TIME REVERSAL OF WAVES IN A PERTURBED RANDOM MEDIUM

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Abstract. In a time reversal experiment, a signal recorded by an array of transducers and sent back time reversed into the same medium approximately refocuses on the original source center. The refocusing resolution is improved in an inhomogeneous medium. In this work we study the effect of changes in the medium, namely, the case when back propagation takes place in a perturbed medium.

Under the paraxial approximation assumption for a medium with weak inhomogeneities we consider a high frequency white noise regime. We show that relatively small perturbations do not affect the stable refocusing (self-averaging) for a localized source, but produces an interesting blurring of the refocused time signal. In some simple situations this effect can be explicitly quantified and related to the statistical model for the medium.

Contents

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1. Introduction

In time reversal experiments a signal emitted by a localized source is recorded by an array of receiver-transducers [also known as Time Reversal Mirror (TRM)], then it is time-reversed and re-emitted into the medium, that is, the tail of the recorded signal is sent back first. In the absence of absorption the signal propagates back and focuses near the source. In Figure 1 a time reversal experiment is schematically illustrated. This phenomenon has numerous applications and has been thoroughly studied, experimentally and theoretically, see e.g. [12, 13, 18, 19]. It has also been the subject of an active mathematical research in the context of wave propagation in random media (some references relevant to this paper are mentioned below).

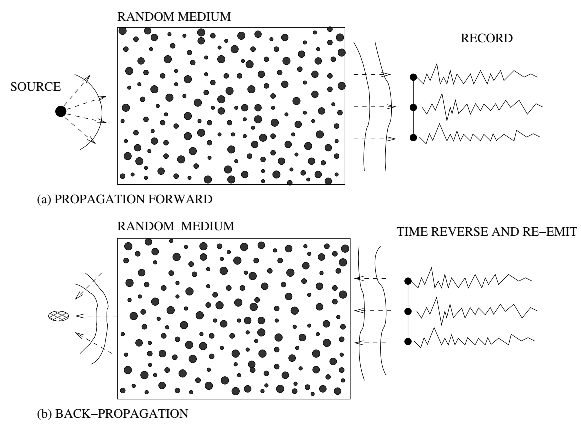

Figure 1. Schematic representation of a time reversal experiment in a random medium.

When time-reversal experiments are carried out in a (random) heterogeneous medium, under appropriate conditions, the size of the refocused spot appears to be smaller than in the case of a homogeneous medium. This surprising effect is a consequence of the multiple scattering in the inhomogeneous medium that creates multipathing and allows the transducers array to captures waves with a larger angular spread. The enhancement of the resolution of the refocused wave by the multipathing is called *super-resolution* [10]. Moreover, the time-reversed back-propagated pulse is also self-averaging so that the refocusing is *statistically stable*, that is, it does not depend on the particular realization of the random medium.

In the context of wave propagation in random media, these properties have been thoroughly studied under the paraxial approximation for several asymptotic regimes in [3, 10, 15, 16, 17, 28, 29]. The full wave equation was considered in [4, 5], see also the review [19]. The case with the full wave equation and layered random media is considered in [20, 21].

The aim of this work is to analyze how medium changes affect time reversal experiments. This is an important issue since in practical applications the medium properties may change during the experiment [11, 14]. Some experimental studies have also shown that the refocused signal is modified as the underlying medium changes [23, 32]. Therefore, we analyze time reversal experiments in a slightly different setting. Namely, we allow the time-reversed wave to propagate back into a medium different from the one upon which the forward propagation took place. [In the schematic representation in Figure 1, this means that the random medium in part (b) is different from the one in part (a).] This *generalized* time reversal procedure leads to more realistic models and presents a wider range of applications than the standard one [32].

Recently, time reversal with medium perturbations has been addressed theoretically and numerically. Bal & Verástegui in [8] considered the transport and diffusive regimes for the full wave equation in two and three dimensions, in their theoretical study some asymptotics for the coherent time-reversed back-propagated wave were obtained. Moreover, their numerical experiments showed the statistical stabilization of the refocused wave.

The parabolic approximation has been used for the study of time reversal of waves since the pioneer paper of Blomgren, Papanicolaou & Zhao (2002) [10]. For the parabolic wave approximation Bal & Ryzhik presented in [6] a rigorous study of the effect of a changing medium. They considered the radiative transfer regime and the white noise limit, the Itô-Schrödinger regime, for the parabolic wave equation. In both cases, the effect of the changing medium is characterized as the combination of an effective absorption and phase modulation of the refocused signal at each frequency. They also established the statistical stabilization of the refocused wave.

Time reversal in the context of medium perturbations was studied in [1] for randomly layered media. The main result then states that the pulse shape is described by a random function, implying that the statistical stabilization property is lost, the refocused pulse is not statistically stable. In this paper we study the effect of medium perturbations on the quality of the refocused signal in the regime of the paraxial approximation in a scaling regime that is different from the one analyzed in [6]. We find again that the lateral medium variation, or diversity, then leads to a situation where the refocused signal is statistically stable also in the case with medium perturbations. We show that for a localized source relatively small perturbations in the medium do not affect the stable refocusing (self-averaging), but produces an interesting blurring of the refocused time signal. In some simple situations this effect can be explicitly quantified and related to the statistical model for the medium.

Some of the results obtained here are closely related to those of the the recent paper [7]. There a diffusion limit for the expectation of the Wigner transform, corresponding to solutions of the Schrödinger equation, is analyzed. In the present work, besides the expectation we also study some second order statistics that allows us to establish the self-averaging of the back-propagated wave. Moreover, the Schrödinger equation treated here is non-autonomous, so we also analyze the effect of the lateral diversity that give rise to a slightly different diffusion regime.

This paper is organized as follows, in Section 2 we describe the time reversal of waves in a perturbed medium, presenting the asymptotic regime we are interesting in and the mathematical description of the time-reversed and back-propagated wave. In Section 3, we consider the asymptotic behavior of the refocused wave. We describe the coherent time-reversed back-propagated limiting wave and establish its statistical stability. Finally, in Section 4 we interpret the obtained results and draw some concluding remarks. In Appendix A, some auxiliary results concerning the diffusion limit of random differential equations used in the main part of the paper are presented.

2. Time reversal of waves in a perturbed medium

In this section we generalize the analysis of time reversal of waves presented in [28] to the situation in which the medium has been subject to small perturbations.

2.1. **Parabolic approximation and asymptotic regime.** Recall that scalar wave propagation in a inhomogeneous medium is modeled by

$$\frac{1}{c^2(\mathbf{x},z)}\frac{\partial^2 v}{\partial t^2} - \Delta_{(\mathbf{x},z)} v = 0 \tag{1}$$

where $c(\mathbf{x},z)$ is the local wave speed, $\mathbf{x}\in\mathbb{R}^d$ with $d\geq 2$ is the transversal coordinate, z represents the longitudinal coordinate, t represents time, $\Delta_{(\mathbf{x},z)} = \Delta + \frac{\partial^2}{\partial z^2}$ is the (full) Laplacian operator and $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ the transversal Laplacian.

Under the assumptions that the wave field has a 'beam-like' structure and that back scattering in the z direction can be neglected, one can apply the parabolic or paraxial approximation [9] and write

$$v(\mathbf{x},z,t) \approx u(\mathbf{x},z,t) = \int e^{ik(z-c_0 t)}\psi(\mathbf{x},z;k)dk \tag{2}$$

where the (complex) wave amplitude $\psi(\mathbf{x},z;k)$ is a solution of the Schrödinger equation

$$2ik\frac{\partial\psi}{\partial z} + \Delta\psi + k^2(n^2(\mathbf{x},z)-1)\psi = 0 \tag{3}$$

with suitable initial conditions and where $n(\mathbf{x},z) = \frac{c_0}{c(\mathbf{x},z)}$ represents the refraction index with respect to the reference speed c_0.

Next, we present a high frequency regime for the parabolic approximation in which small scale separations take place. This regime was introduced in [28] to analyze the statistical stabilization in time reversal of waves. Later on, we will analyze the asymptotic behavior of the wave field under medium perturbations in a successive limit.

Let L_x, L_z be the characteristic propagation distance in the transversal and longitudinal directions, respectively, and k_0 the central carrier wave number of the incident wave. We introduce the scaling

$$k = k_0 k', \quad \mathbf{x} = L_x \mathbf{x}', \quad z = L_z z' \quad \text{and} \quad n^2(\mathbf{x},z) = 1 + 2\sigma\mu\left(\frac{\mathbf{x}}{l_x}, \frac{z}{l_z}\right), \tag{4}$$

and the dimensionless parameters $\epsilon = \frac{l_z}{L_z}$, $\beta = \frac{1}{k_0 l_z}$ and $\theta = \frac{L_z}{k_0 L_x^2}$ (this is the inverse of the so-called Fresnel number). By letting $\frac{l_x}{L_x} = \epsilon^a$ and $\sigma = \frac{\beta\epsilon^{1/2+a}}{\theta}$, substitution in Equation (3) gives (after dropping primes) the scaled Schrödinger equation

$$ik\theta\frac{\partial\psi}{\partial z} + \tfrac{1}{2}\theta^2\Delta\psi + k^2\mu^\epsilon(\mathbf{x},z)\psi = 0 \tag{5}$$

where the fluctuations are given as

$$\mu^\epsilon(\mathbf{x},z) = \epsilon^{a-1/2}\mu(\tfrac{\mathbf{x}}{\epsilon^a}, \tfrac{z}{\epsilon}) \tag{6}$$

with $\mu = O(1)$.

The regime in which we are interested appears when we assume the separation of scales

$$\theta \ll \epsilon \ll 1$$

and the conditions

$$\beta\epsilon \ll \theta \quad \text{and} \quad 0 < a \leq 1,$$

which ensures the validity of the parabolic approximation. The corresponding regime with $a = 1$ was introduced in [28]. Concerning physical parameters these assumptions imply that

$$\frac{1}{k_0 l_z} \ll \left(\frac{L_x}{L_z}\right)^2 \ll 1 \quad \text{and} \quad \frac{l_x}{L_x} = \left(\frac{l_z}{L_z}\right)^a \ll 1. \tag{7}$$

In the first condition, the first inequality corresponds to enforcing a high frequency regime, whereas the second inequality is the usual condition for validity of the paraxial approximation. The second condition states that the refractive index is anisotropic ($a < 1$) and fluctuates rapidly. Furthermore, the presence of a substantial lateral diversity is also enforced.

The refractive index fluctuation $\sigma\mu(\cdot,\cdot)$ in Equation (4) will be modeled as a centered random field over $\mathbb{R}^d$ with l_x, l_z corresponding to its transversal and longitudinal correlation lengths, respectively, and σ to its mean square intensity.

Finally, we remark that in the dimensionless space variables introduced in (4) the approximated wave field $u'(\mathbf{x}', z', t) = u(L_x\mathbf{x}', L_z z', t)$ reads (after dropping primes)

$$u(\mathbf{x}, z, t) = k_0 \int e^{ikk_0(L_z z - c_0 t)} \psi(\mathbf{x}, z; k) dk \tag{8}$$

where $\psi(\mathbf{x}, z; k)$ is now the solution of Eq. (5) with the corresponding initial conditions $\psi(\mathbf{x}, z = z_0; k) = \psi_I(\mathbf{x}; k)$. Furthermore, we have that

$$\psi(\mathbf{x}, z; k) = \int_{\mathbb{R}^d} G^{\theta,\epsilon}(\mathbf{x}, z; \mathbf{y}, z_0; k) \psi_I(\mathbf{y}; k) d\mathbf{y} \tag{9}$$

where $G^{\theta,\epsilon}$ is the Green's function corresponding to the Schrödinger Equation (5) which satisfies the initial condition $G^{\theta,\epsilon}(\mathbf{x}, z = z_0; \mathbf{y}, z_0; k) = \delta(\mathbf{x} - \mathbf{y})$.

2.2. **Time-reversal of waves.** In a standard time reversal experiment an emitted wave pulse is (partially) recorded by means of an array of receivers-transducers (Time-Reversal Mirror [TRM]), then it is time-reversed (corresponding to a phase-conjugation step) and re-emitted back into the medium. The main effect is the refocusing of this back-propagated wave near the location of the initial pulse. The refocusing is not exact because of the finite size of the TRM. It is well-known that the presence of inhomogeneities in the medium allows for a sharper focusing of the wave [10, 29], this property is called *super-resolution*. Furthermore, despite the fact that the refractive index is modeled as a random field, asymptotically the refocused wave does not depend on the random medium realization. This is the celebrated *statistical stabilization* (or self-averaging) property.

2.2.1. *Medium perturbations.* We further assume that the medium in which the wave propagates back after time reversal is different from the one of the forward propagation stage. More specifically, the refractive index fluctuations are given by the random fields $\mu_f^\epsilon(\mathbf{x}, z)$ and $\mu_b^\epsilon(\mathbf{x}, z)$ where the backward medium fluctuations are obtained by perturbations of the fluctuations in the forward medium, i.e. $\mu_b^\epsilon =$

$\mu_f^\epsilon + \eta^\epsilon$. We shall analyze the case with perturbations such that $\eta^\epsilon = O(\frac{\theta}{\sqrt{\epsilon}})$. Thus, we consider

$$\mu_f^\epsilon(\mathbf{x}, z) = \sqrt{\epsilon}\mu\left(\frac{\mathbf{x}}{\epsilon^a}, \frac{z}{\epsilon}\right) \tag{10}$$

$$\eta^\epsilon(\mathbf{x}, z) = \frac{\theta}{\sqrt{\epsilon}}\eta\left(\frac{\mathbf{x}}{\epsilon^a}, \frac{z}{\epsilon}\right) \tag{11}$$

where $\mu = O(1)$, $\eta = O(1)$. Note that the perturbations are relatively small in the sense that $O(\eta^\epsilon(\cdot)) \ll O(\mu^\epsilon(\cdot))$ We also assume that $\mu(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ are stationary, mean-zero, statistically independent, exponentially mixing and sufficiently smooth, isotropic random fields. More specifically, we assume that the isotropic random field $(\nabla_{\mathbf{x}}\mu(\cdot), \eta(\cdot))^t$ satisfies conditions C.1–C.5 in Appendix A (on page 22).

