SCALING LIMITS FOR BEAM WAVE PROPAGATION IN ATMOSPHERIC TURBULENCE

ALBERT C. FANNJIANG* KNUT SOLNA†

Abstract. Description of waves that propagate through the turbulent atmosphere is a fundamental problem, for instance from the point of view of applications to communication and remote sensing. Yet, so far, very little is known about how the wave field interacts with the turbulent or multiscale nature of the refractive index which derives from the multiscale nature of the temperature fluctuations. The parabolic or forward scattering approximation leads to a random Schrödinger equation. Here, we take the parabolic wave equation as our starting point and derive a white noise approximation for this problem. We start with a description where the non-Gaussian multiscale nature of the refractive fluctuations are described by a power law spectrum with prescribed inner and outer scales and analyze the asymptotic limits corresponding respectively to a relatively large outer scale and or small inner scale. The reference scale in our modeling is taken to be the Fresnel length. A main tool used to derive the convergence to a Gaussian Markov limit is the method of multiple scales . From the white noise approximation we derive closed equations for the moments of the wave field.

1. Introduction

The small-scale refractive index variations, called the refractive turbulence, in the atmosphere is the result of small scale fluctuations of temperature, pressure and humidity caused by the turbulence of air velocities. For optical propagation in the atmosphere the influence of the temperature variations on the refractive index field is dominant whereas in the microwave range, the effect of the humidity variations is more important. The refractive turbulence results in the phenomena of beam wander, beam broadening and intensity fluctuation (scintillation). It is important to note that these effects depend on the length scales of the waves as well as the refractive turbulence [16].

The refractive turbulence is modeled on the basis of Kolmogorov theory of turbulence which introduces the notion of the inertial range bounded by the outer scale L_0 (of the order of $100\mathtt{m}-1\mathtt{km}$) and the inner scale ℓ_0 (of the order of $1-10\mathtt{mm}$). Other features of the refractive turbulence in the open clear atmosphere include [19]: (i) small changes (typical value of 3×10^{-4} at sea level) in the refractive index related to small variations in temperature (on the order of $0.1-1^oC$), (ii) small scattering angle which is of the order λ/ℓ_0 and has the typical value $3\times 10^{-4}\mathtt{rad}$ for $\lambda = 0.6\mathtt{mn}$ and $\ell_0 = 2\mathtt{mm}$. Perturbation methods for solving the Maxwell equations are adequate provided that the propagation distance is less than, say, $100\mathtt{m}$, a severe limitation on their applicability to imaging or communication problems. Our motivation is mainly from laser or microwave beams but our consideration and results apply equally well to ultrasound waves in atmospheric turbulence. The results are also relevant in the context of ultrasound waves penetrating through complicated multiscale fluctuating (interface) zones in for instance human tissue.

Under the condition $\lambda = O(\ell_0)$ (including the millimeter and the sub-millimeter range) the depolarization term in the Helmholtz equation for the electric field is negligible [19] and one can

*Department of Mathematics, University of California at Davis, Davis, CA 95616 Internet: fannjian@math.ucdavis.edu. Research is supported by The Centennial Fellowship of American Mathematical Society and UC Davis Chancellor's Fellowship.

† Department of Mathematics, University of California at Irvine, Internet: ksolna@math.uci.edu.

use the (scalar) Helmholtz equation

$$\nabla^2 E + k^2 E = -k^2 \tilde{\epsilon}_a E \tag{1}$$

with appropriate boundary conditions where k is the wavenumber and $\tilde{\epsilon}_a$ the random fluctuation of the atmospheric permittivity field. This is related to the refractive index field n and its fluctuations n' as

$$\tilde{\epsilon}_a = (n^2 - \mathbb{E}[n^2])/\mathbb{E}[n^2] \approx 2n'\mathbb{E}[n]/\mathbb{E}[n^2]$$

with $n' = n - \mathbb{E}[n]$ where we have scaled the wavenumber so that the mean $\bar{\epsilon}_a = 1$. Here and below $\mathbb{E}[\cdot]$ denotes the ensemble averaging.

1.1. **The rescaled parabolic approximation.** The well-known parabolic approximation to equation (1) is applicable in a regime where the variations of the index of refraction are small on the scale of the wavelength so that backscattering is negligible [19]. This is the case when, for instance, the propagation distance L_z satisfies $\lambda^3 L_z \ll \ell_0^4$ or $\lambda L_z \gg L_0^2$. For laser beams, these conditions give a very large range of validity.

In this paper we study the initial value problem for the parabolic wave equation

$$\nabla_\perp^2 \Psi(z,\mathbf{x}) + 2ik\frac{\partial \Psi(z,\mathbf{x})}{\partial z} = -k^2\tilde{\epsilon}_a(z,\mathbf{x})\Psi(z,\mathbf{x}), \quad \Psi(0,\mathbf{x}) = F_0\left(\frac{\mathbf{x}}{a}\right) \in L^2(\mathbb{R}^2) \tag{2}$$

where z is the longitudinal coordinate in the direction of the propagation, $\mathbf{x} = (x_1, x_2)$ is the transverse coordinates, $\nabla_\perp$ is the transverse gradient and Ψ is related to the scalar wave field E by $E = \Psi(z,\mathbf{x})\exp(ikz)$. The initial condition has a typical width a which is the aperture. Below we will drop the perp in denoting the derivatives in the transverse directions.

The difficulty in solving equation (2) lies in the random multiscale nature of $\tilde{\epsilon}_a(z,\mathbf{x})$. First we non-dimensionalize eq. (2) as follows. Let L_z be the propagation distance in the longitudinal direction. Let λ_0 be the characteristic wavelength. The corresponding central wavenumber is $k_0 = 2\pi/\lambda_0$. The Fresnel length L_f is then given by

$$L_f = \sqrt{L_z/k_0}.$$

We introduce dimensionless wave number and coordinates

$$\tilde{k} = k/k_0, \quad \tilde{\mathbf{x}} = \mathbf{x}/L_f, \quad \tilde{z} = z/L_z$$

and rewrite the equation in the form

$$2i\tilde{k}\frac{\partial \Psi}{\partial z} + \Delta\Psi + \tilde{k}^2 k_0 L_z \tilde{\epsilon}_a(zL_z, \mathbf{x}L_f)\Psi = 0, \quad \Psi(0,\mathbf{x}) = F_0(\gamma^{1/2}\mathbf{x}) \in L^2(\mathbb{R}^2) \tag{3}$$

after dropping the tilde in the coordinate variables where

$$\gamma = \left(\frac{L_f}{a}\right)^2$$

is assumed to be $O(1)$, thus the source is supported on the scale determined by the Fresnel length.

