

$$\text{LMMR} = \frac{\log(K/S)}{T-t}.$$

In fact the coefficients m_0 and b_0 are due to the fast volatility factor while the coefficients m_1 and b_1 are due to the slow volatility factor which becomes important for large maturities.

Our method now consists of the following steps:

(I) Given a discrete set of implied volatilities $I(t, S; K_i, T_j)$, we carry out the linear least squares fits, m LMMR $+ b$, with respect to LMMR for each time to maturity $\tau_j = T_j - t$. This is illustrated in Figure 1 for six different maturities and for strikes not far out of the money.

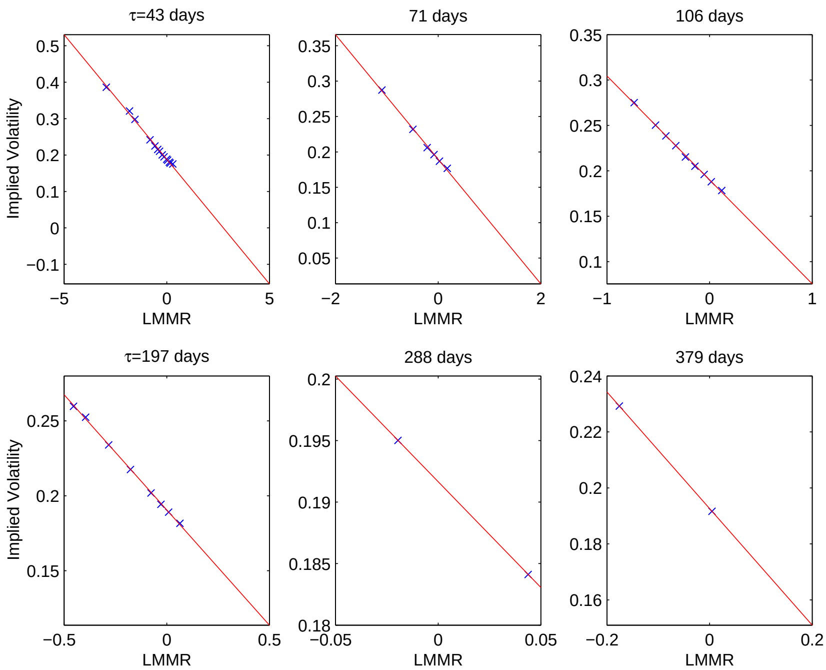

Figure 1: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation (4) for six different maturities.*

We will see in Section 4 that higher order corrections are needed to capture the turn of the skew as illustrated in Figure 8.

Next we estimate the parameters (m_0, b_0), respectively (m_1, b_1), by linear regression with respect to $(T-t)$ of m, respectively b.

In Figure 2 we show the results of these linear regressions on a given day (June 5, 2003) for the S&P 500 implied volatilities.

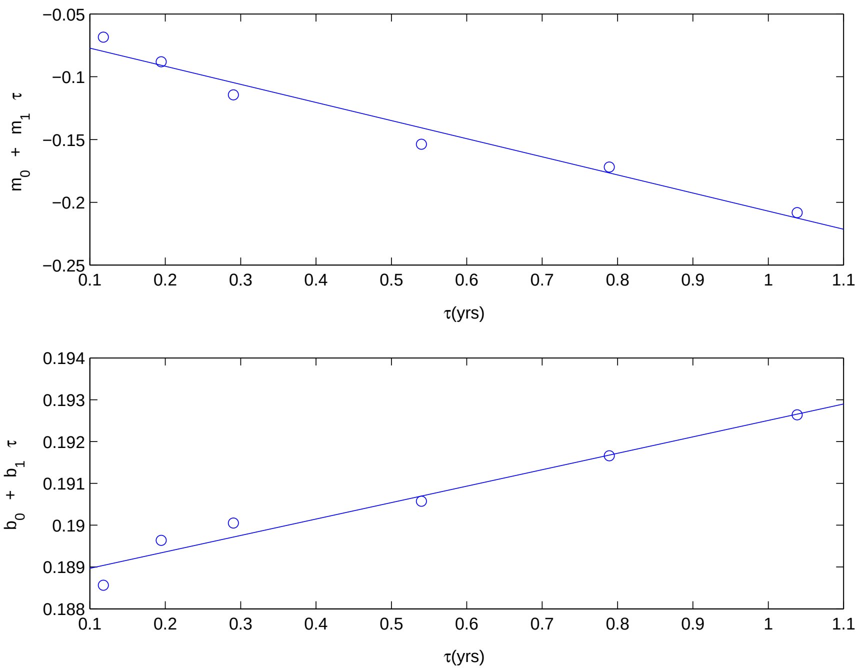

Figure 2: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (rep. top) figure shows the linear regression of* b *(resp.* a*) with respect to time to maturity* $\tau = T - t$.