We represent their covariances by

$$R_\mu(\mathbf{x}, z) = \mathbb{E}\{\mu(\mathbf{x}+\mathbf{x}', z+z')\mu(\mathbf{x}', z')\}, \tag{12}$$

$$R_\eta(\mathbf{x}, z) = \mathbb{E}\{\eta(\mathbf{x}+\mathbf{x}', z+z')\eta(\mathbf{x}', z')\} \tag{13}$$

and their power spectra as

$$\hat{R}_\mu(\mathbf{q}, r) = \int_{\mathbb{R}^{d+1}} e^{-i(\mathbf{x}\cdot\mathbf{q}+z\cdot r)} R_\mu(\mathbf{x}, z) d\mathbf{x} dz, \tag{14}$$

$$\hat{R}_\eta(\mathbf{q}, r) = \int_{\mathbb{R}^{d+1}} e^{-i(\mathbf{x}\cdot\mathbf{q}+z\cdot r)} R_\eta(\mathbf{x}, z) d\mathbf{x} dz. \tag{15}$$

2.2.2. *Back-propagated wave.* We proceed to obtain a representation of the back-propagated wave. We consider a wave emitted from a localized source positioned at the point $(\mathbf{x}_0, 0)$, more specifically we assume initial conditions satisfying the scaling property

$$\psi(\mathbf{x}, 0; k) = \psi_I(\mathbf{x}; k) = \phi_0(\tfrac{\mathbf{x}-\mathbf{x_0}}{\theta}; k)\,. \tag{16}$$

The time-reversal mirror is located in the plane $z = L$. It is characterized by a mirror function $\chi(\mathbf{x})$. For instance, if the array of receiver-transducers is located in the domain $D \subset \mathbb{R}^d$ then the mirror function can be $\chi(\mathbf{x}) = \mathbf{1}_D(\mathbf{x})$, the indicator function of the domain D, or a smooth function rapidly decaying outside D. We also model a blurring of the recorded signal by the mirror. It is modeled by a convolution kernel that satisfies the scaling property $f_\theta(\mathbf{x}) = \theta^{-d} f(\frac{\mathbf{x}}{\theta})$, since the blurring should take place on the scale of the source [3, 4].

The wave arriving at the time reversal mirror is given by

$$u^-(\mathbf{x}, L, t) = k_0 \int e^{ikk_0(LL_z - c_0 t)} \psi_f(\mathbf{x}, L; k) dk, \tag{17}$$

and after recording and time-reversal at the mirror it becomes

$$u^+(\mathbf{x}, L, t) = k_0 \chi(\mathbf{x}) \int \int_{\mathbb{R}^d} e^{ikk_0(LL_z + c_0 t)} f_\theta(\mathbf{x} - \mathbf{y}) \chi(\mathbf{y}) \psi_f(\mathbf{y}, L; k) d\mathbf{y} dk \tag{18}$$

where $\psi_f(\mathbf{x}, z; k)$ is the solution of equation (5) with initial condition $\psi(\mathbf{x}, 0; k) = \psi_I(\mathbf{x}; k)$ and $\mu^\epsilon = \mu_f^\epsilon$. Moreover, the back-propagated wave has the representation

$$u_b(\mathbf{x}, z, t) = k_0 \int e^{ikk_0(L_z z + c_0 t)} \psi_b(\mathbf{x}, z; k) dk \tag{19}$$

where $\psi_b(\mathbf{x}, z; k)$ solves (5) with final condition

$$\psi_b(\mathbf{x}, z = L; k) = \chi(\mathbf{x}) \int_{\mathbb{R}^d} f_\theta(\mathbf{x} - \mathbf{y})\chi(\mathbf{y})\psi_f(\mathbf{y}, L; k) d\mathbf{y}$$

in the situation where $\mu^\epsilon = \mu_b^\epsilon$. Consequently, on the plane $z = 0$ the back-propagated wave is given by

$$u^B(\mathbf{x}, t) = k_0 \int e^{ikk_0c_0t}\psi_b(\mathbf{x}, 0; k) dk.$$

Moreover, using Green's functions we get the representation

$$\begin{aligned}\psi_b(\mathbf{x}, 0; k) &= \int_{\mathbb{R}^d} G_b^{\theta,\epsilon}(\mathbf{x}, 0, \mathbf{y}, L; k)\psi_b(\mathbf{y}, L; k) d\mathbf{y}\\ &= \int_{\mathbb{R}^{2d}} G_b^{\theta,\epsilon}(\mathbf{x}, 0; \mathbf{y}, L; k)\chi(\mathbf{y}) f_\theta(\mathbf{y} - \mathbf{y}')\chi(\mathbf{y}')\psi_f(\mathbf{y}', L; k) d\mathbf{y} d\mathbf{y}'\\ &= \int_{\mathbb{R}^{3d}} G_f^{\theta,\epsilon}(\mathbf{y}', L; \mathbf{y}'', 0; k) G_b^{\theta,\epsilon}(\mathbf{x}, 0; \mathbf{y}, L; k)\\ &\qquad\qquad \chi(\mathbf{y}) f_\theta(\mathbf{y} - \mathbf{y}')\chi(\mathbf{y}')\psi_I(\mathbf{y}''; k) d\mathbf{y} d\mathbf{y}' d\mathbf{y}''\end{aligned}$$

where $G_f^{\theta,\epsilon}$ and $G_b^{\theta,\epsilon}$ represent the Green's functions associated to the Schrödinger equation (5) in the case where $\mu^\epsilon = \mu_f^\epsilon$ and μ_b^ϵ, respectively, i.e. for the forward and backward propagation stages. Next, we will observe this wave through a θ-scaled space window centered at $\mathbf{x}_0$ by introducing the quantities $u_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0, t) = u^B(\mathbf{x}_0 + \theta\boldsymbol{\xi}, t)$ and $\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \psi_b(\mathbf{x}_0 + \theta\boldsymbol{\xi}, 0; k)$. Thus we have that

$$u_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0, t) = k_0 \int e^{ikk_0c_0t}\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k) dk.$$

We next carry out some straightforward transformations

$$\begin{aligned}\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \int_{\mathbb{R}^{3d}} & G_f^{\theta,\epsilon}(\mathbf{y}', L; \mathbf{y}'', 0; k) G_b^{\theta,\epsilon}(\mathbf{x}_0 + \theta\boldsymbol{\xi}, 0; \mathbf{y}, L; k)\\ & \chi(\mathbf{y}) f_\theta(\mathbf{y} - \mathbf{y}')\chi(\mathbf{y}')\phi_0\left(\frac{\mathbf{y}'' - \mathbf{x}_0}{\theta}; k\right) d\mathbf{y} d\mathbf{y}' d\mathbf{y}'',\end{aligned}$$

by setting $\mathbf{y}'' = \mathbf{x}_0 + \theta\boldsymbol{\zeta}$

$$\begin{aligned}\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \int_{\mathbb{R}^{3d}} & G_f^{\theta,\epsilon}(\mathbf{y}', L; \mathbf{x}_0 + \theta\boldsymbol{\zeta}, 0; k) G_b^{\theta,\epsilon}(\mathbf{x}_0 + \theta\boldsymbol{\xi}, 0; \mathbf{y}, L; k)\\ & \chi(\mathbf{y}) f(\frac{\mathbf{y} - \mathbf{y}'}{\theta})\chi(\mathbf{y}')\phi_0(\boldsymbol{\zeta}; k) d\mathbf{y} d\mathbf{y}' d\boldsymbol{\zeta},\end{aligned}$$

and by using the Fourier transform

$$\begin{aligned}\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \int_{\mathbb{R}^{4d}} & G_f^{\theta,\epsilon}(\mathbf{y}', L; \mathbf{x}_0 + \theta\boldsymbol{\zeta}, 0; k) G_b^{\theta,\epsilon}(\mathbf{x}_0 + \theta\boldsymbol{\xi}, 0; \mathbf{y}, L; k)\\ & \chi(\mathbf{y}) e^{-i\mathbf{q}\frac{\mathbf{y}}{\theta}} \hat{f}(\mathbf{q}) e^{i\mathbf{q}\frac{\mathbf{y}'}{\theta}} \chi(\mathbf{y}')\phi_0(\boldsymbol{\zeta}; k) d\mathbf{y} d\mathbf{y}' d\boldsymbol{\zeta} d\mathbf{q}.\end{aligned} \tag{20}$$

For the Green's function the following reciprocity property holds $G(\mathbf{x}, z; \mathbf{x}_0, z_0) = \overline{G(\mathbf{x}_0, z_0; \mathbf{x}, z)}$, where $\overline{H}$ stands for the complex conjugate of H. Indeed, for a solution $\varphi(\mathbf{x}, z)$ of the Schrödinger equation (5) (where for simplicity we consider $k = \theta = \epsilon = 1$) we have

$$\varphi(\mathbf{x}, z) = \int G(\mathbf{x}, z; \mathbf{y}, z_0)\varphi(\mathbf{y}, z_0) d\mathbf{y}.$$

By applying the operator $i\partial/\partial z_0$, we get

$$\int \Big\{ \big(i\frac{\partial}{\partial z_0} G(\mathbf{x},z;\mathbf{y},z_0) - \mu(\mathbf{y},z_0) G(\mathbf{x},z;\mathbf{y},z_0)\big)\varphi(\mathbf{y},z_0) \\ - \tfrac{1}{2} G(\mathbf{x},z;\mathbf{y},z_0)\Delta_{\mathbf{y}}\varphi(\mathbf{y},z_0)\Big\} d\mathbf{y} = 0$$

and after integration by parts it yields

$$\int \Big\{ \big(i\frac{\partial}{\partial z_0} - \tfrac{1}{2}\Delta_{\mathbf{y}} - \mu(\mathbf{y},z_0)\big) G(\mathbf{x},z;\mathbf{y},z_0)\Big\}\varphi(\mathbf{y},z_0) d\mathbf{y} = 0.$$

Since the function $\varphi(\mathbf{y},z_0)$ is arbitrary, it follows that

$$\big(i\frac{\partial}{\partial z_0} - \tfrac{1}{2}\Delta_{\mathbf{y}} - \mu(\mathbf{y},z_0)\big) G(\mathbf{x},z;\mathbf{y},z_0) = 0.$$

On the other hand, notice that $\overline{G(\mathbf{y},z_0;\mathbf{x},z)}$ satisfies a similar equation and also that $\overline{G(\mathbf{y},z;\mathbf{x},z)} = G(\mathbf{x},z;\mathbf{y},z) = \delta(\mathbf{x}-\mathbf{y})$, consequently we have $\overline{G(\mathbf{y},z_0;\mathbf{x},z)} = G(\mathbf{x},z;\mathbf{y},z_0)$.

Using this reciprocity property we can re-write (20) as

$$\phi^B_\theta(\boldsymbol{\xi},\mathbf{x}_0;k) = \int_{\mathbb{R}^{4d}} \hat{f}(\mathbf{q})\, \overline{G^{\theta,\epsilon}_f(\mathbf{x}_0+\theta\boldsymbol{\zeta},0;\mathbf{y}',L;k)}\chi(\mathbf{y}')e^{i\mathbf{q}\frac{\mathbf{y}'}{\theta}} \\ G^{\theta,\epsilon}_b(\mathbf{x}_0+\theta\boldsymbol{\xi},0;\mathbf{y},L;k)\chi(\mathbf{y})e^{-i\mathbf{q}\frac{\mathbf{y}}{\theta}}\phi_0(\boldsymbol{\zeta};k)d\mathbf{y}d\mathbf{y}'d\boldsymbol{\zeta}d\mathbf{q}.$$

Following [3, 6], after introducing $Q^{\theta,\epsilon}_j(\mathbf{x},z;\mathbf{q};k)$ for the solutions of equation (5) with final conditions $Q^{\theta,\epsilon}_j(\mathbf{x},z=L;\mathbf{q};k) = \chi(\mathbf{y})e^{-i\mathbf{q}\frac{\mathbf{x}}{\theta}}$ when $\mu^\epsilon = \mu^\epsilon_j$, for $j=f,b$, we get

$$\phi^B_\theta(\boldsymbol{\xi},\mathbf{x}_0;k) = \int_{\mathbb{R}^{2d}} \hat{f}(\mathbf{q})\, \overline{Q^{\theta,\epsilon}_f(\mathbf{x}_0+\theta\boldsymbol{\zeta},0;\mathbf{q};k)} Q^{\theta,\epsilon}_b(\mathbf{x}_0+\theta\boldsymbol{\xi},0;\mathbf{q},L;k) \\ \phi_0(\boldsymbol{\zeta};k)d\boldsymbol{\zeta}d\mathbf{q}. \tag{21}$$

Recall that the Wigner transform (for a pure state) associated with the family of functions $Q^{\theta,\epsilon}_f$ and $Q^{\theta,\epsilon}_b$ on the scale θ (see details in [22, 27]) is defined as

$$U^{\theta,\epsilon}(\mathbf{x},z,\mathbf{p};\mathbf{q};k) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\boldsymbol{\zeta}}\overline{Q^{\theta,\epsilon}_f(\mathbf{x}+\tfrac{\theta}{2}\boldsymbol{\zeta},z;\mathbf{q};k)} Q^{\theta,\epsilon}_b(\mathbf{x}-\tfrac{\theta}{2}\boldsymbol{\zeta},z;\mathbf{q};k)d\boldsymbol{\zeta},$$

and the corresponding 'mixed' Wigner transform is then given by

$$W^{\theta,\epsilon}(\mathbf{x},z,\mathbf{p};k) = \int_{\mathbb{R}^d} \hat{f}(\mathbf{q})U^{\theta,\epsilon}(\mathbf{x},z,\mathbf{p};k)d\mathbf{q}. \tag{22}$$

After substitution in (21) we obtain the following representations for the back-propagated wave and its amplitude

$$\phi^B_\theta(\boldsymbol{\xi},\mathbf{x}_0,k) = \int_{\mathbb{R}^{2d}} e^{-i\mathbf{p}\cdot(\boldsymbol{\zeta}-\boldsymbol{\xi})}W^{\theta,\epsilon}(\mathbf{x}_0+\tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}),0,\mathbf{p};k)\phi_0(\boldsymbol{\zeta};k)d\mathbf{p}d\boldsymbol{\zeta} \tag{23}$$

and

$$u^B_\theta(\boldsymbol{\xi},\mathbf{x}_0,t) = k_0\int\int_{\mathbb{R}^{2d}} e^{i(kk_0c_0t-\mathbf{p}\cdot(\boldsymbol{\zeta}-\boldsymbol{\xi}))}W^{\theta,\epsilon}(\mathbf{x}_0+\tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}),0,\mathbf{p};k) \\ \phi_0(\boldsymbol{\zeta};k)d\mathbf{p}d\boldsymbol{\zeta}dk. \tag{24}$$

3. Asymptotics for the back-propagated wave

We shall use asymptotic techniques [22, 28] to get information on the behavior of the wave field. We will perform two limiting processes, the first one when θ goes to zero corresponds to the geometrical optics limit (here randomness plays no role) and the second by letting ϵ go to zero. This second step takes advantage of the random formulation and we can readily identify the limiting averaged wave (coherent wave). Furthermore, we will establish that the limiting wave does not depend on the realizations of the refractive index fluctuations and coincides with the coherent back-propagated wave. This is the statistical stabilization or self-averaging property, and in our case it is a consequence of the decorrelation of the limiting Wigner transform at different wave vectors, as in the case of standard time reversal [10, 28].