1.2. **Model spectra.** A widely used model for the structure function of the refractive index field of the atmosphere is based on the Kolmogorov theory of turbulence and has the following modified Von K'arm'an spectral density

$$\Phi_n(\mathbf{k}) = 0.033 C_n^2(|\mathbf{k}|^2 + K_0^2)^{-11/6}\exp(-|\mathbf{k}|^2/K_m^2) \tag{4}$$

where $\mathbf{k} = (\xi, \mathbf{p})$, with $\xi \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^2$ the Fourier variables conjugate to the longitudinal and transversal coordinates, respectively. Here $K_0 = 2\pi/L_0, K_m = 5.92/\ell_0$. This spectrum has the correct behavior only in the inertial subrange, i.e.

$$\Phi_n(\mathbf{k}) \sim |\mathbf{k}|^{-11/3}, \quad |\mathbf{k}| \in (2\pi L_0^{-1}, 2\pi\ell_0^{-1}). \tag{5}$$

Outside of this range, particularly for $|\mathbf{k}| \ll 2\pi L_0^{-1}$ there is no physical basis for their behavior; they are just mathematically convenient expressions of the cutoffs. In particular, if the wave statistics strongly depend on ℓ_0 or L_0, then the problem probably requires more accurate information on the refractive index field outside of the inertial range [5], [10], [11]. Note that the ratio L_0/ℓ_0 grows like $\mathrm{Re}^{3/4}$ as the Reynolds number Re tends to infinity.

There are several variants of (4) arising from modeling more detailed features of the refractive index field. One of them is the Hill spectrum [1], [13] to account for the "bump" at high wave numbers which is known to occur near the inner scale

$$\Phi_n(\mathbf{k}) = 0.033C_n^2\left[1 + 1.802|\mathbf{k}|/K_m - 0.254(|\mathbf{k}|/K_m)^{7/6}\right]\left(|\mathbf{k}|^2 + K_0^2\right)^{-11/6}\exp\left(-|\mathbf{k}|^2/K_m^2\right) \tag{6}$$

where $K_m = 3.3/\ell_0$. The coefficient C_n^2 is itself a random variable that depends on time as well as the altitude. Note that in atmospheric turbulence the inner and outer scales and the exponent in the power law may also have to be modeled as stochastic processes [18]. The temporal dependence is irrelevant for optical propagation; the altitude dependence has a rather permanent, non-universal structure with length scales much greater than the outer scale L_0 [16]. It would complicate the analysis but not the main features of our conclusions. We will consider these issues in a separate paper we will treat it as a (small) constant here.

We will consider a family of power-law type spectra

$$\Phi(\alpha, \mathbf{k}) \sim |\mathbf{k}|^{1-2\alpha}|\mathbf{k}|^{-d}, \quad , d = 2, \quad \text{for } |\mathbf{k}| \in (L_0^{-1}, \ell_0^{-1}), \tag{7}$$

with possibly different coefficients at the two ends of the inertial range as the ratio $\rho \to \infty$ in the high Reynolds number limit. We assume that the spectrum decays sufficiently fast for $|\mathbf{k}| \gg \ell_0^{-1}$ while staying bounded for $|\mathbf{k}| \ll L_0^{-1}$. The details of the spectrum are not important for our analysis, only the exponent α is. In particular, $\alpha = 4/3$ for the Kolmogorov spectrum (5).

1.3. **White noise scaling.** Let us introduce the non-dimensional parameters that are pertinent to our scaling:

$$\varepsilon = \sqrt{\frac{L_f}{L_z}}, \quad \eta = \frac{L_f}{L_0}, \quad \rho = \frac{L_f}{\ell_0}.$$

In terms of the parameters and the power-law spectrum in (7) we rewrite (3) as

$$2i\tilde{k}\frac{\partial \Psi^\varepsilon}{\partial z} + \Delta\Psi^\varepsilon + \frac{\tilde{k}^2}{\varepsilon}\sigma_\alpha \mathcal{V}\left(\frac{z}{\varepsilon^2}, \mathbf{x}\right)\Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, \mathbf{x}) = F_0(\gamma^{1/2}\mathbf{x}) \in L^2(\mathbb{R}^2). \tag{8}$$

with

$$\sigma_\alpha = \frac{L_f^{\alpha-1}}{\varepsilon^3}\tilde{C}_n \tag{9}$$

where $\tilde{C}_n$ the total structure parameter in the original refractive index spectrum. The spectrum for the (normalized) process $\mathcal{V}$ is given by

$$\Phi_{(\eta,\rho)}(\alpha, \mathbf{k}) \sim |\mathbf{k}|^{1-2\alpha}|\mathbf{k}|^{-d}, \quad \text{for } |\mathbf{k}| \in (\eta, \rho), \tag{10}$$

as in (7). We only require that (10) holds for $1 \ll |\mathbf{k}| \le \rho$ (Theorem 1) and/or $1 \gg |\mathbf{k}| \ge \eta$ (Theorem 2), possibly with different $O(1)$ constants, and that $\Phi_{(\eta,\rho)}$ decays fast for $|\mathbf{k}| \ge \rho$ and levels off for $|\mathbf{k}| \le \eta$. For high Reynolds number one has $L_0/\ell_0 = \rho/\eta \gg 1$ which is always the case in our study.

In the beam approximation one has $\varepsilon \ll 1$. The beam approximation is well within the range of validity of the parabolic approximation. The white-noise scaling then corresponds to $\sigma_\alpha = O(1)$.

We set it to unity by absorbing the constant into $\mathcal{V}$ since the constants in (10) are unspecified. This implies relatively weak fluctuations of the index field, i.e.

$$\tilde{C_n} \sim L_f^{5/2-\alpha} L_z^{-3/2} \ll 1, \quad \text{as } L_z \to \infty$$

in view of the fact that $\alpha \in (1,2)$ and $\varepsilon \ll 1$.

In the present paper we first study the case $\rho \to \infty$, but η fixed, as $\varepsilon \to 0$ (Theorem 1). This means that the Fresnel length is comparable to the outer scale. Second, we study the narrow beam regime $\eta \ll 1$ where the Fresnel length is in the middle of the inertial subrange (Theorem 2). For the proof, we adopt the approach of [9] where the turbulent transport of passive scalars is studied.

2. Formulation and main results

2.1. **Martingale formulation.** We consider the weak formulation of the equation:

$$2i\tilde{k}\left[\langle \Psi_z^\varepsilon, \theta\rangle - \langle \Psi_0, \theta\rangle\right] \;=\; -\int_0^z \langle \Psi_s^\varepsilon, \Delta\theta\rangle\, ds - \frac{\tilde{k}^2}{\varepsilon}\int_0^z \left\langle \Psi_s^\varepsilon, \mathcal{V}(\frac{s}{\varepsilon^2}, \cdot)\cdot\theta\right\rangle ds \tag{11}$$

for any test function $\theta \in C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact support. The tightness result (Section 4.1) implies that for L^2 initial data the limiting measure $\mathbb{P}$ is supported in $L_w^2([0, z_0]; L_w^2(\mathbb{R}^d))$.