(II) The parameters $\sigma^\star, v_0, v_1$ and v_3 that are needed for pricing are given *explicitly* by the following formulas:

$$\begin{aligned} \sigma^\star &= b_0 + m_0 \left(r - \frac{b_0^2}{2} \right) \\ v_0 &= b_1 + m_1 \left(r - \frac{b_0^2}{2} \right) \\ v_1 &= m_1 b_0^2 \\ v_3 &= m_0 b_0^3 \end{aligned} \tag{5}$$

Observe that in the regime that our approximation is valid the parameters v_0, v_1 and v_3 are expected to be small, while $\sigma^\star$ is the leading order magnitude of volatility. Here, r is the short rate which we assume to be known and constant.

3.3 Pricing Equations

We explain some of the background for the above results and relate this to deriving pricing equations for rather general contracts. The price approximation given by the right hand side of (3) can be written

$$P_{BS}(\sigma^\star) + P_1(\sigma^\star)$$

where the correction $P_1(\sigma^\star)$ is given by:

$$P_1(\sigma^\star) = (T-t)v_0\mathcal{V} + \{(T-t)v_1 + (v_3/\sigma^\star)\}\, S\Delta(\mathcal{V}).$$

The leading order term $P_{BS}(\sigma^\star)$ is the classical Black-Scholes price at the constant volatility level $\sigma^\star$. It is the solution of the PDE problem

$$\mathcal{L}_{BS}(\sigma^\star)P_{BS} = 0$$

with the terminal condition $P_{BS}(T,S) = h(S)$ where h is the payoff function for the European vanilla option that we consider. Recall that the Black-Scholes operator is given by

$$\mathcal{L}_{BS}(\sigma^\star) = \frac{\partial}{\partial t} + \frac{1}{2}(\sigma^\star)^2 S^2 \frac{\partial^2}{\partial S^2} + r\left(S\frac{\partial}{\partial S} - \cdot\right).$$

The price correction $P_1(\sigma^\star)$ solves the following partial differential equation

$$\mathcal{L}_{BS}(\sigma^\star)P_1(\sigma^\star) = -\left(2v_0\frac{\partial P_{BS}}{\partial \sigma} + 2v_1 S\frac{\partial^2 P_{BS}}{\partial S \partial \sigma} + v_3 S\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 P_{BS}}{\partial S^2}\right)\right)(\sigma^\star), \quad (6)$$

with a zero terminal condition $P_1(\sigma^\star)(T,S) = 0$. In terms of the Greeks introduced in (2) this equation reads

$$\mathcal{L}_{BS}(\sigma^\star)P_1(\sigma^\star) = -\Big(2v_0\mathcal{V} + 2v_1 S\Delta(\mathcal{V}) + v_3 S\Delta(S^2\Gamma)\Big) \quad (7)$$

where again the Greeks are evaluated at the effective volatility $\sigma^\star$.

3.4 Pricing Exotic Contracts

We are now in a position to carry out our main task, that is, with the parameters calibrated from the smile we will price more general contracts than just the vanilla cases considered above. The pricing procedure is simply:

1. Compute the **leading order** (Black-Scholes) price $P_0(\sigma^\star)$ which is the price of the contract at the **constant** volatility level $\sigma^\star$ defined in (5). This involves solving partial differential equations with appropriate boundary and terminal conditions.

2. Compute the **Greeks** $\mathcal{V}, S\Delta(\mathcal{V}), S\Delta(S^2\Gamma)$ of the price $P_0(\sigma^\star)$ of the exotic contract.

3. Compute the price correction $P_1(\sigma^\star)$ by solving the same pricing problem as in Step 1 for $P_0(\sigma^\star)$ with the constant volatility $\sigma^\star$, but with a **zero payoff** and with a **source**, as in (7), defined in terms of the computed Greeks and the three parameters v_0, v_1 and v_3 that are calibrated from the skew as explained in section 3.2. The source is defined exactly as in (7), and the partial differential equations depend on the nature of the contract.

4. The price is now given by correcting the leading order price:

$$P \approx P_0(\sigma^\star) + P_1(\sigma^\star).$$

We present next some remarks regarding the above procedure.

- For complicated contracts, computing the price $P_0(\sigma^\star)$ along with the Greeks usually requires numerical methods (finite differences, Monte Carlo,..) depending on the nature of the contract. We do not comment on the details of these numerical methods. These methods are well documented elsewhere (see for instance [24]), what is important to note is that in this framework they only need to be applied in a setting with a **constant** volatility.