3.1. **Limiting back-propagated wave.** Important statistical information on the back-propagated wave can be obtained by considering the asymptotics of its coherent component. In the regime we are analysing here, this reduces to the calculation of the limits

$$\bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \lim_{(\theta \ll \epsilon) \to 0} \mathbb{E}\{\phi_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0; k)\},$$
$$\bar{u}^B(\boldsymbol{\xi}, \mathbf{x}_0, t) = \lim_{(\theta \ll \epsilon) \to 0} \mathbb{E}\{u_\theta^B(\boldsymbol{\xi}, \mathbf{x}_0, t)\}$$

where by $\lim_{(\theta \ll \epsilon) \to 0}$ we mean the result of successively passing to the limit, as $\theta \to 0$ (with ϵ fixed) and after that as $\epsilon \to 0$.

According to the representations obtained in the previous section this leads us to the calculation of the limiting averaged Wigner transform

$$W^0(\mathbf{x}, z = 0, \mathbf{p}; k) = \lim_{(\theta \ll \epsilon) \to 0} \mathbb{E}\{W^{\theta,\epsilon}(\mathbf{x}, 0, \mathbf{p}; k)\}.$$

These limiting quantities are related through the equations

$$\bar{u}^B(\boldsymbol{\xi}, \mathbf{x}_0, t) = k_0 \int e^{ikk_0 c_0 t} \bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k) dk \tag{25}$$

$$\bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\boldsymbol{\xi}} W^0(\mathbf{x}_0, 0, \mathbf{p}; k) \tilde{\phi}_0(\mathbf{p}; k) d\mathbf{p} \tag{26}$$

where $\tilde{\phi}_0(\mathbf{p}; k)$ represents the Fourier transform of $\phi_0(\mathbf{x}; k)$ with respect to $\mathbf{x}$. Hence the computation of the coherent limiting wave is reduced to the calculation of the limiting averaged Wigner transform.

To obtain more detailed information one should proceed to the computation of higher order statistical moments of the back-propagated wave and its amplitude. However, in the present case, we have that the limiting wave amplitude and the back-propagated wave are statistically stable (or self-averaging), i.e. that their limits coincide with their averages.

In the next sections we carry out the corresponding asymptotic analysis in order to rigorously establish the mentioned properties and clearly stating in which sense the convergence outlined above holds.

3.2. **Wigner transform and the high frequency limit.** Recall that the 'pure' Wigner transform $U^{\theta,\epsilon}$ associated to the functions $Q_f^{\theta,\epsilon}$ and $Q_b^{\theta,\epsilon}$ are bounded in a closed subspace of the (Schwartz) space of tempered distributions $\mathcal{S}'(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$.

Moreover, the corresponding 'mixed' Wigner transform $W^{\theta,\epsilon}$ appears as more regular, it is actually bounded in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ (see [27]). This extra regularity, that comes from the blurring effect of the mirror, has been used to establish some interesting results for Wigner transforms in random media (see for instance [3, 2]). We will also take advantage of this property.

The Wigner transform $W^{\theta,\epsilon}(\mathbf{x}, z, \mathbf{p}; k)$ satisfies, in the sense of distributions, the following equation (see [22, 27])

$$k\frac{\partial W^{\theta,\epsilon}}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^{\theta,\epsilon} = \frac{ik^2}{(2\pi)^d\theta}\int_{\mathbb{R}^d} e^{i\mathbf{q}\cdot\mathbf{x}}\big(\tilde{\mu}^\epsilon_b(\mathbf{q},z)W^{\theta,\epsilon}(\mathbf{x},z,\mathbf{p}-\tfrac{\theta}{2}\mathbf{q};k) - \tilde{\mu}^\epsilon_f(\mathbf{q},z)W^{\theta,\epsilon}(\mathbf{x},z,\mathbf{p}+\tfrac{\theta}{2}\mathbf{q};k)\big)d\mathbf{q} \tag{27}$$

with the final condition

$$W^\theta(\mathbf{x}, z = L, \mathbf{p}; k) = W^\theta_L(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\boldsymbol{\zeta}} f(\boldsymbol{\zeta})\chi(\mathbf{x}+\tfrac{\theta}{2}\boldsymbol{\zeta})\chi(\mathbf{x}-\tfrac{\theta}{2}\boldsymbol{\zeta})d\boldsymbol{\zeta},$$

where $\tilde{g}(\mathbf{q}, z)$ represents the (partial) Fourier transform of $g(\mathbf{x}, z)$ with respect to $\mathbf{x}$.

Equation (27) preserves the L^2-norm (this can be established as in [27]). Therefore, $W^{\theta,\epsilon}(\mathbf{x}, z, \mathbf{p}; k)$ is uniformly bounded in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ (with respect to θ, ϵ and the randomness).

It was proved in [3] that $W^{\theta,\epsilon}(\mathbf{x}, L, \mathbf{p}; k) \to W_L(\mathbf{x}, \mathbf{p}; k)$ (strongly) in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$, where

$$W_L(\mathbf{x},\mathbf{p};k) = \hat{f}(\mathbf{p})\chi^2(\mathbf{x}). \tag{28}$$

As a consequence, in the (random) geometrical optics limit (i.e. when $\theta \to 0$ for each fixed realization of the random field), for any sequence θ_n (after possibly extracting a subsequence) $W^{\theta_n,\epsilon}$ converges (strongly) in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ as $\theta_n \to 0$. Furthermore, a straightforward calculation shows that its limit W^ϵ satisfies the following (random) transport equation [22, 27]

$$k\frac{\partial W^\epsilon}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}} W^\epsilon + \frac{k^2}{\sqrt{\epsilon}}\nabla_\perp\mu(\frac{\mathbf{x}}{\epsilon^a},\frac{z}{\epsilon})\cdot\nabla_{\mathbf{p}} W^\epsilon - \frac{ik^2}{\sqrt{\epsilon}}\eta(\frac{\mathbf{x}}{\epsilon^a},\frac{z}{\epsilon})W^\epsilon = 0 \tag{29}$$

with final condition

$$W^\epsilon(\mathbf{x}, z = L, \mathbf{p}; k) = W_L(\mathbf{x},\mathbf{p};k) = \hat{f}(\mathbf{p})\chi^2(\mathbf{x}), \tag{30}$$

where $\nabla_\perp\mu(\mathbf{x},\cdot) = \nabla_{\mathbf{x}}\mu(\mathbf{x},\cdot)$. Notice that equation (29) preserves the L^2-norm, this implies uniqueness in L^2, and as a consequence the convergence as $\theta \to 0$ is guaranteed without choosing any subsequence. Additionally, the functions W^ϵ are uniformly bounded in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ (with respect to ϵ and the randomness).

The following theorem presents the high-frequency asymptotics of the back-propagated wave. In particular, it states that one can drop the term $\frac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta})$ in the argument of $W^{\theta,\epsilon}$ in expressions (23) and (24), and use the corresponding asymptotics of the Wigner transform.

Theorem 3.1. *(Random geometrical optics asymptotics)*
Let ϵ and the medium realization be fixed. Assume that the source function $\phi_0 \in L^2(\mathbb{R}^d_{\boldsymbol{\zeta}} \times \mathbb{R}_k)$. Then for every $\boldsymbol{\xi} \in \mathbb{R}^d$ and almost every k, the complex amplitude $\phi^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0; k)$ converges (strongly) in $L^2(\mathbb{R}^d_{\mathbf{x}_0})$, as $\theta \to 0$, to the function

$$\phi^B_\epsilon(\boldsymbol{\xi},\mathbf{x}_0;k) = \int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\boldsymbol{\xi}} W^\epsilon(\mathbf{x}_0, 0, \mathbf{p}; k)\tilde{\phi}_0(\mathbf{p};k)d\mathbf{p}, \tag{31}$$

where $W^\epsilon(\mathbf{x}, z, \mathbf{p}; k)$ *is the solution of equation* (29)-(30).

Furthermore, the back-propagated wave $u^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0, t)$ *converges (strongly) in* $L^2(\mathbb{R}^d_{\mathbf{x}_0} \times \mathbb{R}_t)$ *to*

$$u^B_\epsilon(\boldsymbol{\xi}, \mathbf{x}_0, t) = k_0 \int e^{ikk_0c_0t} \phi^B_\epsilon(\boldsymbol{\xi}, \mathbf{x}_0; k) dk. \tag{32}$$

Proof. In the first part of the proof, in order to simplify notation, we suppress the dependence on k. We have to prove that

$$\int d\mathbf{x} \Big| \int e^{-i\mathbf{p}(\boldsymbol{\xi}-\boldsymbol{\zeta})} \big(W^{\theta,\epsilon}(\mathbf{x} + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), 0, \mathbf{p}) - W^\epsilon(\mathbf{x}, 0, \mathbf{p})\big) \phi_0(\boldsymbol{\zeta}) d\boldsymbol{\zeta} d\mathbf{p} \Big|^2 \to 0. \tag{33}$$

From the decomposition

$$\begin{aligned} W^{\theta,\epsilon}(\mathbf{x}_0 + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), \cdot) - W^\epsilon(\mathbf{x}_0, \cdot) &= W^{\theta,\epsilon}(\mathbf{x}_0 + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), \cdot) - W^\epsilon(\mathbf{x}_0 + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), \cdot) \\ &\quad + W^\epsilon(\mathbf{x}_0 + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), \cdot) - W^\epsilon(\mathbf{x}_0, \cdot), \end{aligned}$$

follows that it is enough to prove the corresponding convergence result for each term of this decomposition.

Using the Fourier transform, we can write the integral corresponding to the first term as

$$\begin{aligned} I_1 &= \int d\mathbf{x} \Big| \int e^{-i\mathbf{p}(\boldsymbol{\xi}-\boldsymbol{\zeta})} \big(W^{\theta,\epsilon}(\mathbf{x} + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), 0, \mathbf{p}) \\ &\qquad\qquad - W^\epsilon(\mathbf{x} + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), 0, \mathbf{p})\big) \phi_0(\boldsymbol{\zeta}) d\boldsymbol{\zeta} d\mathbf{p} \Big|^2 \\ &= (2\pi)^d \int d\mathbf{q} \Big| \int e^{-i(\mathbf{p}-\frac{\theta}{2}\mathbf{q})\boldsymbol{\xi}} \big(\tilde{W}^{\theta,\epsilon}(\mathbf{q}, 0, \mathbf{p}) - \tilde{W}^\epsilon(\mathbf{q}, 0, \mathbf{p})\big) \overline{\hat{\phi}_0(\mathbf{p} + \tfrac{\theta}{2}\mathbf{q})} d\mathbf{p} \Big|^2 \end{aligned}$$

and using the Cauchy-Schwartz inequality we get the estimate

$$\begin{aligned} |I_1| &\le (2\pi)^d \|\hat{\phi}_0\|^2_{L^2} \int |\tilde{W}^{\theta,\epsilon}(\mathbf{q}, 0, \mathbf{p}) - \tilde{W}^\epsilon(\mathbf{q}, 0, \mathbf{p})|^2 d\mathbf{q} d\mathbf{p} \\ &\le \|\hat{\phi}_0\|^2_{L^2(\mathbb{R}^d_{\mathbf{p}})} \|W^{\theta,\epsilon}(0, \cdot) - W^\epsilon(0, \cdot)\|^2_{L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})} \to 0. \end{aligned}$$

For the integral corresponding to the second term, after using the Fourier transform, we have

$$\begin{aligned} I_2 &= \int d\mathbf{x} \Big| \int e^{-i\mathbf{p}(\boldsymbol{\xi}-\boldsymbol{\zeta})} \big(W^\epsilon(\mathbf{x} + \tfrac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta}), 0, \mathbf{p}) - W^\epsilon(\mathbf{x}, 0, \mathbf{p})\big) \phi_0(\boldsymbol{\zeta}) d\boldsymbol{\zeta} d\mathbf{p} \Big|^2 \\ &= (2\pi)^d \int d\mathbf{q} \Big| \int (e^{i\frac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta})} - 1) \overline{\hat{W}^\epsilon(\mathbf{q}, 0, \boldsymbol{\zeta}-\boldsymbol{\xi})} \phi_0(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \Big|^2 \\ &\le (2\pi)^d \int I^2_\theta(\mathbf{q}) d\mathbf{q} \end{aligned}$$

where

$$I_\theta(\mathbf{q}) = \int |(e^{i\frac{\theta}{2}(\boldsymbol{\xi}+\boldsymbol{\zeta})} - 1)| |\overline{\hat{W}^\epsilon(\mathbf{q}, 0, \boldsymbol{\zeta}-\boldsymbol{\xi})} \phi_0(\boldsymbol{\zeta})| d\boldsymbol{\zeta}.$$

Note that $I_\theta(\mathbf{q}) \to 0$ for almost every $\mathbf{q}$, as $\theta \to 0$. Indeed, its integrand converges to zero pointwise. Furthermore, it is bounded by $2|\overline{\hat{W}^\epsilon(\mathbf{q}, 0, \boldsymbol{\zeta}-\boldsymbol{\xi})}\phi_0(\boldsymbol{\zeta})|$ which is integrable with respect to $\boldsymbol{\zeta}$ for almost every $\mathbf{q}$, as a direct consequence of the Cauchy-Schwartz inequality and the fact that $\hat{W}^\epsilon(0, \cdot) \in L^2(\mathbb{R}^d_{\mathbf{q}} \times \mathbb{R}^d_{\boldsymbol{\zeta}})$. By applying the dominated convergence theorem, the result follows.