For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $\mathcal{V}_z^\varepsilon \equiv \mathcal{V}(z/\varepsilon^2, \cdot)$, $\mathcal{F}_z^\varepsilon$ the σ-algebras generated by $\{\mathcal{V}_s^\varepsilon,\, s \le z\}$ and $\mathbb{E}_z^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Let $\mathcal{M}^\varepsilon$ be the space of measurable functions adapted to $\{\mathcal{F}_z^\varepsilon, \forall z\}$ such that $\sup_{z<z_0}\mathbb{E}|f(z)| < \infty$. We say $f(\cdot) \in \mathcal{D}(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f = g$ if $f, g \in \mathcal{M}^\varepsilon$ and for $f^\delta(z) \equiv \delta^{-1}[\mathbb{E}_z^\varepsilon f(z+\delta) - f(z)]$ we have

$$\begin{aligned} \sup_{z,\delta} \mathbb{E}|f^\delta(z)| \;&<\; \infty \\ \lim_{\delta\to 0} \mathbb{E}|f^\delta(z) - g(z)| \;&=\; 0, \quad \forall z. \end{aligned}$$

Consider the special class of admissible functions $f(z) = \phi(\langle \Psi_z^\varepsilon, \theta\rangle), f'(z) = \phi'(\langle \Psi_z^\varepsilon, \theta\rangle), \forall \phi \in C^\infty(\mathbb{R})$, then we have the following expression from (11) and the chain rule

$$\mathcal{A}^\varepsilon f(z) \;=\; \frac{i}{2\tilde{k}} f'(z)\left[\langle \Psi_z^\varepsilon, \Delta\theta\rangle + \frac{\tilde{k}^2}{\varepsilon}\langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\theta\rangle\right]. \tag{12}$$

A main property of $\mathcal{A}^\varepsilon$ is that

$$f(z) - \int_0^z \mathcal{A}^\varepsilon f(s) ds \quad \text{is a } \mathcal{F}_z^\varepsilon\text{-martingale}, \quad \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{13}$$

Also,

$$\mathbb{E}_s^\varepsilon f(z) - f(s) = \int_s^z \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f(\tau) d\tau \quad \forall s < z \quad \text{a.s.} \tag{14}$$

(see [14]). We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $\mathcal{V}_z(\cdot) = \mathcal{V}(z, \cdot)$.

Define

$$\Gamma^{(1)}(\mathbf{x},\mathbf{y}) = \int\int\int_0^\infty \cos\left((\mathbf{x}-\mathbf{y})\cdot\mathbf{p}\right)\cos\left(s\xi\right)\Phi_{(\eta,\rho)}(\alpha,\xi,\mathbf{p})\ ds\,d\xi\,d\mathbf{p} \tag{15}$$
$$= \pi\int \cos\left((\mathbf{x}-\mathbf{y})\cdot\mathbf{p}\right)\Phi_{(\eta,\rho)}(\alpha,0,\mathbf{p})\ d\mathbf{p}$$
$$\overline{\Gamma}(\mathbf{x},\mathbf{y}) = \lim_{\rho\to\infty}\Gamma^{(1)}(\mathbf{x},\mathbf{y}) \tag{16}$$
$$\overline{\Gamma}_0(\mathbf{x}) = \overline{\Gamma}(\mathbf{x},\mathbf{x}) \tag{17}$$

where we have written the wavevector $\mathbf{k}\in\mathbb{R}^3$ as $\mathbf{k}=(\xi,\mathbf{p})$ with $\mathbf{p}\in\mathbb{R}^2$.

Now we formulate the solutions for the Gaussian Markovian model (for Theorem 1) as the solutions to the corresponding martingale problem: Find a measure $\mathbb{P}$ (of Ψ_z) on the subspace of $D([0,\infty);L^2_w(\mathbb{R}^d))$ whose elements have the initial condition $F_0(\gamma^{1/2}\mathbf{x})$ such that

$$f(\langle\Psi_z,\theta\rangle) - \int_0^z \left\{ f'(\langle\Psi_s,\theta\rangle)\left[\frac{i}{2\tilde{k}}\left\langle\Psi_s,\Delta\theta\right\rangle - \frac{\tilde{k}^2}{4}\left\langle\Psi_s,\overline{\Gamma}_0\theta\right\rangle\right] - \frac{\tilde{k}^2}{4}f''(\langle\Psi_s,\theta\rangle)\left\langle\theta,\overline{\mathcal{K}}_{\Psi_s}\theta\right\rangle\right\} ds$$

is a martingale w.r.t. the filtration of a cylindrical Wiener process, for each $f\in C^\infty(\mathbb{R})$

where

$$\overline{\mathcal{K}}_{\Psi_s}\theta = \int \Psi_s(\mathbf{x})\Psi_s(\mathbf{y})\overline{\Gamma}(\mathbf{x},\mathbf{y})\theta(\mathbf{y})\,d\mathbf{y}. \tag{18}$$

The Gaussian Markovian model has been extensively studied for beam wander, broadening and scintillation effects in the literature (see, e.g. [4], [12]). It can also been written as the Itô's equation

$$d\Psi_z = \left(\frac{i}{2\tilde{k}}\Delta - \frac{\tilde{k}^2}{4}\overline{\Gamma}_0\right)\Psi_z\,dz + \frac{i\tilde{k}}{\sqrt{2}}\left(\overline{\mathcal{K}}_{\Psi_z}\right)^{1/2}\ dW_z$$
$$= \left(\frac{i}{2\tilde{k}}\Delta - \frac{\tilde{k}^2}{4}\overline{\Gamma}_0\right)\Psi_z\,dz + \frac{i\tilde{k}}{\sqrt{2}}\Psi_z d\overline{W}_z,\quad \Psi_0(\mathbf{x}) = F_0(\gamma^{1/2}\mathbf{x})$$

where $\overline{W}_z$ is the Brownian field with the spatial covariance $\overline{\Gamma}(\mathbf{x},\mathbf{y})$.

As we let $\eta=\eta(\varepsilon)\to 0$ (Theorem 2) the limiting Gaussian, Markovian model has different covariance structure $\overline{\Gamma'}$. as defined below. We introduce the new fields

$$\mathcal{V}'(z,\mathbf{x}) = \mathcal{V}(z,\mathbf{x}) - \mathcal{V}(z,0) \tag{19}$$
$$= \int\left[\exp\left(i\mathbf{p}\cdot\mathbf{x}\right)-1\right]\hat{\mathcal{V}}(z,d\mathbf{p}) \tag{20}$$

in view of the (partial) spectral representation for $\mathcal{V}$

$$\mathcal{V}(z,\mathbf{x}) = \int \exp\left(i\mathbf{p}\cdot\mathbf{x}\right)\hat{\mathcal{V}}(z,d\mathbf{p})$$

where the process $\hat{\mathcal{V}}(z,d\mathbf{p})$ is the (partial) spectral measure of orthogonal increments over $\mathbf{p}$.

Let

$$\overline{\Gamma'}(\mathbf{x},\mathbf{y}) = \lim_{\rho\to\infty,\eta\to 0}\mathbb{E}\left[\mathcal{V}'_z(\mathbf{y})\int_z^\infty \mathbb{E}_z\left[\mathcal{V}'_s(\mathbf{x})\right]\ ds\right] \tag{21}$$
$$= \pi\int (e^{i\mathbf{x}\cdot\mathbf{p}}-1)(e^{-i\mathbf{y}\cdot\mathbf{p}}-1)\Phi_0^\infty(\alpha,\mathbf{p})d\mathbf{p}$$

and

$$\overline{\Gamma'}_0(\mathbf{x}) = \overline{\Gamma'}(\mathbf{x},\mathbf{x})$$

where

$$\Phi_0^\infty(\alpha,\mathbf{p}) = \lim_{\rho\to\infty,\eta\to 0}\Phi_{(\eta,\rho)}(\alpha,0,\mathbf{p}).$$

Note that the limit $\eta\to 0$ in (21) is convergent only if

$$\alpha < 3/2;$$

in particular, the limit exists for the Kolmogorov spectrum $\alpha = 4/3$.