- Solving the problem for the correction P_1 requires generalizations of these methods to the case with a source term. The authors have explicitly considered some of these problems (Asian, Barriers, American,...) in [9] with only the fast scale, and in [13] and the forthcoming book

[14] with both fast and slow scales. Note that for American options the free boundary is determined by solving the problem for $P_0(\sigma^\star)$ and it is then used as a fixed boundary in the problem with a source that determines $P_1(\sigma^\star)$.

- A variation of this approach consists of formulating a closed problem for a slightly modified price $P^\star$ with the same order of accuracy as for $P_0(\sigma^\star) + P_1(\sigma^\star)$. The new problem derives from (6) by replacing P_{BS} in the source by the unknown function $P^\star$:

$$\mathcal{L}_{BS}(\sigma^\star)P^\star + \left(2v_0\frac{\partial P^\star}{\partial\sigma} + 2v_1 S\frac{\partial^2 P^\star}{\partial S\partial\sigma} + v_3 S\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 P^\star}{\partial S^2}\right)\right) = 0.$$

  This gives a third order linear partial differential equation for the modified price with respect the two variables S and σ. We essentially replace two one-dimensional problems with one two-dimensional problem, but the advantage of this approach is that by using implicit methods we can bypass the expensive computation of the Greeks in Step 2.

4 Further Corrections

Observe that above we used a leading order expansion of the price in the context of a multifactor stochastic volatility to obtain a connection between the implied volatility skew and pricing formulas. The mathematical tools underlying the approximation (6) consists in writing first a class of stochastic volatility models containing fast and slow volatility factors. We then expand the corresponding pricing equations with respect to the small parameters defining these two time scales: one parameter being the time scale of the fast factor and the other being the reciprocal of the time scale of the slow factor. The formulas above constitute the first order approximation with respect to these parameters.

A natural extension of this approach is to include further terms of the asymptotic expansion. In particular, as the first-order terms describe affine skews (as a function of log-moneyness), but often we observe slight turns (or wings) at extreme strikes, we consider the next set of terms, which turn out to allow for skews that are quartic polynomials in log-moneyness. By including these terms we improve the quality of the fit to the skew and the accuracy of the pricing formulas. Indeed the number of parameters increases (from four to eleven), higher order Greeks are involved (up to sixth order derivatives) and consequently the computational cost also increases.

The upshot of a long calculation that includes the next three (second-order) terms in the combined fast and slow scales expansion, is that, outside of a small terminal layer (very close to expiration), implied volatilities are approximated by

$$I \approx \sum_{j=0}^{4} a_j(\tau)\,(LM)^j + \frac{1}{\tau}\Phi_t, \tag{8}$$

where τ denotes the time-to maturity $T-t$, LM denotes the moneyness $\log(K/S)$, and Φ_t is a rapidly changing component that varies with the fast volatility factor. In (8) we choose to separate the log-moneyness and the maturity dependence. Alternatively we could have written the implied volatility as a polynomial in LMMR as we did in (4) for the first order approximation.

Again, this calibration formula is employed in a two-stage fitting procedure that recognizes the thinness of data in the maturity dimension, relative to the many available strikes. On each day, the skew for each available maturity is fit to a quartic polynomial in log-moneyness to obtain estimates of $a_1(\tau)$, $a_2(\tau)$, $a_3(\tau)$ and $a_4(\tau)$ for those τ that are observed on that day. The a_0 estimates include the small component Φ_t, and we discuss only the $a_1, \cdots, a_4$ fits here.

Figure 3 shows some typical quartic fits of S&P 500 implied volatilities for a few maturities. Here we use a wider range of strikes than in the linear fit shown in Figure 4, in particular in the out of the money direction. We see from these plots that the quartic produced by second order approximation becomes important in capturing the turn of the skew. In these fits, it is important to fit the main body of the skew to an affine function of log-moneyness first (corresponding to the first order approximation presented in Section 3.2), and then fit the remainder

$$\frac{I - (a_0 + a_1(LM))}{(LM)^2}$$

to a quadratic in moneyness LM (in practice, LM is shifted to $LM + 1$ to avoid divide-by-zero issues). This split procedure is necessary because a free one-stage fit often uses the freedom of the quartic to catch stray data points, leading to large estimates of a_3 and a_4. By viewing the wings as small corrections to the linear skew, we avoid "tail wagging the dog" phenomena.