On the other hand, using the Cauchy-Schwartz inequality, we have the estimate

$$I_\theta^2(\mathbf{q}) \le 4\|\phi_0\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\zeta}})}\|\hat{W}^\epsilon(\mathbf{q},0,\cdot)\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\zeta}})},$$

and the function in the righthand-side is integrable. Consequently, from the dominated convergence theorem we conclude that $I_2 \to 0$ as $\theta \to 0$.

For the proof of the second part of the theorem we explicitly write the dependence on k.

We prove the equivalent statement that, as $\theta \to 0$,

$$\int dk \int_{\mathbb{R}^d} |\phi_\theta^B(\boldsymbol{\xi},\mathbf{x};k) - \phi_\epsilon^B(\boldsymbol{\xi},\mathbf{x};k)|^2\, d\mathbf{x} = \int I(k)dk \to 0.$$

Note that the first part of the theorem states that $I(k) \to 0$, for $k \neq 0$. Furthermore, similarly as in that part of the proof, one can get the estimates

$$|I(k)| \le \|\phi_0(\cdot;k)\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\zeta}})}(C_1\|W^{\theta,\epsilon}(0,\cdot;k)\|^2_{L^2(\mathbb{R}^d_{\mathbf{x}}\times\mathbb{R}^d_{\mathbf{p}})} + C_2\|W^\epsilon(0,\cdot;k)\|^2_{L^2(\mathbb{R}^d_{\mathbf{x}}\times\mathbb{R}^d_{\mathbf{p}})}),$$

where C_1, C_2 are real constants. Additionally, since equations (27) and (29) conserve the L^2-norm and W_L^θ does not depend on k, we get that there is a real constant C_3, such that $|I(k)| \le C_3\|\phi_0(\cdot;k)\|^2_{L^2(\mathbb{R}^d_{\boldsymbol{\zeta}})}$. Finally, to conclude the proof we use the dominated convergence theorem and the fact that $\phi_0 \in L^2(\mathbb{R}^d_{\boldsymbol{\zeta}} \times \mathbb{R}_k)$. □

3.3. **Diffusion limit for the Wigner transform.** In this section, we characterize the limit as $\epsilon \to 0$ of the (random) geometrical optics Wigner transform W^ϵ. More specifically, we prove that $\lim_{\epsilon\to 0} W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k) = W^0(\mathbf{x}, z = 0, \mathbf{p}; k)$, where convergence is understood in the weak sense and the function W^0 solves a deterministic transport-diffusion equation. This highlights the self-averaging property of the Wigner transform.

3.3.1. *Convergence of the expectation.* We start by characterizing the asymptotics as $\epsilon \to 0$ of the averaged geometrical optics Wigner transform $E\{W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k)\}$. This goal is achieved in part because of the 'white noise' scaling in the coefficients of equation (29), their regularity and mixing properties.

Observe that the solution of equation (29)–(30) has the representation

$$W^\epsilon(\mathbf{x},0,\mathbf{p};k) = W_L(\mathbf{X}^\epsilon(L;\mathbf{x},\mathbf{p}),\mathbf{P}^\epsilon(L;\mathbf{x},\mathbf{p});k)e^{-iQ^\epsilon(L;\mathbf{x},\mathbf{p})} \tag{34}$$

where $\mathbf{X}^\epsilon(s) = \mathbf{X}^\epsilon(s;\mathbf{x},\mathbf{p})$, $\mathbf{P}^\epsilon(s) = \mathbf{P}^\epsilon(s;\mathbf{x},\mathbf{p})$, $Q^\epsilon(s) = Q^\epsilon(L;\mathbf{x},\mathbf{p})$ solves the characteristics ODEs corresponding to equation (29)

$$\begin{cases} \dfrac{d}{ds}\mathbf{X}^\epsilon = \dfrac{\mathbf{P}^\epsilon}{k}, \\ \dfrac{d}{ds}\begin{pmatrix}\mathbf{P}^\epsilon \\ Q^\epsilon\end{pmatrix} = \dfrac{k}{\sqrt{\epsilon}}\begin{pmatrix}\nabla_\perp\mu(\frac{\mathbf{X}^\epsilon}{\epsilon^a},\frac{s}{\epsilon}) \\ \eta(\frac{\mathbf{X}^\epsilon}{\epsilon^a},\frac{s}{\epsilon})\end{pmatrix}, \end{cases} \tag{35}$$

and satisfy the initial conditions $\mathbf{X}^\epsilon(0) = \mathbf{x}$, $\mathbf{P}^\epsilon(0) = \mathbf{p}$, $Q^\epsilon(0) = 0$.

These characteristics ODEs coincide with the type of ODEs that we study in appendix A in the particular case where $m = d$ and $l = d + 1$. As a consequence, for a sufficiently smooth function W_L, for instance $W_L \in C_b^2(\mathbb{R}^d_{\mathbf{x}}\times\mathbb{R}^d_{\mathbf{p}})$, from theorem A.1, we get that $E\{W^\epsilon(\mathbf{x},0,\mathbf{p})\}$ converges pointwise to $W^0(\mathbf{x},0,\mathbf{p})$ as $\epsilon \to 0$. The function $W^0(\mathbf{x},z,\mathbf{p};k)$ satisfies in the weak sense the equation

$$k\frac{\partial W}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}W + \tfrac{k^3}{2}\mathcal{L}^a_{\mathbf{p}}W = 0 \tag{36}$$

with final condition

$$W(\mathbf{x}, z = L, \mathbf{p}; k) = W_L(\mathbf{x}, \mathbf{p}). \tag{37}$$

The differential operator $\mathcal{L}^a_{\mathbf{p}}$ is defined as

$$\mathcal{L}^a_{\mathbf{p}} W = \begin{cases} \nabla_{\mathbf{p}} \cdot \big(D(\mathbf{0})\nabla_{\mathbf{p}} W\big) - A(\mathbf{0})W, & \text{if } 0 < a < 1 \\ \nabla_{\mathbf{p}} \cdot \big(D(\frac{\mathbf{p}}{k})\nabla_{\mathbf{p}} W\big) - A(\frac{\mathbf{p}}{k})W, & \text{if } a = 1 \end{cases} \tag{38}$$

where the positive-definite diffusion matrix $D(\tilde{\mathbf{p}})$ has elements

$$D_{ij}(\tilde{\mathbf{p}}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{R}_\mu(\mathbf{q}, \tilde{\mathbf{p}} \cdot \mathbf{q}) q_i q_j d\mathbf{q}, \quad i, j = 1, \ldots, d$$

and the attenuation coefficient $A(\tilde{\mathbf{p}})$ is given by

$$A(\tilde{\mathbf{p}}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{R}_\eta(\mathbf{q}, \tilde{\mathbf{p}} \cdot \mathbf{q}) d\mathbf{q} \geq 0.$$

This leads to the following result.

Theorem 3.2. *Assume that $k \neq 0$ and the isotropic random vector field $(\nabla_\perp \mu(\cdot), \eta(\cdot))^t$ satisfies conditions C.1–C.5 of Appendix A (page 22) with $m = d$ and $l = d + 1$.*

If the function $W_L(\mathbf{x}, \mathbf{p}) = \hat{f}(\mathbf{p})\chi^2(\mathbf{x}) \in C^2_b(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}}) \cap L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$. Then, for almost every $\mathbf{x}$, the averaged Wigner transform $E\{W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k)\}$ converges weakly in $L^2(\mathbb{R}^d_{\mathbf{p}})$, as $\epsilon \to 0$, to the function $W^0(\mathbf{x}, z = 0, \mathbf{p}; k)$ that solves (36)–(37).

Proof. We have to establish that for any $\lambda \in L^2(\mathbb{R}^d_{\mathbf{p}})$, we have that for almost every $\mathbf{x}$

$$\lim_{\epsilon \to 0} E\{\langle W^\epsilon(\mathbf{x}, z = 0, \cdot), \lambda(\cdot)\rangle\} = \langle W^0(\mathbf{x}, z = 0, \cdot), \lambda(\cdot)\rangle,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $L^2(\mathbb{R}^d_{\mathbf{p}})$. Note, that it is enough to consider $\lambda \in C^\infty_0(\mathbb{R}^d_{\mathbf{p}})$.

We have that

$$\langle E\{W^\epsilon(\mathbf{x}, 0, \cdot)\} - W^0(\mathbf{x}, 0, \cdot), \lambda(\cdot)\rangle = \int_{\mathbb{R}^d} (E\{W^\epsilon(\mathbf{x}, 0, \mathbf{p})\} - W^0(\mathbf{x}, 0, \mathbf{p}))\overline{\lambda(\mathbf{p})} d\mathbf{p}.$$

Moreover, the following estimate is valid

$$|\big(E\{W^\epsilon(\mathbf{x}, 0, \mathbf{p})\} - W^0(\mathbf{x}, 0, \mathbf{p})\big)\overline{\lambda(\mathbf{p})}| \leq 2(\max |W_L|)|\lambda(\mathbf{p})|.$$

From Theorem A.1, we know that $E\{W^\epsilon(\mathbf{x}, 0, \mathbf{p})\} \to W^0(\mathbf{x}, 0, \mathbf{p})$ pointwise. Thus, the function in the lefthand-side above converges to zero pointwise. Finally, since the function on the righthand-side is integrable, the dominated convergence theorem ensures that $E\{\langle W^\epsilon(\mathbf{x}, 0, \cdot), \lambda(\cdot)\rangle\} \to \langle W^0(\mathbf{x}, 0, \cdot), \lambda(\cdot)\rangle$ for almost every $\mathbf{x}$. □

In the general case, when we consider a non-smooth function W_L, the results of Appendix A do not apply directly. However, a simple approximation argument allows us to obtain a weaker version of the previous theorem.

Theorem 3.3. *Assume that $k \neq 0$ and the isotropic random vector field $(\nabla_\perp \mu(\cdot), \eta(\cdot))^t$ satisfies conditions C.1–C.5 of Appendix A (page 22) with $m = d$ and $l = d + 1$.*

If the function $W_L(\mathbf{x}, \mathbf{p}) = \hat{f}(\mathbf{p})\chi^2(\mathbf{x}) \in L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$. Then the averaged Wigner transform $E\{W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k)\}$ converges weakly in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$, as $\epsilon \to 0$, to the function $W^0(\mathbf{x}, z = 0, \mathbf{p}; k)$ that solves (36)–(37).

Proof. It is enough to prove it for any $\lambda \in C_0^\infty(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ we have that

$$\lim_{\epsilon \to 0} E\{\langle W^\epsilon(z=0,\cdot), \lambda(\cdot)\rangle\} = \langle W^0(z=0,\cdot), \lambda(\cdot)\rangle, \tag{39}$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$.

We first prove this, in the case of a smooth data. More specifically, consider a function $W_1 \in C_0^\infty(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$. Let $W_1^\epsilon(\mathbf{x}, z, \mathbf{p})$ and $W_1^0(\mathbf{x}, z, \mathbf{p})$ be the solutions of equations (29) and (36), respectively, with final conditions $W_1^\epsilon(z = L, \cdot) = W_1^0(z = L, \cdot) = W_1(\cdot)$. Then

$$\lim_{\epsilon \to 0} E\{\langle W_1^\epsilon(z=0,\cdot), \lambda(\cdot)\rangle\} = \langle W_1^0(z=0,\cdot), \lambda(\cdot)\rangle.$$

Indeed, we have that

$$\langle E\{W_1^\epsilon(0,\cdot) - W_1^0(0,\cdot), \lambda(\cdot)\rangle = \int_{\mathbb{R}^{2d}} (E\{W_1^\epsilon(\mathbf{x},0,\mathbf{p})\} - W_1^0(\mathbf{x},0,\mathbf{p}))\overline{\lambda(\mathbf{x},\mathbf{p})} d\mathbf{x} d\mathbf{p}.$$

Therefore, we get the following estimate for the integrand function above

$$|\big(E\{W_1^\epsilon(\mathbf{x},0,\mathbf{p})\} - W_1^0(\mathbf{x},0,\mathbf{p})\big)\overline{\lambda(\mathbf{x},\mathbf{p})}| \le 2(\max|W_1|)\,|\lambda(\mathbf{x},\mathbf{p})|.$$

From theorem A.1, we know that $E\{W_1^\epsilon(\mathbf{x},0,\mathbf{p})\} \to W_1^0(\mathbf{x},0,\mathbf{p})$ pointwise. Thus, the integrand converges to zero pointwise. On the other hand, since λ is compactly supported, we have that the function on the righthand-side is integrable. Consequently, the dominated convergence theorem assures that $E\{\langle W_1^\epsilon(0,\cdot), \lambda(\cdot)\rangle\} \to \langle W_1^0(0,\cdot), \lambda(\cdot)\rangle$.

Next, we consider the general case. Let $\delta > 0$, and choose $W_1 \in C_0^\infty(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ such that $\|W_L - W_1\|_{L^2} < \frac{\delta}{4\|\lambda\|_{L^2}}$.