2.2. **Uniqueness.** To identify the limit we need the uniqueness result for the limiting martingale problem. Because of the non-smoothness of the white-noise potential the approach of [7] does not apply here.

Take the function $f(r) = r^n$ in the martingale formulation, we arrive after some algebra at the following equation

$$\frac{\partial F_z^{(n)}}{\partial z} = \mathcal{C}_1 F_z^{(n)} + \mathcal{C}_2 F_z^{(n)} \tag{22}$$

for the $n-$point correlation function

$$F_z^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) \equiv \mathbb{E}\left[\Psi_z(\mathbf{x}_1)\cdots\Psi_z(\mathbf{x}_n)\right]$$

where

$$\mathcal{C}_1 = \frac{i}{2\tilde{k}}\sum_{j=1}^n \Delta_{\mathbf{x}_j} \tag{23}$$

$$\mathcal{C}_2 = -\frac{\tilde{k}^2}{4}\sum_{j,k=1}^n \overline{\Gamma}(\mathbf{x}_j,\mathbf{x}_k), \quad \text{with } \overline{\Gamma}(\mathbf{x},\mathbf{x}) = \overline{\Gamma}_0(\mathbf{x}) \tag{24}$$

$$\text{or } \mathcal{C}_2 = -\frac{\tilde{k}^2}{4}\sum_{j,k=1}^n \overline{\Gamma}'(\mathbf{x}_j,\mathbf{x}_k), \quad \text{with } \overline{\Gamma}'(\mathbf{x},\mathbf{x}) = \overline{\Gamma}'_0(\mathbf{x}). \tag{25}$$

We will now establish the uniqueness for eq. (22) with the initial data

$$F_0^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbb{E}\left[\Psi_0(\mathbf{x}_1)\cdots\Psi_0(\mathbf{x}_n)\right], \quad \Psi_0 \in L^2(\mathbb{R}^2).$$

In the former case (24) $\mathcal{C}_2$ is a *bounded*, Hölder continuous function, we rewrite eq. (22) in the mild formulation

$$F_z^{(n)} = \exp(z\mathcal{C}_1)F_0^{(n)} + \int_0^z \exp\left[(z-s)\mathcal{C}_1\right]\mathcal{C}_2 F_s^{(n)}\, ds$$

whose local existence and uniqueness can be easily established by straightforward application of the contraction mapping principle. By linearity, local well-posedness can be extended to global well-posedness.

In the latter case (25) $\mathcal{C}_2$ is unbounded, Hölder continuous function with sub-Lipschitz growth. We first note that $\mathcal{C}_2$ is non-positive everywhere since

$$\sum_{j,k=1}^n \overline{\Gamma}'(\mathbf{x}_j,\mathbf{x}_k) = \pi\int \sum_j (e^{i\mathbf{x}_j\cdot\mathbf{p}}-1)\overline{\sum_k (e^{i\mathbf{x}_k\cdot\mathbf{p}}-1)}\Phi_0^\infty(\alpha,\mathbf{p})d\mathbf{p} \geq 0.$$

Hence both $\mathcal{C}_1$ and $\mathcal{C}_2$ are generators of one-parameter contraction semigroups on $L^2(\mathbb{R}^{2n})$, thus by the product formula (Theorem 3.30, [6]) we have

$$\lim_{m\to\infty}\left[\exp\left(\frac{z}{m}\mathcal{C}_1\right)\exp\left(\frac{z}{m}\mathcal{C}_2\right)\right]^m F = \exp\left[z(\mathcal{C}_1+\mathcal{C}_2)\right]F$$

for all $F\in L^2(\mathbb{R}^{2n})$, which then gives rise to a unique semigroup on $L^2(\mathbb{R}^{2n})$.

2.3. **Main assumptions and theorems.** Assume that the new random field

$$\tilde{\mathcal{V}}_z(\mathbf{x}) \quad = \quad \int_z^\infty \mathbb{E}_z\left[\mathcal{V}_s(\mathbf{x})\right]\, ds \tag{25}$$

is well-defined. This holds, for instance, when the mixing coefficient of $\mathcal{V}_z$ is integrable [8]. It is easy to see that

$$\mathcal{A}\tilde{\mathcal{V}}_z = -\mathcal{V}_z$$

and that $\tilde{\mathcal{V}}_z(\mathbf{x})$ has a spectral density like (10) with an exponent $\alpha+1$ such that

$$\Gamma^{(1)}(\mathbf{x},\mathbf{y}) = \mathbb{E}\left[\tilde{\mathcal{V}}_z(\mathbf{x})\mathcal{V}_z(\mathbf{y})\right].$$

In addition to (10) we also assume that up to the fourth moment of the field $\mathcal{V}$ can be estimated in terms of (10) as in the case of Gaussian fields. We call this the *fourth-order scale-invariance* property. We assume that for $\eta > 0$ fixed,

$$\sup_{z<z_0} \|\theta\mathcal{V}(\frac{z}{\varepsilon^2},\cdot)\|_2 \quad = \quad o\left(\frac{1}{\varepsilon}\right), \quad \varepsilon\to 0, \quad \forall\theta\in C_c^\infty(\mathbb{R}^d) \tag{26}$$

with a random constant of finite moment. We assume that

$$\sup_{z<z_0} \|\theta\mathcal{V}'(\frac{z}{\varepsilon^2},\cdot)\|_2^2 \quad = \quad O(\varepsilon^{-1}), \quad \text{independent of } \eta \text{ and } \rho, \quad \forall\theta\in C_c^\infty(\mathbb{R}^d) \tag{27}$$

with a random constant of finite moment. Note that the refractive index field $\mathcal{V}$ loses regularity as $\rho\to\infty$ and homogeneity as $\eta\to 0$.

In the case of *Gaussian* refractive index fields conditions (26) and (27) are always satisfied

$$\sup_{z<z_0} \|\theta\mathcal{V}(\frac{z}{\varepsilon^2},\cdot)\|_2 \quad \leq \quad C_1 \log\left[\frac{z_0}{\varepsilon^2}\right] \tag{28}$$

$$\sup_{z<z_0} \|\theta\mathcal{V}'(\frac{z}{\varepsilon^2},\cdot)\|_2 \quad \leq \quad C_2 \log\frac{1}{\varepsilon} \tag{29}$$

where the random constants C_1, C_2, C_3 have a Gaussian-like tail by Chernoff's bound.

Theorem 1. *Let $\mathcal{V}$ satisfy (10), (25), (26) and the fourth-order scale-invariance property. Let $\rho\to\infty$ as $\varepsilon\to 0$ while η is fixed such that*

$$\lim_{\varepsilon\to 0} \varepsilon\rho^{2-\alpha} = 0. \tag{30}$$

Then the weak solution Ψ^ε of (11) converges in the space of $D([0,\infty); L^2(\mathbb{R}^d))$ to that of the Gaussian, Markovian model with the covariance functions $\overline{\Gamma}$ and $\overline{\Gamma}_0$.