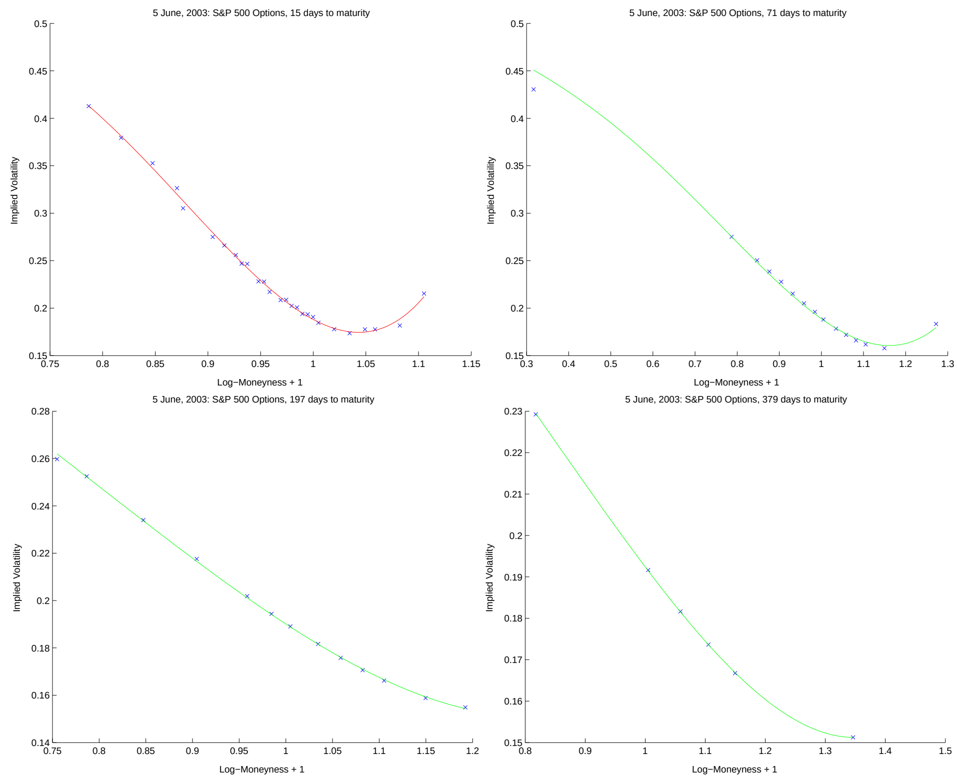

Figure 3: *S&P* 500 *Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.*

Then, we fit the quartic coefficients to the following term-structure formulas coming from the asymptotics:

$$\begin{aligned} a_1(\tau) &= \sum_{k=-1}^{2} a_{1,k}\tau^k \\ a_2(\tau) &= \sum_{k=-2}^{1} a_{2,k}\tau^k \\ a_3(\tau) &= \sum_{k=-1}^{0} a_{3,k}\tau^k \\ a_4(\tau) &= \sum_{k=-2}^{-1} a_{4,k}\tau^k. \end{aligned} \quad (9)$$

The calibrated parameters $\{a_{j,k}\}$ play the role played by (b_0, b_1, m_0, m_1) in the first-order theory.

Figure 4 shows the fits of the $a(\tau)$'s to their term-structure formulas for S&P 500 data on June 5, 2003.