We have that

$$\begin{aligned} \Delta_\lambda^\epsilon W &= |E\{\langle W^\epsilon(0,\cdot), \lambda(\cdot)\rangle\} - \langle W^0(\cdot), \lambda(\cdot)\rangle| \\ &= |E\{\langle W^\epsilon(0,\cdot) - W(0,\cdot), \lambda(\cdot)\rangle\}| \\ &\le |E\{\langle W^\epsilon(0,\cdot) - W_1^\epsilon(0,\cdot), \lambda(\cdot)\rangle\}| + |E\{\langle W_1^\epsilon(0,\cdot) - W_1^0(0,\cdot), \lambda(\cdot)\rangle\}| \\ &\quad + |E\{\langle W_1^0(0,\cdot) - W(0,\cdot), \lambda(\cdot)\rangle\}|. \end{aligned} \tag{40}$$

First, observe that since equation (29) preserves the L^2-norm, we have that

$$\begin{aligned} |E\{\langle W^\epsilon(0,\cdot) - W_1^\epsilon(0,\cdot), \lambda(\cdot)\rangle\}| &\le \|W^\epsilon(0,\cdot) - W_1^\epsilon(0,\cdot)\|_{L^2}\|\lambda\|_{L^2} \\ &= \|W_L - W_1\|_{L^2}\|\lambda\|_{L^2} < \frac{\delta}{4}. \end{aligned} \tag{41}$$

Moreover, the L^2-norm of a solution of equation (36) is a non-decreasing function of z, then

$$\begin{aligned} |E\{\langle W_1^0(0,\cdot) - W(0,\cdot), \lambda(\cdot)\rangle\}| &= |\langle W_1^0(0,\cdot) - W(0,\cdot), \lambda(\cdot)\rangle| \\ &\le \|W_L - W_\delta\|_{L^2}\|\lambda\|_{L^2} < \frac{\delta}{4}. \end{aligned} \tag{42}$$

Second, from the first part of this proof, we know that $E\{\langle W_1^\epsilon(0,\cdot) - W_1^0(0,\cdot), \lambda(\cdot)\rangle\} \to 0$ as $\epsilon \to 0$. Thus, there is $\epsilon' > 0$ such that for any $\epsilon < \epsilon'$,

$$|E\{\langle W_1^\epsilon(0,\cdot) - W_1^0(0,\cdot), \lambda(\cdot)\rangle\}| < \frac{\delta}{2}. \tag{43}$$

Finally, from estimates (40)–(43), one gets that for any $\epsilon < \epsilon'$, $\Delta_\lambda^\epsilon W < \delta$, and (39) follows. □

3.3.2. *Statistical stability in the diffusion limit.* To complete the characterization of the asymptotics, as $\epsilon \to 0$, for the geometrical optics Wigner transform we establish in this section the celebrated self-averaging property.

For a sufficiently smooth function W_L, for instance $W_L \in C_b^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$, from Theorem A.2, we know that the limiting averaged Wigner transform decorrelates. More specifically, we have the pointwise convergence

$$E\{W^\epsilon(\mathbf{x}_1, 0, \mathbf{p}_1)\overline{W^\epsilon(\mathbf{x}_2, 0, \mathbf{p}_2)}\} \to W^0(\mathbf{x}_1, 0, \mathbf{p}_1)\overline{W^0(\mathbf{x}_2, 0, \mathbf{p}_2)}, \quad \text{for } \mathbf{p}_1 \neq \mathbf{p}_2$$

as $\epsilon \to 0$. Using this result for $\mathbf{x}_1 = \mathbf{x}_2$, one can establish the convergence of the variance in a similar way as we did in Theorem 3.2. Thus, the statistical stabilization of the limiting Wigner transform follows.

This property can also be established in a more general situation. Namely, we have the following result.

Theorem 3.4. *Assume that $d \geq 2$, $k \neq 0$ and the isotropic random vector field $(\nabla_\perp \mu(\cdot), \eta(\cdot))^t$ satisfies conditions C.1–C.5 of Appendix A (page 22) with $m = d$ and $l = d + 1$.*

If the function $W_L(\mathbf{x}, \mathbf{p}) = \hat{f}(\mathbf{p})\chi^2(\mathbf{x}) \in L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$. Then the Wigner transform $W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k)$ given by equations (29)–(30) *converges in probability and weakly in $L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$, as $\epsilon \to 0$, to the function $W^0(\mathbf{x}, z = 0, \mathbf{p}; k)$ that solves* (36)–(37). *More precisely, for any fixed test function $\lambda \in L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ the random variable $\langle W^\epsilon(0, \cdot), \lambda(\cdot)\rangle$ converges in probability to $\langle W^0(0, \cdot), \lambda(\cdot)\rangle$ as $\epsilon \to 0$.*

Furthermore, when $W_L(\mathbf{x}, \mathbf{p}) \in C_b^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}}) \cap L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ the Wigner transform $W^\epsilon(\mathbf{x}, z = 0, \mathbf{p}; k)$ converges in probability and weakly in $L^2(\mathbb{R}^d_{\mathbf{p}})$, for almost every $\mathbf{x}$.

The proof is similar to that of Theorem 3.3.

Proof. It is enough to establish that for any $\lambda \in L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{p}})$ we have that

$$\lim_{\epsilon \to 0} E\{|\langle W^\epsilon(0, \cdot), \lambda(\cdot)\rangle|^2\} = |\langle W^0(0, \cdot), \lambda(\cdot)\rangle|^2. \tag{44}$$

Note that

$$|\langle W^\epsilon(0, \cdot), \lambda(\cdot)\rangle|^2 = \langle (W^\epsilon \otimes \overline{W^\epsilon})(0, \cdot), (\lambda \otimes \overline{\lambda})(\cdot)\rangle$$
$$|\langle W^\epsilon(0, \cdot), \lambda(\cdot)\rangle|^2 = \langle (W^0 \otimes \overline{W^0})(0, \cdot), (\lambda \otimes \overline{\lambda})(\cdot)\rangle$$

where

$$(V_1 \otimes V_2)(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = V_1(\mathbf{x}_1, \mathbf{p}_1)V_2(\mathbf{x}_2, \mathbf{p}_2)$$

represents the tensor product of functions and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^d_{\mathbf{x}_1} \times \mathbb{R}^d_{\mathbf{p}_1} \times \mathbb{R}^d_{\mathbf{x}_2} \times \mathbb{R}^d_{\mathbf{p}_2})$.

Let $\mathcal{T}^\epsilon_{\mathbf{x},\mathbf{p}}$, $\tilde{\mathcal{T}}^\epsilon_{\mathbf{x},\mathbf{p}}$ represent the transport operator appearing in equation (29) and its complex conjugate, respectively, and $\mathcal{D}_{\mathbf{x},\mathbf{p}}$ the transport-diffusion operator appearing in equation (36). We have that $W^\epsilon \otimes \overline{W^\epsilon}$ satisfies the equation

$$(k\frac{\partial}{\partial z} + \mathcal{T}^\epsilon_{\mathbf{x}_1,\mathbf{p}_1} + \tilde{\mathcal{T}}^\epsilon_{\mathbf{x}_2,\mathbf{p}_2})V = 0 \tag{45}$$

with the final condition $V(z = L, \cdot) = (W_L \otimes \overline{W_L})(\cdot)$. Additionally, $W^0 \otimes \overline{W^0}$ solves equation

$$(k\frac{\partial}{\partial z} + \mathcal{D}_{\mathbf{x}_1,\mathbf{p}_1} + \mathcal{D}_{\mathbf{x}_2,\mathbf{p}_2})\bar{V} = 0 \tag{46}$$

with the additional condition $\bar{V}(z = L, \cdot) = (W_L \otimes \overline{W_L})(\cdot)$.

On one hand, as we mentioned above, from Theorem A.2 follows: The solution $V^\epsilon(z, \cdot)$ of equation (45) with a sufficiently smooth final condition $(W_1 \otimes \overline{W_1})(\cdot)$, satisfies that $E\{V^\epsilon(0, \cdot)\}$ converges pointwise to $\bar{V}(0, \cdot)$, which solves equation (46) with the final condition $(W_1 \otimes \overline{W_1})(\cdot)$.

On the other hand, one can prove that equation (45) preserves the L^2-norm, while the solutions of (46) have a L^2-norm non-increasing with z.

Using these facts, the proof can be completed in the same way as in Theorem 3.3. □

3.4. **Statistical stability of the back-propagated wave.** From the results obtained in the previous section, one can establish the statistical stabilization (or self-averaging) of the limiting time-reversed back-propagated wave. This property can be stated as follows

$$\phi^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0; k) \to \bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k) \quad \text{as } (\theta \ll \epsilon) \to 0 \text{ for } k \neq 0,$$
$$u^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0, t) \to \bar{u}^B(\boldsymbol{\xi}, \mathbf{x}_0, t) \quad \text{as } (\theta \ll \epsilon) \to 0,$$

where the convergence is considered in probability.

Moreover, from the previous section, it is apparent that the convergence of the (complex) wave amplitude should be in a weak sense since we need to average with respect to $\mathbf{x}_0$. Concerning, the back-propagated wave, we have a similar situation with respect to $\mathbf{x}_0$ while in the time domain we still can have a stronger convergence. More specifically, we say that a sequence of functions $f_n \in L^2(\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}_t)$ converges *semi-weakly* to f, if for every $\lambda \in L^2(\mathbb{R}^d_{\mathbf{x}})$

$$\|\langle f_n - f, \lambda\rangle\|_{L^2(\mathbb{R}_t)} \to 0.$$

Theorem 3.5. *Let $d \geq 2$ and assume that the isotropic random vector field $(\nabla_\perp \mu(\cdot), \eta(\cdot))^t$ satisfies conditions C.1–C.5 of Appendix A (page 22) with $m = d$ and $l = d + 1$. For each $\boldsymbol{\xi} \in \mathbb{R}^d$ and almost every $k \neq 0$, the amplitude of the back-propagated wave $\phi^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0; k)$ given by (23) converges in probability and weakly in $L^2(\mathbb{R}^d_{\mathbf{x}_0})$, as $(\epsilon \ll \theta) \to 0$, to the deterministic function $\bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k)$ given by (26) where $W^0(\mathbf{x}_0, 0, \mathbf{p}; k)$ is the solution of the equation (36) with final condition (37).*

Additionally, the back-propagated wave $u^B_\theta(\boldsymbol{\xi}, \mathbf{x}_0, t)$ converges in probability and semi-weakly in $L^2(\mathbb{R}^d_{\mathbf{x}_0} \times \mathbb{R}_t)$, to the deterministic wave $\bar{u}^B(\boldsymbol{\xi}, \mathbf{x}_0, t)$ given by equation (25).

Proof. Using theorem 3.1, the convergence of the amplitude of the back-propagated wave follows by applying Theorem 3.4 with test functions of the form $\lambda_1(\mathbf{x}, \mathbf{p}; k) = e^{i\mathbf{p}\boldsymbol{\xi}} \lambda(\mathbf{x}) \overline{\tilde{\phi}_0(\mathbf{p}; k)}$.

For the back-propagated wave, notice first that it is enough to establish that

$$I = E\Big\{\int |\langle \phi^B_\epsilon(\boldsymbol{\xi}, \cdot; k) - \bar{\phi}^B(\boldsymbol{\xi}, \cdot; k), \lambda(\cdot)\rangle|^2 dk\Big\}$$

converges to zero. Re-writing I as

$$I = \int E\{|\langle W^\epsilon(0, \cdot; k) - W^0(0, \cdot; k), \lambda_1(\cdot; k)\rangle|^2\} dk,$$

we know from theorem 3.4 that the integrand function above converges to zero for almost every $k \neq 0$, and since W^ϵ and W^0 solve equations (29)–(30) and (36)–(37), respectively, it is bounded by the integrable function $C_1\|\lambda\|^2_{L^2}\|\tilde{\phi}_0(\cdot;k)\|^2_{L^2(\mathbb{R}^d_{\mathbf{p}})}$. Finally, using the dominated convergence theorem it follows that $I \to 0$. □

4. Time-reversal super-focusing and stability

The results presented above give us a complete asymptotic characterization of the back-propagated wave. It is remarkable that despite the perturbations the time-reversed back-propagated wave remains statistically stable and the refocused spot has a better resolution than the one corresponding to the homogeneous medium and we describe this phenomenon next.

4.1. **Perturbation effects on the refocused wave.** First, note that when there is no perturbation, the unperturbed limiting Wigner transform W^0_{unp} satisfies the equation

$$k\frac{\partial W}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}W + \tfrac{k^3}{2}\bar{\mathcal{L}}^a_{\mathbf{p}}W = 0$$

with final condition

$$W(\mathbf{x}, z = L, \mathbf{p}; k) = W_L(\mathbf{x}, \mathbf{p}; k),$$

where

$$\bar{\mathcal{L}}^a_{\mathbf{p}}W = \begin{cases} \nabla_{\mathbf{p}}\cdot\big(D(\mathbf{0})\nabla_{\mathbf{p}}W\big), & \text{if } 0 < a < 1\\ \nabla_{\mathbf{p}}\cdot\big(D(\frac{\mathbf{p}}{k})\nabla_{\mathbf{p}}W\big), & \text{if } a = 1.\end{cases} \tag{47}$$

The associated limiting unperturbed (complex) wave amplitude and back-propagated wave are then given by

$$\bar{\phi}^B_{\text{unp}}(\boldsymbol{\xi}, \mathbf{x}_0; k) = \int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\boldsymbol{\xi}} W^0_{\text{unp}}(\mathbf{x}_0, 0, \mathbf{p}; k)\tilde{\phi}_0(\mathbf{p}; k)d\mathbf{p}$$

and

$$\bar{u}^B_{\text{unp}}(\boldsymbol{\xi}, \mathbf{x}_0, t) = k_0 \int e^{ikk_0c_0t}\bar{\phi}^B_{\text{unp}}(\boldsymbol{\xi}, \mathbf{x}_0; k)dk.$$

This is in complete agreement with the analysis presented in [28], since when $a = 1$ we recover the advection-diffusion equation derived there.

Therefore, the effect of perturbations corresponds to the additional reaction term in (36) characterized by the attenuation coefficient $A(\cdot)$. Furthermore, in the case where $0 < a < 1$, $A(\cdot) \equiv A_0 = \text{const}$. Consequently, one can readily quantify the effect of perturbations since the limiting Wigner distributions satisfy the equation

$$W^0(\mathbf{x}, 0, \mathbf{p}; k) = W^0_{\text{unp}}(\mathbf{x}, 0, \mathbf{p}; k)e^{-\frac{k^2}{2}A_0L}. \tag{48}$$

For the limiting (complex) wave amplitude and back-propagated wave we get

$$\bar{\phi}^B(\boldsymbol{\xi}, \mathbf{x}_0; k) = \bar{\phi}^B_{\text{unp}}(\boldsymbol{\xi}, \mathbf{x}_0; k)e^{-\frac{k^2}{2}A_0L} \tag{49a}$$

$$\bar{u}^B(\boldsymbol{\xi}, \mathbf{x}_0, t) = \big(G(\cdot) \star \bar{u}^B_{\text{unp}}(\boldsymbol{\xi}, \mathbf{x}_0, \cdot)\big)(t) \tag{49b}$$

where $\star$ stands for convolution in time and G is given by the Gaussian kernel

$$G(t) = \frac{1}{\sqrt{2\pi}\sigma_G}\, e^{-\frac{t^2}{2\sigma_G^2}},$$

where $\sigma_G = \frac{\sqrt{A_0 L}}{k_0 c_0}$. Note that A_0 represents a statistical measure of the perturbations intensity. We have

$$A_0 = \int R_\eta(\mathbf{0}, z) dz,$$

so that this parameter is the longitudinal correlation length in the case with anisotropic medium fluctuations.