Remark 1. *The power-law spectrum (10) plays no role in the proofs of Theorem 1 and 2. A similar theorem holds for the random media with a finite correlation length such that their fourth moments can be controlled by their second moments and that the roughness of the random media can be controlled as in for instance (30).*

Remark 2. *Since both the limiting and pre-limiting equations preserve the L^2-norm of the initial data it suffices to prove the convergence in the space $D([0,\infty); L^2_w(\mathbb{R}^d))$ where the weak topology is used.*

Note that in the limiting model the white-noise velocity field has transverse regularity of Hölder exponent $\alpha - 1/2$.

Next we let η tend to zero as well, but this would induce uncontrollable large scale fluctuation which should be factored out first. Thus we consider the solution of the form

$$\Psi^\varepsilon(z,\mathbf{x}) = \tilde{\Psi}^\varepsilon(z,\mathbf{x}) \exp\left(\frac{i\tilde{k}}{2\varepsilon}\int_0^z \mathcal{V}^\varepsilon_s(0)\, ds\right)$$

and the resulting equation

$$2i\tilde{k}\frac{\partial\tilde{\Psi}^\varepsilon}{\partial z}+\Delta\tilde{\Psi}^\varepsilon+\frac{\tilde{k}^2}{\varepsilon}\mathcal{V}'(\frac{z}{\varepsilon^2},\mathbf{x})\tilde{\Psi}^\varepsilon=0 \tag{30}$$

where $\mathcal{V}'$ is defined by (19).

Theorem 2. *Let $\alpha<3/2$. Let the assumptions of Theorem 1 be satisfied. Additionally, assume (27) and $\eta=\eta(\varepsilon)\to 0$ such that*

$$\lim_{\varepsilon\to 0}\varepsilon\eta^{2-2\alpha}=0.$$

Then the weak solution $\tilde{\Psi}^\varepsilon$ of (2.3) converges in the space of $D([0,\infty);L^2(\mathbb{R}^d))$ to that of the Gaussian, Markovian model with the covariance functions $\overline{\Gamma'}$ and $\overline{\Gamma'}_0$.

Because of $\alpha<3/2$, the limiting model is only Hölder continuous in the transverse coordinates.

The convergence of the white-noise limit has been established in [2] and [3] under more stringent conditions. In particular, their limit theorems do not allow $\rho\to\infty,\eta\to 0$ among other restrictions. In [17] the random media studied has a finite bandwidth. Also, the geometric optics limit is first taken, then the white-noise limit and the broad beam limit $\eta\to\infty$ are taken subsequently.

3. Proof of Theorem 1

3.1. **Tightness.** In the sequel we will adopt the following notation

$$f(z)\equiv f(\langle\Psi^\varepsilon_z,\theta\rangle),\quad f'(z)\equiv f'(\langle\Psi^\varepsilon_z,\theta\rangle),\quad f''(z)\equiv f''(\langle\Psi^\varepsilon_z,\theta\rangle),\qquad \forall f\in C^\infty(\mathbb{R}).$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not z) of f,f' etc.

A family of processes $\{\Psi^\varepsilon,0<\varepsilon<1\}\subset D([0,\infty);L^2_w(\mathbb{R}^d))$ is tight if and only if the family of processes $\{\langle\Psi^\varepsilon,\theta\rangle,0<\varepsilon<1\}\subset D([0,\infty);L^2_w(\mathbb{R}^d))$ is tight for all $\theta\in C^\infty_c(\mathbb{R}^d)$. We use the tightness criterion of [15] (Chap. 3, Theorem 4), namely, we will prove: Firstly,

$$\lim_{N\to\infty}\limsup_{\varepsilon\to 0}\mathbb{P}\{\sup_{z<z_0}|\langle\Psi^\varepsilon_z,\theta\rangle|\geq N\}=0,\quad\forall z_0<\infty. \tag{31}$$

Secondly, for each $f\in C^\infty(\mathbb{R})$ there is a sequence $f^\varepsilon(z)\in\mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $z_0<\infty$ $\{\mathcal{A}^\varepsilon f^\varepsilon(z),0<\varepsilon<1,0<z<z_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0}\mathbb{P}\{\sup_{z<z_0}|f^\varepsilon(z)-f(\langle\Psi^\varepsilon,\theta\rangle)|\geq\delta\}=0,\quad\forall\delta>0. \tag{32}$$

Then it follows that the laws of $\{\langle\Psi^\varepsilon,\theta\rangle,0<\varepsilon<1\}$ are tight in the space of $D([0,\infty);L^2_w(\mathbb{R}^d))$

Condition (31) is satisfied because the L^2-norm is preserved. Let

$$f_1^\varepsilon(z)\equiv\frac{i\tilde{k}}{2\varepsilon}\int_z^\infty\mathbb{E}^\varepsilon_z\,f'(z)\,\langle\Psi^\varepsilon_z,\mathcal{V}^\varepsilon_s\theta\rangle\,ds$$

be the 1-st perturbation of $f(z)$. Let

$$\tilde{\mathcal{V}}^\varepsilon_z=\frac{1}{\varepsilon^2}\int_z^\infty\mathbb{E}^\varepsilon_z\mathcal{V}^\varepsilon_s\,ds.$$

We obtain

$$f_1^\varepsilon(z)=\frac{i\tilde{k}\varepsilon}{2}f'(z)\left\langle\Psi^\varepsilon_z,\tilde{\mathcal{V}}^\varepsilon_z\theta\right\rangle. \tag{32}$$

Proposition 1.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_1^\varepsilon(z)|=0,\quad\lim_{\varepsilon\to 0}\sup_{z<z_0}|f_1^\varepsilon(z)|=0\quad\textit{in probability}$$

.

Proof. We have

$$\mathbb{E}[|f_1^\varepsilon(z)|] \le \varepsilon \|f'\|_\infty \|\Psi_0\|_2 \mathbb{E}\|\theta\tilde{\mathcal{V}}_z^\varepsilon\|_2 \tag{33}$$

and

$$\sup_{z<z_0} |f_1^\varepsilon(z)| \le \varepsilon \|f'\|_\infty \|\Psi_0\|_2 \sup_{z<z_0} \|\theta\tilde{\mathcal{V}}_z^\varepsilon\|_2. \tag{34}$$

The right side of (33) is $O(\varepsilon)$ while that of (34) is $o(1)$ in probability by (26). Proposition 1 now follows from (33), (34) and (26). □

Set $f^\varepsilon(z) = f(z) + f_1^\varepsilon(z)$. A straightforward calculation yields

$$\begin{aligned}
\mathcal{A}^\varepsilon f_1^\varepsilon &= -\frac{\varepsilon}{4} f''(z) \left[\langle \Psi_z^\varepsilon, \Delta\theta\rangle + \frac{\tilde{k}^2}{\varepsilon} \langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\theta\rangle \right] \left\langle \Psi_z^\varepsilon, \tilde{\mathcal{V}}_z^\varepsilon\theta \right\rangle \\
&\quad -\frac{\varepsilon}{4} f'(z) \left[\left\langle \Psi_z^\varepsilon, \Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)\right\rangle + \frac{\tilde{k}^2}{\varepsilon} \left\langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle \right] - \frac{i\tilde{k}}{2\varepsilon} f'(z) \langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\theta\rangle
\end{aligned}$$

and, hence

$$\begin{aligned}
\mathcal{A}^\varepsilon f^\varepsilon(z) &= \frac{i}{2\tilde{k}} f'(z) \langle \Psi_z^\varepsilon, \Delta\theta\rangle - \frac{\tilde{k}^2}{4} f'(z) \left\langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle - \frac{\tilde{k}^2}{4} f''(z) \langle \Psi_z^\varepsilon, \mathcal{V}_z^\varepsilon\theta\rangle \left\langle \Psi_z^\varepsilon, \tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle \\
&\quad -\frac{\varepsilon}{4} \left[f'(z) \left\langle \Psi_z^\varepsilon, \Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)\right\rangle + f''(z) \langle \Psi_z^\varepsilon, \Delta\theta\rangle \left\langle \Psi_z^\varepsilon, \tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle \right] \\
&= A_1^\varepsilon(z) + A_2^\varepsilon(z) + A_3^\varepsilon(z) + A_4^\varepsilon(z)
\end{aligned} \tag{33}$$

where $A_2^\varepsilon(z)$ and $A_3^\varepsilon(z)$ are the $O(1)$ statistical coupling terms.