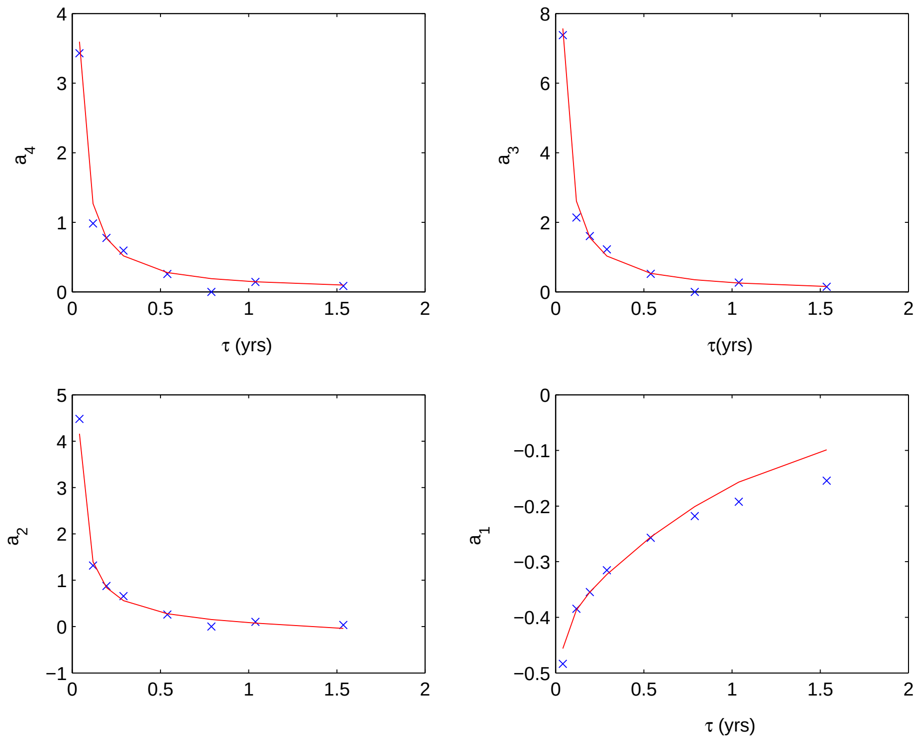

Figure 4: *S&P* 500 *Term-Structure Fit using second order approximation. Data from June 5, 2003.*

As discussed in the introduction, one of the main issues in volatility calibration is the stability with respect to t of the parameter estimates. To illustrate this point we carried out the quartic fits on S&P 500 implied volatilities collected over the course of a month, We obtain estimates of $a_1(\tau)$, $a_2(\tau)$, $a_3(\tau)$ and $a_4(\tau)$ for those τ that are observed over this period. Figure 5 shows the fits of $a_1, \cdots, a_4$ to their corresponding term-structure formulas given in (9). The reasonable fits using a month's data demonstrate the stability of the approximation over some time. We remark that the a_1 estimates become less structured at small maturities because of a periodic maturity cycle component due to the option expiration ('witching') dates the

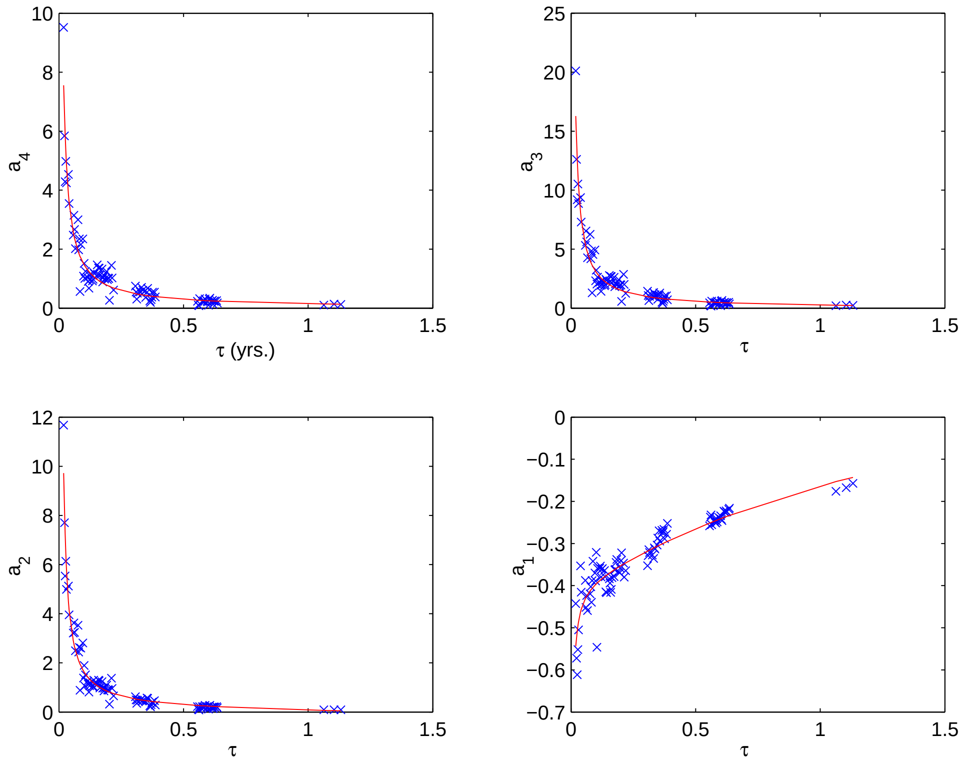

Figure 5: *S&P 500 Term-Structure Fit. Data from every trading day in May 2003.*

third Friday of each month. This is studied in detail in [10].

The final step is to recover the parameters needed for pricing from the estimates of $\{a_{j,k}\}$, the analog of (5) in the first-order theory. However, these relations are no longer linear in the second-order theory, and a nonlinear inversion algorithm is required. This aspect has to be treated case by case in order to take advantage of the particular features of the market under study. For instance in FX markets, the correlation between the underlying and its volatility tend to be zero which reduces the complexity of the implementation of the second order theory.

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