It is remarkable that the perturbations produce an exponential attenuation of the wave amplitude, when compared with its counterpart from the unperturbed case. At a fixed frequency the perturbations result in a re-scaling factor that does not depend on the space variable, and does not affect the space resolution of the wave amplitude. Thus, the **perturbations produce a time domain smearing effect** on the back-propagated pulse when compared to the time reversal in an unperturbed medium. In particular, if the initial pulse is a Gaussian time-pulse with carrier frequency $\omega_0 = k_0 c_0$ and width σ_t then the back-propagated signal in the perturbed medium coincides with the re-compressed wave in the unperturbed medium generated by an effective initial pulse. This effective initial pulse is a Gaussian with carrier frequency $\tilde{\omega}_0 = \frac{\omega_0}{\gamma}$ and pulse width $\tilde{\sigma}_t = \sigma_t \gamma$, re-scaled by the factor $\gamma \exp(-A_0 L / 2\gamma)$, where $\gamma = 1 + \sigma_G^2/\sigma_t^2$. This is a manifestation of the attenuation of the back-propagated wave, caused by perturbations that has been observed in physical experiments [32] and is also valid in other asymptotic regimes [4, 6].

Concerning space resolution, it is apparent from equations (48)–(49), that **perturbations do not affect the space super-resolution** of the back-propagated wave when compared with time reversal in an unperturbed medium. This means that the wave pulse $\bar{u}_0^B$ has a tighter support in the case of a random medium than in the absence of medium fluctuations and multipathing, even though the back-propagation takes place in a slightly modified random medium. In other words, relatively weak perturbations do not eliminate the super-focusing of the re-compressed wave. Furthermore, the re-compressed wave is statistically stable, i.e. its shape does not depend on the medium and perturbation realizations.

Super-focusing in the case without medium perturbations is discussed, for instance in [10, 28, 29]. In particular, in [28] the asymptotic regime corresponding to the case where $a = 1$ is treated. Therefore, their observations concerning the effective aperture of the TRM are also valid for our case. Furthermore, when $a < 1$ and the diffusion matrix $D_{ij} = D\delta_{ij}$ with a constant diffusion coefficient D we have that the effective aperture of the TRM is given by $a_{TRM}^{eff} = \sqrt{a_{TRM}^2 + \frac{DL^3}{3}}$, where a_{TRM} represents the actual aperture of the TRM and L the distance from the source to the TRM.

4.2. **Numerical results.** The conclusions in the previous section are obtained under the condition $a < 1$. The results of the numerical simulations, carried out for the case where $a = 1$, show that this restriction is not fundamental. More specifically, we conclude in this situation and in the presence of perturbations, that the re-compressed wave remains statistically stable (i.e the back-propagated wave is self-averaging) and its space super-resolution is not affected, but the wave amplitude experience a frequency dependent attenuation.

All the numerical examples correspond to the $2D$ setting. We set the propagation velocity $c_0 = 1$ and consider the central wave length $\lambda_0 = 2\pi/k_0 = 2\pi/\omega_0$, where

k_0 and ω_0 are the central wave number and frequency, respectively. Consider the following values for the characteristic lengths involved: the TRM aperture $a_{TRM} = 250\lambda_0$, the distance from the TRM to the source $L = 5000\lambda_0$ and let the transversal and longitudinal correlation lengths of the medium fluctuations be $l_x = 5\lambda_0$ and $l_z = 100\lambda_0$, respectively. Furthermore, we ensure a high frequency regime by taking $\omega_0 = k_0 = 2\pi$. Note that this set of values satisfy the restrictions corresponding to the asymptotic regime introduced in section 2.1 in the case when $a = 1$. The random fluctuations are stationary centered Gaussian random fields with an exponential autocorrelation function, they are constructed spectrally. The maximum contrast is 10% for the forward medium, and the incremented independent perturbations represents 5% of the backward medium. As initial source we take $\phi_0(\mathbf{x}; k_0) = \exp\{-\frac{|\mathbf{x}|^2}{2\sigma_s^2}\}$ with $\sigma_s = 3\lambda_0$.

In the numerical simulations the Schrödinger equation is solved using a FD code, to model the infinite medium a *perfectly matched layer* that allows plane wave absorption through the (computational) lateral boundaries is introduced. The computational domain has a $4a_{TRM}$ width and its discretization uses $\Delta x = 0.25\lambda_0$ and $\Delta z = 0.5\lambda_0$.

In figure 2 we show the super-resolution of the back-propagated wave. First, we compare the re-compressed wave amplitude for the homogeneous medium and an individual realization of the unperturbed random medium. The left plot shows that a sharper re-compression is obtained in the later situation. Thus, the presence of inhomogeneities and the multipathing enhances the resolution of the re-compressed wave. In the right plot we compare the re-compressed wave amplitude for individual realizations of the unperturbed and perturbed random media. The resolution in both cases is similar, while the maximum amplitude is higher for the unperturbed case. As predicted, the perturbations do not affect the spatial resolution of the back-propagated wave, but attenuate its amplitude.

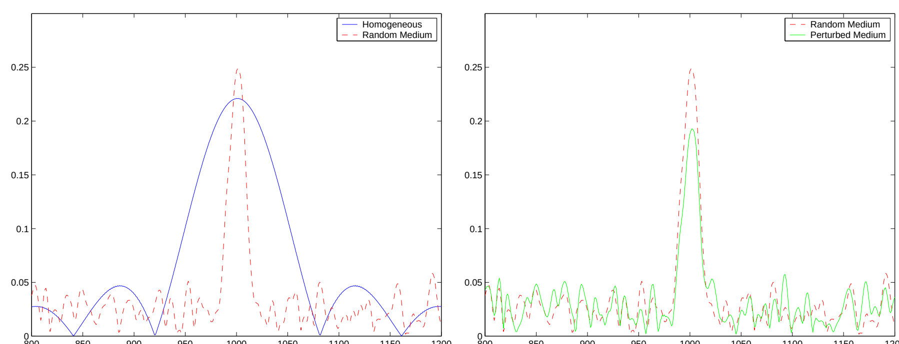

FIGURE 2. Super-focusing of the back-propagated wave. The absolute value of the re-compressed wave amplitude at the central wave number $k_0 = 2\pi$ is shown. Left plot: Homogeneous vs. (unperturbed) random media, the former is shown as a solid line and the latter with a dotted line. Right plot: Unperturbed vs. perturbed random media, the former is shown as a dotted line and the latter with a solid one.

In figure 3 (left) we illustrate the statistical stabilization (or self-averaging) of the re-compressed wave. The results from a set of 10 time reversal experiments are plotted. It is remarkable that for all medium realizations the resolution of the back-propagated wave are quite similar. It is also remarkable that nearby the center the re-compressed waves for different realizations vary less than at the sides. Although we only show the results of 10 realizations, we remark that other numerical simulations exhibit analogous results.

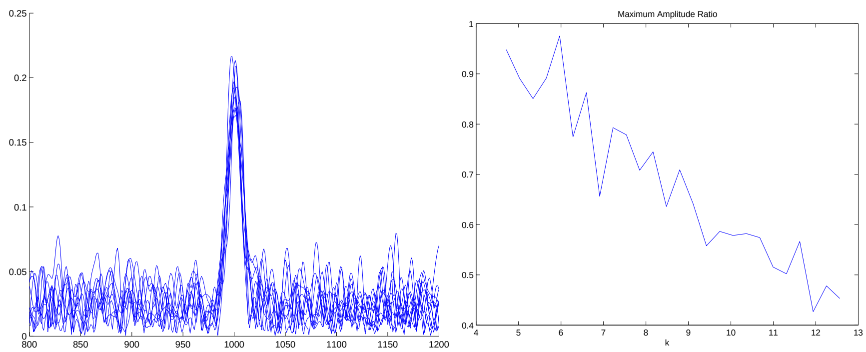

Figure 3. Left plot: Self-averaging of the back-propagated wave in a perturbed random media. We show the absolute value of the Wave Amplitude at the central wave number $k_0 = 2\pi$ for 10 realizations of the (perturbed) random media. Right plot: Wave Amplitude Ratio of perturbed to unperturbed random media at the central position for different wave numbers in the interval $[3\pi/2, 4\pi]$.

In the right plot of figure 3, we illustrate the attenuation effect of the perturbations on the re-compressed wave amplitude for different wave numbers. We present the resulting maximum wave amplitude ratio of the perturbed to unperturbed random media, corresponding to 26 values of the wave number k uniformly located in the interval $[3\pi/2, 4\pi]$. It is apparent from this plot that the wave amplitude decays as frequency increases, leading to the attenuation of the back-propagated pulse.

4.3. **Lateral diversity and stability.** Generally speaking, in the parabolic regime, the statistical stability of the back-propagated wave comes from the decorrelation of the Wigner transform in the frequency domain or the phase space. Then, integration in the frequency domain or the phase space, averages out the random variations of the Wigner transform. It is worth noticing that in the situation analyzed in this work, as well as in other high-frequency regimes (see for instance [10, 29]) the underlying physical mechanism leading to the mentioned decorrelation is the diffusion in the phase space. In fact, the high-frequency asymptotics of the Wigner transform is related to a Liouville equation. Under appropriate conditions, the trajectories associated with the Liouville equation have a diffusive behavior and two trajectories starting at different points of the phase space remain well separated. Finally, if the random fluctuations of the medium at the points of the trajectories decorrelates sufficiently fast, then the decorrelation of the corresponding solutions of the Liouville equation follows.

A fast decorrelation of the medium fluctuations is enforced by considering a large lateral diversity, i.e. the typical width of the beam is large compared to the corresponding correlation length. The self-averaging, in time reversal without perturbations, that results from lateral diversity is analyzed in [29, 30].

For time reversal without perturbations, the results obtained in [25] lead to the conclusion that in the high-frequency regime the lateral diversity is a necessary condition for the statistical stability. Indeed, there it is shown that in the regime corresponding to $a = 0$ in the scaling (6) the Wigner transform does not decorrelate. The results in the present work and in [29, 30] show that as soon as some lateral diversity is present, a statistically stable re-compressed wave is obtained. Moreover, we proved here that relatively weak perturbations do not affect the self-averaging.

4.4. **Concluding remarks.** We have analyzed and explained the effect of medium perturbations during a time reversal experiment. Our analysis was carried out for a high frequency regime where the TRM size is much smaller than the propagation distance. The medium is anisotropic and its longitudinal and transversal correlation lengths are much smaller than the propagation distance and the TRM size, respectively, and the first ratio is smaller than or comparable to the second. Furthermore, the fluctuations of the refractive index are weak and the perturbations are relatively small and we considered an initial pulse generated by a localized source.

We proved that perturbations do not affect the main properties of the re-compressed wave, namely, super-resolution and statistical stability but produces a pulse smeared in time. Moreover, the influence of the perturbations can be quantified through the statistical properties of the medium.

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Appendix A. Diffusion limit of the characteristic ODEs

In this appendix, we study the weak convergence of stochastic processes associated with a system of ODEs with a special form. These auxiliary results are used in the main part of the paper to study the characteristic ODEs (35). Actually, we will study a system of ODEs slightly more general than these equations. Our analysis follows the same lines as in [25, 26].

This appendix is organized as follows, in A.1 we present the conditions satisfied by the random field associated with the right-hand side of the system of ODEs. In Section A.2, we study the one-particle problem and characterize the weak limit of the solution to the ODEs. Section A.3 is concerned with the two-particle problem, that is, we analyze the weak convergence of the joint solution to the ODEs that start at different points.

A.1. **Introduction .** Let $(\Omega, \mathcal{F}, P)$ be a probability space and the vector field $F : \mathbb{R}^{m+1} \times \Omega \to \mathbb{R}^l$ be jointly measurable with respect to $\mathcal{B}^{m+1} \times \mathcal{F}$, where $\mathcal{B}^{m+1}$ is the σ-algebra of Borel sets in $\mathbb{R}^{m+1}$ and assume that $m \leq l$. We also assume that the random field $F(\mathbf{y}) = F(\mathbf{y}, \omega)$ is stationary and sufficiently smooth with respect to $\mathbf{y}$. We use the representation $\|f\|_\infty = \text{ess-sup}_{\omega\in\Omega} |f(\cdot, \omega)|$.

More specifically, we assume that the random field F satisfies the following conditions:

(C.1) The random field F is strictly stationary and has mean zero, i.e. $E\{F(\mathbf{y})\} = \int_\Omega F(\mathbf{y},\omega)P(d\omega) = 0$, $\forall \mathbf{y} \in \mathbb{R}^{m+1}$

(C.2) The derivatives $\frac{\partial}{\partial \mathbf{y}^\alpha}F(\mathbf{y},\omega) = \frac{\partial^{|\alpha|}}{\partial y_0^{\alpha_0}\ldots\partial y_m^{\alpha_m}}F(\mathbf{y},\omega)$ exist for P almost all ω and satisfy the continuity condition

$$\lim_{\substack{\mathbf{h}\in\mathbb{R}^m \\ |\mathbf{h}|\to 0}} \|\frac{\partial}{\partial \mathbf{y}^\alpha}F(s,\mathbf{h}) - \frac{\partial}{\partial \mathbf{y}^\alpha}F(s,\mathbf{0})\|_\infty = 0, \tag{50}$$

when the index $\alpha = (0,\alpha_1,\ldots,\alpha_m)$ and $|\alpha| = \sum_{i=0}^m \alpha_i \leq 3$.