For the tightness criterion stated in the beginnings of the section, it remains to show

Proposition 2. $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable and*

$$\lim_{\varepsilon\to 0} \sup_{z<z_0} \mathbb{E}|A_4^\varepsilon(z)| = 0$$

.

Proof. We show that $\{A_i^\varepsilon\}, i = 1, 2, 3, 4$ are uniformly integrable. To see this, we have the following estimates.

$$\begin{aligned}
|A_1^\varepsilon(z)| &\le \frac{1}{2\tilde{k}} \|f'\|_\infty \|\Psi_0\|_2 \|\Delta\theta\|_2 \\
|A_2^\varepsilon(z)| &\le \frac{\tilde{k}^2}{4} \|f'\|_\infty \|\Psi_0\|_2 \|\mathcal{V}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\|_2 \\
|A_3^\varepsilon(z)| &\le \frac{\tilde{k}^2}{4} \|f''\|_\infty \|\Psi_0\|_2^2 \|\mathcal{V}_z^\varepsilon\theta\|_2 \|\tilde{\mathcal{V}}_z^\varepsilon\theta\|_2.
\end{aligned}$$

For fixed η, the second moments of the right hand side of the above expressions are uniformly bounded as $\varepsilon \to 0, \rho \to \infty$ and hence $A_1^\varepsilon(z), A_2^\varepsilon(z), A_3^\varepsilon(z)$ are uniformly integrable.

$$|A_4^\varepsilon| \le \frac{\varepsilon}{4} \left[\|f''\|_\infty \|\Psi_0\|_2^2 \|\Delta\theta\|_2 \|\tilde{\mathcal{V}}_z^\varepsilon\theta\|_2 + \|f'\|_\infty \|\Psi_z^\varepsilon\|_2 \|\Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)\|_2 \right].$$

The only term in (28) that is not bounded $\rho \to \infty$ is

$$\|\Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)\|_2, \tag{28}$$

whose second moment is $O(\rho^{2(2-\alpha)})$. By assumption, $\varepsilon\rho^{2-\alpha} \to 0$, A_4^ε is therefore uniformly integrable. Finally, it is clear that

$$\lim_{\varepsilon\to 0} \sup_{z<z_0} \mathbb{E}|A_4^\varepsilon(z)| = 0.$$

□

3.2. **Identification of the limit.** Once the tightness is established we can use another result in [15] (Chapter 3, Theorem 2) to identify the limit. Let $\bar{\mathcal{A}}$ be a diffusion or jump diffusion operator such that there is a unique solution ω_z in the space $D([0,\infty); L^2_w(\mathbb{R}^d))$ such that

$$f(\omega_z) - \int_0^z \bar{\mathcal{A}} f(\omega_s)\, ds \tag{28}$$

is a martingale. We shall show that for each $f \in C^\infty(\mathbb{R})$ there exists $f^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that

$$\sup_{z<z_0,\varepsilon} \mathbb{E}|f^\varepsilon(z) - f(\langle \Psi^\varepsilon_z, \theta\rangle)| \;<\; \infty \tag{29}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}|f^\varepsilon(z) - f(\langle \Psi^\varepsilon_z, \theta\rangle)| \;=\; 0, \quad \forall z < z_0 \tag{30}$$

$$\sup_{z<z_0,\varepsilon} \mathbb{E}|\mathcal{A}^\varepsilon f^\varepsilon(z) - \bar{\mathcal{A}} f(\langle \Psi^\varepsilon_z, \theta\rangle)| \;<\; \infty \tag{31}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}|\mathcal{A}^\varepsilon f^\varepsilon(z) - \bar{\mathcal{A}} f(\langle \Psi^\varepsilon_z, \theta\rangle)| \;=\; 0, \quad \forall z < z_0. \tag{32}$$

Then the aforementioned theorem implies that any tight processes $\langle \Psi^\varepsilon_z, \theta\rangle$ converges in law to the unique process generated by $\bar{\mathcal{A}}$. As before we adopt the notation $f(z) = f(\langle \Psi^\varepsilon_z, \theta\rangle)$.

For this purpose, we introduce the next perturbations $f_2^\varepsilon, f_3^\varepsilon$. Let

$$A_2^{(1)}(\phi) \;\equiv\; \int\int \theta(\mathbf{x})\phi(\mathbf{x})\Gamma^{(1)}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\theta(\mathbf{y})\; d\mathbf{x}\, d\mathbf{y} \tag{33}$$

$$A_3^{(1)}(\phi) \;\equiv\; \int \Gamma^{(1)}(\mathbf{x},\mathbf{x})\phi(\mathbf{x})\theta(\mathbf{x})\; d\mathbf{x} \tag{34}$$

where

$$\Gamma^{(1)}(\mathbf{x},\mathbf{y}) \;\equiv\; \mathbb{E}\left[\mathcal{V}^\varepsilon_z(\mathbf{x})\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{y})\right]. \tag{35}$$

It is easy to see that

$$A_2^{(1)}(\phi) = \mathbb{E}\left[\langle \phi, \mathcal{V}^\varepsilon_z\theta\rangle \left\langle \phi, \tilde{\mathcal{V}}^\varepsilon_z\theta\right\rangle\right]. \tag{35}$$

Define

$$f_2^\varepsilon(z) \equiv \frac{\tilde{k}^2}{4} f''(z) \int_z^\infty \mathbb{E}^\varepsilon_z\left[\langle \Psi^\varepsilon_z, \mathcal{V}^\varepsilon_s\theta\rangle \left\langle \Psi^\varepsilon_z, \tilde{\mathcal{V}}^\varepsilon_s\theta\right\rangle - A_2^{(1)}(\Psi^\varepsilon_z)\right] ds$$
$$f_3^\varepsilon(z) \equiv \frac{\tilde{k}^2}{4} f'(z) \int_z^\infty \mathbb{E}^\varepsilon_z\left[\left\langle \Psi^\varepsilon_z, \mathcal{V}^\varepsilon_s(\tilde{\mathcal{V}}^\varepsilon_s\theta)\right\rangle - A_3^{(1)}(\Psi^\varepsilon_z)\right] ds.$$

Let

$$\Gamma^{(2)}(\mathbf{x},\mathbf{y}) \equiv \mathbb{E}\left[\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x})\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{y})\right],$$

and

$$A_2^{(2)}(\phi) \;\equiv\; \int\int \theta(\mathbf{x})\phi(\mathbf{x})\Gamma^{(2)}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\theta(\mathbf{y})\; d\mathbf{x}\, d\mathbf{y} \tag{36}$$

$$A_3^{(2)}(\phi) \;\equiv\; \int \Gamma^{(2)}(\mathbf{x},\mathbf{x})\phi(\mathbf{x})\theta(\mathbf{x})\; d\mathbf{x}, \tag{37}$$

we then have

$$f_2^\varepsilon(z) \;=\; \frac{\varepsilon^2\tilde{k}^2}{8} f''(z)\left[\left\langle \Psi^\varepsilon_z, \tilde{\mathcal{V}}^\varepsilon_z\theta\right\rangle^2 - A_2^{(2)}(\Psi^\varepsilon_z)\right] \tag{38}$$

$$f_3^\varepsilon(z) \;=\; \frac{\varepsilon^2\tilde{k}^2}{8} f'(z)\left[\left\langle \Psi^\varepsilon_z, \tilde{\mathcal{V}}^\varepsilon_z\tilde{\mathcal{V}}^\varepsilon_z\theta\right\rangle - A_3^{(2)}(\Psi^\varepsilon_z)\right]. \tag{39}$$

Proposition 3.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_2^\varepsilon(z)|=0,\quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_3^\varepsilon(z)|=0.$$

Proof. We have the bounds

$$\begin{aligned}\sup_{z<z_0}\mathbb{E}|f_2^\varepsilon(z)| &\leq \sup_{z<z_0}\frac{\varepsilon^2\tilde{k}^2}{4}\|f''\|_\infty\left[\|\Psi_0\|_2^2\mathbb{E}\|\tilde{\mathcal{V}}_z^\varepsilon\theta\|_2^2+\mathbb{E}[A_2^{(2)}(\Psi_z^\varepsilon)]\right]\\ \sup_{z<z_0}\mathbb{E}|f_3^\varepsilon(z)| &\leq \sup_{z<z_0}\frac{\varepsilon^2\tilde{k}^2}{4}\|f'\|_\infty\left[\|\Psi_0\|_2\mathbb{E}\|\tilde{\mathcal{V}}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\|_2+\mathbb{E}[A_3^{(2)}(\Psi_z^\varepsilon)]\right];\end{aligned}$$

both of them tend to zero. □

We have

$$\begin{aligned}\mathcal{A}^\varepsilon f_2^\varepsilon(z) &= \frac{\tilde{k}^2}{4}f''(z)\left[-\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\theta\rangle\left\langle\Psi_z^\varepsilon,\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle+A_2^{(1)}(\Psi_z^\varepsilon)\right]+R_2^\varepsilon(z)\\ \mathcal{A}^\varepsilon f_3^\varepsilon(z) &= \frac{\tilde{k}^2}{4}f'(z)\left[-\left\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon(\tilde{\mathcal{V}}_z^\varepsilon\theta)\right\rangle+A_3^{(1)}(\Psi_z^\varepsilon)\right]+R_3^\varepsilon(z)\end{aligned}$$

with

$$\begin{aligned}R_2^\varepsilon(z) &= \frac{i\varepsilon^2\tilde{k}}{8}\frac{f'''(z)}{2}\left[\langle\Psi_z^\varepsilon,\Delta\theta\rangle+\frac{\tilde{k}^2}{\varepsilon}\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\theta\rangle\right]\left[\left\langle\Psi_z^\varepsilon,\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle^2-A_2^{(2)}(\Psi_z^\varepsilon)\right]\\ &\quad+\frac{i\varepsilon^2\tilde{k}}{4}f''(z)\left\langle\Psi_z^\varepsilon,\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle\left[\left\langle\Psi_z^\varepsilon,\Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)\right\rangle+\frac{\tilde{k}^2}{\varepsilon}\left\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle\right]\\ &\quad-\frac{i\varepsilon^2\tilde{k}}{4}f''(z)\left[\left\langle\Psi_z^\varepsilon,\Delta(G_\theta^{(2)}\Psi_z^\varepsilon)\right\rangle+\frac{\tilde{k}^2}{\varepsilon}\left\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon G_\theta^{(2)}\Psi_z^\varepsilon\right\rangle\right] \qquad (34)\end{aligned}$$

where $G_\theta^{(2)}$ denotes the operator

$$G_\theta^{(2)}\phi\equiv\int\theta(\mathbf{x})\Gamma^{(2)}(\mathbf{x},\mathbf{y})\theta(\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y}.$$

Similarly

$$\begin{aligned}R_3^\varepsilon(z) &= \frac{i\varepsilon^2\tilde{k}}{8}f'(z)\left[\left\langle\Psi_z^\varepsilon,\Delta(\tilde{\mathcal{V}}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta)\right\rangle+\frac{\tilde{k}^2}{\varepsilon}\left\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle\right]\\ &\quad+\frac{i\varepsilon^2\tilde{k}}{8}f''(z)\left[\langle\Psi_z^\varepsilon,\Delta\theta\rangle+\frac{\tilde{k}^2}{\varepsilon}\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\theta\rangle\right]\left[\left\langle\Psi_z^\varepsilon,\tilde{\mathcal{V}}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta\right\rangle-A_3^{(2)}(\Psi_z^\varepsilon)\right]\\ &\quad-\frac{i\varepsilon^2\tilde{k}}{8}f'(z)\left[\left\langle\Psi_z^\varepsilon,\Delta(\Gamma_0^{(2)}\theta)\right\rangle+\frac{\tilde{k}^2}{\varepsilon}\left\langle\Psi_z^\varepsilon,\mathcal{V}_z^\varepsilon\Gamma_0^{(2)}\theta\right\rangle\right]\end{aligned}$$

where

$$\Gamma_0^{(2)}(\mathbf{x})\equiv\Gamma^{(2)}(\mathbf{x},\mathbf{x}).$$

Proposition 4.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_2^\varepsilon(z)|=0,\quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_3^\varepsilon(z)|=0.$$

The argument is entirely analogous to that for Proposition 3. The most severe factors involve $\Delta(\tilde{\mathcal{V}}_z^\varepsilon\theta)$ and $\Delta(\tilde{\mathcal{V}}_z^\varepsilon\tilde{\mathcal{V}}_z^\varepsilon\theta)$, both of which also have the prefactor ε^2. Therefore they do not require any more conditions than what we have needed so far.