(C.3) F is exponentially ϕ-mixing. More precisely, for $\Lambda \subset \mathbb{R}^{m+1}$ set

$$\mathcal{G}_\Lambda = \sigma\{F(\mathbf{y},\cdot) : \mathbf{y} \in \Lambda\}$$

and for $\Lambda_j \subset \mathbb{R}^{m+1}$, $j = 1,2$ define

$$\phi(\Lambda_1,\Lambda_2) = \sup_{A\in\mathcal{G}_{\Lambda_1}, B\in\mathcal{G}_{\Lambda_2}} |P(B) - P(B|A)|.$$

Define the uniform mixing rate as

$$\phi(\varrho) = \sup\{\phi(\Lambda_1,\Lambda_2) : \Lambda_j \in \mathcal{B}^{m+1},\, d(\Lambda_1,\Lambda_2) \geq \varrho\} \tag{51}$$

where $d(\Lambda_1,\Lambda_2) = \inf\{|\mathbf{y}_1 - \mathbf{y}_2| : \mathbf{y}_i \in \Lambda_i\}$, then there exists a constant $C_1 > 0$ such that

$$\phi(\varrho) \leq 2e^{-C_1\varrho}, \quad \varrho > 0. \tag{52}$$

(C.4) We let

$$R(\mathbf{y}) = E\{F(\mathbf{y}) \otimes F(\mathbf{0})\}, \quad \mathbf{y} \in \mathbb{R}^{m+1}$$

be the covariance matrix of the vector field $F(\cdot)$. Note that (52) implies that there exists a constant C_2 such that

$$|\frac{\partial}{\partial \mathbf{y}^\alpha}R_{ij}(\mathbf{y})| \leq C_2 e^{-\frac{|\mathbf{y}|_2}{C_2}}, \quad \forall \mathbf{y} \in \mathbb{R}^{m+1}, \quad i,j = 1,\ldots,l \tag{53}$$

where $|\mathbf{y}|_2$ represents the euclidean norm of the vector $\mathbf{y}$ and the index $\alpha = (0,\alpha_1,\cdots,\alpha_m)$ with $|\alpha| \leq 2$.

(C.5) (a) Assume that $R_{ij} \in C^\infty(\mathbb{R}^{m+1})$, $i,j = 1,\ldots,l$. Consider the $l \times l$ matrix $A(\tilde{\mathbf{q}})$ and the l-dimensional vector $b(\tilde{\mathbf{q}})$ with elements

$$A_{ij}(\tilde{\mathbf{q}}) = a_{ij}(\tilde{\mathbf{q}}) + a_{ji}(\tilde{\mathbf{q}}) \quad \text{and} \quad b_i(\tilde{\mathbf{q}}) = \sum_{j=1}^m \frac{\partial a_{ij}}{\partial q_j}(\tilde{\mathbf{q}})$$

where

$$a_{ij}(\tilde{\mathbf{q}}) = \int_0^\infty R_{ij}(s, s\tilde{\mathbf{q}})ds.$$

(b) Suppose that the matrix $A(\tilde{\mathbf{q}})$ is positive-definite and let $C(\tilde{\mathbf{q}})$ be its positive symmetric square root. We further assume that the $m \times m$ sub-matrix $\tilde{C}(\tilde{\mathbf{q}})$ with elements $\tilde{C}_{ij}(\tilde{\mathbf{q}}) = C_{ij}(\tilde{\mathbf{q}})$, $i,j = 1,\ldots,m$ is non-singular $\forall\, \tilde{\mathbf{q}} \in \mathbb{R}^m$.

A.2. **One-particle problem .** For any $0 < \epsilon \le 1$ define the stochastic process $(\mathbf{x}^\epsilon(s), \mathbf{q}^\epsilon(s)) = (\mathbf{x}^\epsilon(s,\omega), \mathbf{q}^\epsilon(s,\omega))$ with values in $\mathbb{R}^m \times \mathbb{R}^l$, $s \ge 0$, $\omega \in \Omega$ as the solution of the system of ODEs

$$\begin{cases} \dfrac{d\mathbf{x}^\epsilon}{ds} = \tilde{\mathbf{q}}^\epsilon \\ \dfrac{d\mathbf{q}^\epsilon}{ds} = \dfrac{1}{\sqrt{\epsilon}} F(\dfrac{s}{\epsilon}, \dfrac{\mathbf{x}^\epsilon}{\epsilon^a}) \\ (\mathbf{x}^\epsilon(0), \mathbf{q}^\epsilon(0)) = (\mathbf{x}_0, \mathbf{q}_0) \end{cases} \tag{54}$$

where $(\mathbf{x}_0, \mathbf{q}_0)$ is non-random and $\tilde{\mathbf{q}}^\epsilon$ represents the $\mathbb{R}^m$-projection of $\mathbf{q}^\epsilon$.

Next, we focus on the weak convergence of the stochastic process defined above as ϵ goes to zero.

Consider the diffusion operator

$$\mathcal{L}^a f(\mathbf{q}) = \begin{cases} \frac{1}{2}\sum_{i,j=1}^l A_{ij}(\tilde{\mathbf{q}}) \frac{\partial^2 f}{\partial q_i \partial q_j}(\mathbf{q}) + \sum_{i=1}^l b_i(\tilde{\mathbf{q}}) \frac{\partial f}{\partial q_i}(\mathbf{q}), & \text{if } a = 1 \\ \frac{1}{2}\sum_{i,j=1}^l A_{ij}(\mathbf{0}) \frac{\partial^2 f}{\partial q_i \partial q_j}(\mathbf{q}) + \sum_{i=1}^l b_i(\mathbf{0}) \frac{\partial f}{\partial q_i}(\mathbf{q}), & \text{if } 0 < a < 1 \end{cases} \tag{55}$$

for all $f \in C_b^2(\mathbb{R}^l)$. From the conditions above follows that this operator is well-defined. Furthermore, if the random field $F(\cdot)$ is isotropic then we have that

$$(\mathcal{L}^a f)(\mathbf{q}) = \begin{cases} \frac{1}{2}\nabla_{\mathbf{q}} \cdot \big(A(\tilde{\mathbf{q}})\nabla_{\mathbf{q}} f\big), & \text{if } a = 1 \\ \frac{1}{2}\nabla_{\mathbf{q}} \cdot \big(A(\mathbf{0})\nabla_{\mathbf{q}} f\big), & \text{if } 0 < a < 1 \end{cases}$$

and

$$A_{ij}(\tilde{\mathbf{q}}) = \int_{-\infty}^{+\infty} R_{ij}(s, s\tilde{\mathbf{q}}) ds.$$

For a positive integer n, let us represent by $\mathcal{C}^n = C([0,\infty);\mathbb{R}^n)$, the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^n$. The following theorem is a generalization of the results in [25, 26].

Theorem A.1. *Suppose that conditions C.1–C.3 and C.5(a) above are fulfilled. Then the stochastic process* $(\mathbf{x}^\epsilon(s), \mathbf{q}^\epsilon(s))$ *converges weakly in* $\mathcal{C}^{m+l}$ *as* $\epsilon \to 0$, *to the process* $(\mathbf{x}(s), \mathbf{q}(s))$ *such that*

$$\mathbf{x}(s) = \mathbf{x}_0 + \int_0^s \tilde{\mathbf{q}}(s') ds'$$

and $\mathbf{q}(s)$ *is the diffusion process in* $\mathcal{C}^l$ *with generator* $\mathcal{L}^a$ *defined by* (55) *starting from* $\mathbf{q}_0$.

The proof of the above theorem is very similar to that of Theorem 3 in [25, page 105] and consequently we only highlight the main ideas. Let R^ϵ denote the probability measure induced by $\mathbf{q}^\epsilon(s)$ on $\mathcal{C}^l$. It is enough to establish that R^ϵ weakly converges to $R_{\mathbf{q}_0}$, the probability measure of a diffusion in $\mathcal{C}^l$ with generator $\mathcal{L}^a$ and starting from $\mathbf{q}_0$. The main idea of the proof is to study a truncated process whose dynamic up to a certain stopping time coincides with that of the original process. Furthermore, the weak limits of the truncated process can be identified along with some relevant properties. Finally, by using a measure theoretic argument the weak convergence of the original process is established.

We remark that for the study of the one-particle problem one can assume less restrictive conditions. Namely, we can assume that the random field $F(\cdot)$ has the following uniform mixing properties. Let

$$\tilde{\mathcal{G}}_s^t = \sigma\{F(u,\mathbf{y},\cdot) : s \le u \le t,\ \mathbf{y} \in \mathbb{R}^m\}$$

and define the uniform mixing rate as

$$\tilde{\phi}(s) = \sup_{r\ge 0}\sup\{|P(B) - P(B|A)| \ : \ A \in \mathcal{G}_0^r, B \in \mathcal{G}_{r+s}^\infty\}. \tag{56}$$

And further, assume that

$$\int_0^\infty s^3 \tilde{\phi}^{1/2}(s)ds < \infty. \tag{57}$$

In particular, if the condition C.3 presented early holds then the mixing condition (57) is satisfied.

Moreover, the condition C.5(a) is used in the proof only to ascertain the uniqueness of the corresponding diffusion process and weaker conditions that assures this property can be found in [31].

A.2.1. *Mixing Lemmas.* For estimating some integrals in this section we will use two mixing lemmas which are slightly modified variants of the Lemmas 1 and 2 in [25, pp. 109 and 112].

Consider the random fields $U, V : [0,+\infty) \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ that are strictly stationary and satisfy the continuity conditions

$$\lim_{|\mathbf{h}|\to 0} \|U(s,\mathbf{h}) - U(s,\mathbf{0})\|_\infty = \lim_{|\mathbf{h}|\to 0} \|V(s,\mathbf{h}) - V(s,\mathbf{0})\|_\infty = 0 \tag{58}$$

for any $s \ge 0$.

Lemma A.1. *Assume that $U(\frac{\tau}{\epsilon},\mathbf{y})$ is $\tilde{\mathcal{G}}_0^{\tau/\epsilon}$ measurable and $E\{U(\frac{\tau}{\epsilon},\mathbf{y})\} = 0$ for each fixed $\mathbf{y} \in \mathbb{R}^n$. Further, let $Z(\frac{\sigma}{\epsilon})$, $\sigma \le \tau$ be a $\tilde{\mathcal{G}}_0^{\sigma/\epsilon}$ measurable random variable. Then for any ϱ, $0 \le \varrho \le \sigma \le \tau$ and a $\tilde{\mathcal{G}}_0^{\varrho/\epsilon}$ measurable random variable $\mathbf{y}_\varrho$ with values in $\mathbb{R}^n$*

$$\big|E\big\{U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)Z(\frac{\sigma}{\epsilon})\big\}\big| \le 2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty E\big\{\big|Z(\frac{\sigma}{\epsilon})\big|\big\}. \tag{59}$$

Proof. The proof is standard and is based on an approximation argument and a well-known mixing result of Ibragimov & Linnik [24].

Let $\mathbf{l} = (l_1,\dots,l_n) \in \mathbb{Z}^n$ and M_1 be a positive fixed integer, we introduce the notations $C_\mathbf{l} = [l_1/M_1,(l_1+1)/M_1) \times \cdots \times [l_n/M_1,(l_n+1)/M_1) \subset \mathbb{R}^n$, $\bar{\mathbf{y}}_\mathbf{l} = ((2l_1+1)/M_1),\dots,(2l_n+1)/M_1) \in \mathbb{R}^n$, the event $A_\mathbf{l} = \{\omega : \mathbf{y}_\varrho \in C_\mathbf{l}\}$ and its indicator function $I_\mathbf{l} = \mathbf{1}_{A_\mathbf{l}}$.

Choose an arbitrary $\delta > 0$ and select M_1 sufficiently large such that $\|U(\frac{\tau}{\epsilon},\mathbf{y}) - U(\frac{\tau}{\epsilon},\bar{\mathbf{y}}_\mathbf{l})\|_\infty \le \delta$ for every $\mathbf{l}$ and $\mathbf{y} \in C_\mathbf{l}$. Furthermore, we have the following estimates

$$\begin{aligned}
|E\{U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)Z(\frac{\sigma}{\epsilon})\}| &= |\sum_\mathbf{l} E\{U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)Z(\frac{\sigma}{\epsilon})I_\mathbf{l}\}| \\
&= |\sum_\mathbf{l}\big\{E\{(U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho) - U(\frac{\tau}{\epsilon},\bar{\mathbf{y}}_\mathbf{l}))Z(\frac{\sigma}{\epsilon})I_\mathbf{l}\} + E\{U(\frac{\tau}{\epsilon},\bar{\mathbf{y}}_\mathbf{l})Z(\frac{\sigma}{\epsilon})I_\mathbf{l}\}\big\}| \\
&\le \sum_\mathbf{l}\big\{E\{|U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho) - U(\frac{\tau}{\epsilon},\bar{\mathbf{y}}_\mathbf{l})||Z(\frac{\sigma}{\epsilon})|I_\mathbf{l}\} + |E\{U(\frac{\tau}{\epsilon},\bar{\mathbf{y}}_\mathbf{l})Z(\frac{\sigma}{\epsilon})I_\mathbf{l}\}|\big\}
\end{aligned}$$

The first term above is estimated by using the approximation assumption and the second one by using the results from [24, Theorem 17.2.3, pp. 309–310] to get

$$\begin{aligned}|E\{U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)Z(\frac{\sigma}{\epsilon})\}| &\le \sum_{\mathbf{l}}\big(\delta+2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\big)E\{|Z(\frac{\sigma}{\epsilon})|I_{\mathbf{l}}\}\\ &\le \big(\delta+2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\big)E\{|Z(\frac{\sigma}{\epsilon})|\}.\end{aligned}$$

Since δ is arbitrary the conclusion of the lemma follows. □

The following result is a useful variant of Lemma A.1.