Consider the test function $f^\varepsilon(z) = f(z) + f_1^\varepsilon(z) - f_2^\varepsilon(z) - f_3^\varepsilon(z)$. We have

$$\mathcal{A}^\varepsilon f^\varepsilon(z) = \frac{i}{2\tilde{k}} f'(z) \langle \Psi_z^\varepsilon, \Delta\theta \rangle - \frac{\tilde{k}^2}{4} f''(z) A_2^{(1)}(\Psi_z^\varepsilon) - \frac{\tilde{k}^2}{4} f' A_3^{(1)}(\Psi_z^\varepsilon) - R_2^\varepsilon(z) - R_3^\varepsilon(z) + A_4^\varepsilon(z). \tag{31}$$

Set

$$R^\varepsilon(z) = R_1^\varepsilon(z) - R_2^\varepsilon(z) - R_3^\varepsilon(z), \quad \text{with } R_1^\varepsilon(z) = A_4^\varepsilon(z). \tag{31}$$

It follows from Propositions 3 and 5 that

$$\lim_{\varepsilon\to 0} \sup_{z<z_0} \mathbb{E}|R^\varepsilon(z)| = 0.$$

Recall that

$$\begin{aligned}
M_z^\varepsilon(\theta) &= f^\varepsilon(z) - \int_0^z \mathcal{A}^\varepsilon f^\varepsilon(s)\, ds \\
&= f(z) + f_1^\varepsilon(z) - f_2^\varepsilon(z) - f_3^\varepsilon(z) - \int_0^z \frac{i}{2\tilde{k}} f'(z) \langle \Psi_z^\varepsilon, \Delta\theta \rangle\, ds \\
&\quad + \int_0^z \frac{\tilde{k}^2}{4} \left[f''(s) A_2^{(1)}(\Psi_s^\varepsilon) + f'(s) A_3^{(1)}(\Psi_s^\varepsilon) \right] ds - \int_0^z R^\varepsilon(s)\, ds
\end{aligned}$$

is a martingale. Now that (29)-(32) are satisfied we can identify the limiting martingale to be

$$M_z(\theta) = f(z) - \int_0^z \left\{ f'(s) \left[\frac{i}{2\tilde{k}} \langle \Psi_s, \Delta\theta \rangle - \frac{\tilde{k}^2}{4} \bar{A}_3(\Psi_s) \right] - \frac{\tilde{k}^2}{4} f''(s) \bar{A}_2(\Psi_s) \right\} ds \tag{28}$$

where

$$\bar{A}_2(\phi) = \lim_{\rho\to\infty} A_2^{(1)}(\phi), \quad \bar{A}_3(\phi) = \lim_{\rho\to\infty} A_3^{(1)}(\phi).$$

Since $\langle \Psi_z^\varepsilon, \theta \rangle$ is uniformly bounded

$$|\langle \Psi_z^\varepsilon, \theta \rangle| \le \|\Psi_0\|_2 \|\theta\|_2$$

we have the convergence of the second moment

$$\lim_{\varepsilon\to 0} \mathbb{E}\left\{ \langle \Psi_z^\varepsilon, \theta \rangle^2 \right\} = \mathbb{E}\left\{ \langle \Psi_z, \theta \rangle^2 \right\}.$$

Use $f(r) = r$ and r^2 in (3.2)

$$M_z^{(1)}(\theta) = \langle \Psi_z, \theta \rangle - \int_0^z \left[\frac{i}{2\tilde{k}} \langle \Psi_s, \Delta\theta \rangle - \frac{\tilde{k}^2}{4} \bar{A}_3(\Psi_s) \right] ds$$

is a martingale with the quadratic variation

$$\left[M^{(1)}(\theta), M^{(1)}(\theta) \right]_z = \frac{-\tilde{k}^2}{2} \int_0^z \bar{A}_2(\Psi_s)\, ds = \frac{-\tilde{k}^2}{2} \int_0^z \left\langle \theta, \overline{\mathcal{K}}_{\Psi_s}\theta \right\rangle ds$$

where

$$\overline{\mathcal{K}}_{\Psi_s}\theta = \int \Psi_s(\mathbf{x}) \overline{\Gamma}(\mathbf{x}, \mathbf{y}) \Psi_s(\mathbf{y}) \theta(\mathbf{y})\, d\mathbf{y}.$$

Therefore,

$$M_z^{(1)} = \frac{i\tilde{k}}{\sqrt{2}} \int_0^z \sqrt{\overline{\mathcal{K}}_{\Psi_s}}\, dW_s$$

where W_s is a real-valued, cylindrical Wiener process (i.e. $dW_z(\mathbf{x})$ is a space-time white noise field) and $\sqrt{\mathcal{K}_{\Psi_s}}$ is the square-root of the positive-definite operator given in (18).

4. Proof of Theorem 2

We turn to eq. (2.3). We will use the same notation

$$\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x}) = \frac{1}{\varepsilon^2}\int_z^\infty \mathbb{E}^\varepsilon_z \mathcal{V}'(\frac{s}{\varepsilon^2}, \mathbf{x})\, ds.$$

The proof of the uniform integrability of $\mathcal{A}^\varepsilon[f(z) - f^\varepsilon_1(z)]$ breaks down. The problem is related to the divergence of the second moment of $\tilde{\mathcal{V}}^\varepsilon_z$, an $O(\eta^{2-2\alpha})$ quantity. In this case, we work with the perturbed test function

$$f^\varepsilon(z) = f(z) + f^\varepsilon_1(z) - f^\varepsilon_2(z) - f^\varepsilon_3(z)$$

for both tightness and identification.

Proposition 5.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f^\varepsilon_j(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}|f^\varepsilon_j(z)| = 0 \quad \textit{in probability}, \quad \forall j = 1, 2, 3. \tag{28}$$

Proof. The argument for the case of $f^\varepsilon_1(z)$ is the same as Proposition 1. For $f^\varepsilon_2(z)$ and $f^\varepsilon_3(z)$ we have the bounds

$$\sup_{z<z_0}\mathbb{E}|f^\varepsilon_2(z)| \leq C_1\varepsilon^2\eta^{2-2\alpha} \tag{29}$$

$$\sup_{z<z_0}\mathbb{E}|f^\varepsilon_3(z)| \leq C_2\varepsilon^2\eta^{2-2\alpha} \tag{30}$$

both of which are $O(\varepsilon^2)$ under the assumptions of the theorem.

As for estimating $\sup_{z<z_0}|f^\varepsilon_j(z)|, j = 2, 3$, we can use

$$M^d\int_{|\mathbf{x}|<M}|\tilde{\mathcal{V}}^\varepsilon_z|^2(\mathbf{x})\; d\mathbf{x} \quad \text{in place of} \quad \int_{|\mathbf{x}|<M}\mathbb{E}|\tilde{\mathcal{V}}^\varepsilon_z|^2(\mathbf{x})\, d\mathbf{x}$$

in the above bounds and obtain by assumption the desired estimate which have a similar order of magnitude with an additional factor of $1/\varepsilon$ and a random constant possessing a finite moment. □

Proposition 6.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R^\varepsilon_j(z)| = 0, \quad j = 1, 2, 3.$$

Proof. The proof is similar to that of Proposition 5 with the additional consideration due to $\eta \to 0$. These additional terms can all be estimated by

$$C_1\varepsilon\int_{|\mathbf{x}|<M}\mathbb{E}\left[\left|\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x})\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x})\right|\right]\; d\mathbf{x} \leq C_2\varepsilon\eta^{2-2\alpha}$$

which tends to zero under the assumptions of the theorem. □

For the tightness it remains to show

Proposition 7. $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable.*

Proof. We shall prove that each term in the expression (3.2) is uniformly integrable. The analysis of Proposition 2 still works here except for additional considerations in connection to the limit $\eta \to 0$. The most severe term arising from this is

$$\varepsilon\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x})\tilde{\mathcal{V}}^\varepsilon_z(\mathbf{x}) \tag{30}$$

in the expression for R^ε_3 whose second moment behaves like $\varepsilon^2\eta^{4-4\alpha}$ and vanishes in the limit. □

Now we have all the estimates needed to identify the limit as in the proof of Theorem 1.

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