Lemma A.2. *Assume that* $U(\frac{\tau}{\epsilon},\mathbf{y})$, $V(\frac{\sigma}{\epsilon},\mathbf{y}))$ *are* $\tilde{\mathcal{G}}^{\tau/\epsilon}_{\tau/\epsilon}$ *and* $\tilde{\mathcal{G}}^{\sigma/\epsilon}_{\sigma/\epsilon}$ *measurable, respectively, for each fixed* $\mathbf{y}\in\mathbb{R}^n$. *Assume also that* $E\{U(\frac{\tau}{\epsilon},\mathbf{y})\}=0$ *and set*

$$W(\tau/\epsilon,\sigma/\epsilon,\mathbf{y})=E\big\{U(\tau/\epsilon,\mathbf{y})V(\sigma/\epsilon,\mathbf{y})\big\}.$$

Furthermore, let $Z(\frac{\varrho}{\epsilon})$, $\mathbf{y}_\varrho$ *be a real and* $\mathbb{R}^n$*-valued, respectively,* $\tilde{\mathcal{G}}^{\varrho/\epsilon}_0$ *measurable random variables. Then for any* ϱ, $0\le\varrho\le\sigma\le\tau$

$$\begin{aligned}&\big|E\big\{Z(\frac{\varrho}{\epsilon})\big[U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)V(\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)-W(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)\big]\big\}\big|\le\\ &\qquad 4\tilde{\phi}^{\frac12}(\frac{\tau-\sigma}{\epsilon})\tilde{\phi}^{\frac12}(\frac{\sigma-\varrho}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty E\big\{\big|Z(\frac{\varrho}{\epsilon})\big|\big\}.\end{aligned}\tag{60}$$

Proof. First we apply Lemma A.1 with $UV-W$ substituted for U to show that

$$\begin{aligned}&\big|E\big\{Z(\frac{\varrho}{\epsilon})\big[U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)V(\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)-W(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)\big]\big\}\big|\le\\ &\qquad 4\tilde{\phi}(\frac{\sigma-\varrho}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty E\big\{\big|Z(\frac{\varrho}{\epsilon})\big|\big\}\end{aligned}\tag{61}$$

since $\|U(\frac{\tau}{\epsilon},\cdot)V(\frac{\sigma}{\epsilon},\cdot)-W(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\cdot)\|_\infty\le 2\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty$. Next we apply this lemma again, now with ZV substituted for Z to obtain

$$\big|E\big\{Z(\frac{\varrho}{\epsilon})U(\frac{\tau}{\epsilon},\mathbf{y}_\varrho)V(\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)\big\}\big|\le 2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty E\big\{\big|Z(\frac{\varrho}{\epsilon})\big|\big\}.\tag{62}$$

For a fixed $\mathbf{y}$, by applying [24, Theorem 17.2.3, pp. 309–310] one gets

$$\big|E\big\{W(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\mathbf{y})\big\}\big|\le 2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty,$$

and consequently

$$\big|E\big\{Z(\frac{\varrho}{\epsilon})W(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\mathbf{y}_\varrho)\big\}\big|\le 2\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})\|U(\frac{\tau}{\epsilon},\cdot)\|_\infty\|V(\frac{\sigma}{\epsilon},\cdot)\|_\infty E\big\{\big|Z(\frac{\varrho}{\epsilon})\big|\big\}.\tag{63}$$

By combining the inequalities (61)–(63) the desired estimate follows. □

A.2.2. *Truncated process.* Let $M>|\mathbf{q}_0|$, consider a cutoff function $\varphi_M:\mathbb{R}^l\to[0,1]$, $\varphi_M\in C^\infty(\mathbb{R}^l)$ such that

$$\varphi_M(\mathbf{q})=\begin{cases}0, & |\mathbf{q}|\ge 2M\\ 1, & |\mathbf{q}|\le M\end{cases},\tag{64}$$

set $G^M(\mathbf{y},\mathbf{q}) = \varphi_M(\mathbf{q})F(\mathbf{y})$ and define the truncated system of ODEs:

$$\begin{cases} \dfrac{d\mathbf{x}^{\epsilon,M}}{ds} = \tilde{\mathbf{q}}^{\epsilon,M} \\ \dfrac{d\mathbf{q}^{\epsilon,M}}{ds} = \dfrac{1}{\sqrt{\epsilon}} G^M(\dfrac{s}{\epsilon}, \dfrac{\mathbf{x}^{\epsilon,M}}{\epsilon^a}, \mathbf{q}^{\epsilon,M}) \end{cases} \tag{65}$$

with initial conditions $(\mathbf{x}_0, \mathbf{q}_0)$.

For each fixed M, we will prove now the tightness of the family of measures $R^{\epsilon,M}$ induced by the truncated process $\mathbf{q}^{\epsilon,M}(s)$ in $\mathcal{D}^l = D([0,\infty);\mathbb{R}^l)$, the space of 'càdlàg' functions from $[0,\infty)$ to $\mathbb{R}^l$ with the Skorohod topology. Further, we simplify the notation by suppressing the index M.

The proof relays on the approach of [25, pp.107–108]. We will obtain that for $0 \le t \le u \le T < \infty$ there exists $C = C(T)$ independent of $\epsilon \in (0,1]$ such that

$$E\{|\mathbf{q}^\epsilon(u) - \mathbf{q}^\epsilon(t)|^2 \Phi\} \le C(u-t)E\{\Phi\} \tag{66}$$

for $\Phi = \Phi(s,t) = |\mathbf{q}^\epsilon(t) - \mathbf{q}^\epsilon(s)|^r$, $0 \le s \le t$ and $r = 0, 2$.

We have that

$$\begin{aligned} |\mathbf{q}^\epsilon(u) - \mathbf{q}^\epsilon(t)|^2 &= \frac{2}{\sqrt{\epsilon}} \sum_{i=1}^{l} \int_t^u \int_t^\tau \frac{d}{d\sigma}\Big[G_i(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma))\big(q_i^\epsilon(\sigma) - q_i^\epsilon(t)\big)\Big] d\tau d\sigma \\ &= \frac{2}{\epsilon} \sum_{i=1}^{l} \int_t^u \int_t^\tau \Big\{ \sum_{j=1}^{l} G_j(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) \Big[\big(q_i^\epsilon(\sigma) - q_i^\epsilon(t)\big) \\ &\times \frac{\partial G_i}{\partial q_j}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) + \delta_{ij} G_i(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma))\Big] \\ &+ \frac{\tau - \sigma}{\epsilon^a} \sum_{k=1}^{m} \big(q_k^\epsilon(\sigma) - q_k^\epsilon(t)\big) \frac{\partial G_i}{\partial y_k}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) \\ &\times G_k(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) \Big\} d\tau d\sigma \end{aligned}$$

where $\mathbf{l}^\epsilon(\sigma;\tau) = \mathbf{x}^\epsilon(\sigma) + (\tau - \sigma)\tilde{\mathbf{q}}^\epsilon(\sigma)$ and consequently

$$E\{|\mathbf{q}^\epsilon(u) - \mathbf{q}^\epsilon(t)|^2\, \Phi\} = \int_t^u \sum_{i=1}^{l} \big(\sum_{j=1}^{l} (I_{ij}^{(1)}(\tau) + \delta_{ij} I_i^{(1)}(\tau)) + \sum_{k=1}^{m} I_{ik}^{(2)}(\tau) \big) d\tau$$

where

$$\begin{aligned} I_{ij}^{(1)}(\tau) &= \frac{1}{\epsilon} E\{\int_t^\tau \frac{\partial G_i}{\partial q_j}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) G_j(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) \\ &\qquad \times \big(q_i^\epsilon(\sigma) - q_i^\epsilon(t)\big) d\sigma\, \Phi\} \\ I_i^{(1)}(\tau) &= \frac{1}{\epsilon} E\{\int_t^\tau G_i(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) G_i(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) d\sigma\, \Phi\} \\ I_{ik}^{(2)}(\tau) &= \frac{1}{\epsilon^{1+a}} E\{\int_t^\tau \frac{\partial G_i}{\partial y_k}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^\epsilon(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) G_k(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a}, \mathbf{q}^\epsilon(\sigma)) \\ &\qquad \times \big(q_i^\epsilon(\sigma) - q_i^\epsilon(t)\big)(\tau - \sigma) d\sigma\, \Phi\} \end{aligned}$$

A straightforward application of the mixing Lemma A.1, leads to the following inequalities

$$\begin{aligned} I_{ij}^{(1)}(\tau) &\leq \|\nabla\varphi_M\|_\infty \|F\|_\infty^2 E\{|\Phi|\} \frac{8M}{\epsilon}\int_t^\tau \tilde{\phi}(\frac{\tau-\sigma}{\epsilon})d\sigma \leq C_1 E\{|\Phi|\} \\ I_{i}^{(1)}(\tau) &\leq \|F\|_\infty^2 E\{|\Phi|\} \frac{2}{\epsilon}\int_t^\tau \tilde{\phi}(\frac{\tau-\sigma}{\epsilon})d\sigma \leq C_2 E\{|\Phi|\} \\ I_{ik}^{(2)}(\tau) &\leq \|F\|_\infty \|\nabla F\|_\infty E\{|\Phi|\} \frac{8M}{\epsilon^{1+a}}\int_t^\tau (\tau-\sigma)\tilde{\phi}(\frac{\tau-\sigma}{\epsilon})d\sigma \leq C_3 \epsilon^{1-a} E\{|\Phi|\} \end{aligned}$$

from which the estimate (66) immediately follows with $C = l(C_1 l + C_2 + C_3 m)$.

We only prove the first inequality as the others can be obtained in a similar way. Let $n = m$, set

$$\mathbf{y}_\varrho = \mathbf{l}^\epsilon(\varrho;\tau), \quad U(\frac{\tau}{\epsilon},\mathbf{y}) = F_i(\frac{\tau}{\epsilon},\frac{\mathbf{y}}{\epsilon^a}) \quad \text{and}$$

$$V(\frac{\sigma}{\epsilon}) = \frac{\partial\varphi_M}{\partial q_j}(\mathbf{q}^\epsilon(\sigma))\varphi_M(\mathbf{q}^\epsilon(\sigma))(q_i^\epsilon(\sigma)-q_i^\epsilon(t))F_j(\frac{\sigma}{\epsilon},\frac{\mathbf{x}^\epsilon(\sigma)}{\epsilon^a})\Phi(t,s).$$

These functions satisfy the assumptions of the lemma since for any $s \geq 0$ we have that $\mathbf{x}^\epsilon(s)$, $\mathbf{q}^\epsilon(s)$ are $\tilde{\mathcal{G}}_0^{s/\epsilon}$ measurable, and as a consequence $\mathbf{l}^\epsilon(s;\tau)$ is also. Furthermore, $|q_i^\epsilon(\sigma) - q_i^\epsilon(t)| \leq 4M$ and $\|\nabla\varphi_M\|_\infty < \infty$ therefore, after substitution in (59) one gets the desired result since $E\{|V(\frac{\sigma}{\epsilon})|\} \leq 4M\|\nabla\varphi_M\|_\infty\|F\|_\infty E\{|\Phi|\}$.

A.2.3. *Limit identification.* Our next step consists in the identification of the limit points of R^ϵ. We proceed as in [25] by first identifying the limit points of $R^{\epsilon,M}$ in $\mathcal{D}^l$. Representing by $X(t)$ the t-coordinate function in $\mathcal{D}^l$ the corresponding σ-fields of subsets are given by

$$\mathcal{M}_u^v = \sigma\{X(t) : u \leq t \leq v\}.$$

Assume that $\{\epsilon_n\}$ is a sequence of positive numbers approaching zero such that the sequence of measures $\{R^{\epsilon_n,M}\}$ weakly converges to R^M on $\mathcal{D}^l$ as $n \to \infty$. For any $f \in C^\infty(\mathbb{R}^l)$ with compact support define

$$\mathcal{L}^M f(\mathbf{q}) = \frac{1}{2}\sum_{i,j=1}^{l} A_{ij}^M(\mathbf{q})\frac{\partial^2 f}{\partial q_i \partial q_j} + \sum_{i=1}^{l} b_i^M(\mathbf{q})\frac{\partial f}{\partial q_j} \tag{67}$$

where

$$A_{ij}^M(\mathbf{q}) = \varphi_M^2(\mathbf{q}) \times \begin{cases} A_{ij}(\tilde{\mathbf{q}}), & \text{if } a = 1 \\ A_{ij}(\mathbf{0}), & \text{if } 0 < a < 1 \end{cases}$$

$$b_i^M(\mathbf{q}) = \varphi_M(\mathbf{q})\sum_{j=1}^{l}\begin{cases} \frac{\partial}{\partial q_j}\big(\varphi_M(\mathbf{q})a_{ij}(\tilde{\mathbf{q}})\big), & \text{if } a = 1 \\ \frac{\partial\varphi_M(\mathbf{q})}{\partial q_j}a_{ij}(\mathbf{0}) + \varphi_M(\mathbf{q})\frac{\partial a_{ij}}{\partial q_j}(\mathbf{0}), & \text{if } 0 < a < 1. \end{cases}$$

We show that $f(X(t)) - \int_0^t \mathcal{L}^M f(X(s))ds$ is a $(R^M, \mathcal{M}_0^t)$ martingale. To establish this fact it suffices to prove for any integer $n > 0$, bounded continuous function $\Phi : (\mathbb{R}^l)^n \to \mathbb{R}$ and $s_1 \leq \cdots \leq s_n \leq t < u$ that

$$E^M\{(f(X(u)) - f(X(t)))\Phi(X(s_1),\cdots,X(s_n))\} = E^M\{\int_t^u \mathcal{L}^M f(X(s))\Phi(X(s_1),\cdots,X(s_n))\} \tag{68}$$

where $E^M\{\cdot\}$ represents expectation with respect to R^M.

We further study the quantity

$$
\begin{aligned}
I &= E^M\{(f(X(u)) - f(X(t)))\Phi(X(s_1), \cdots, X(s_n))\} \\
&= \lim_{n\to\infty} E\{(f(\mathbf{q}^{\epsilon_n,M}(u)) - f(\mathbf{q}^{\epsilon_n,M}(t)))\Phi(\mathbf{q}^{\epsilon_n,M}(s_1), \cdots, \mathbf{q}^{\epsilon_n,M}(s_n))\}.
\end{aligned} \tag{69}
$$

To simplify notations, we drop the index n and let ϵ go to zero through the sequence $\{\epsilon_n\}$.

We have that $\delta f(\mathbf{q}^{\epsilon,M})(u,t) = f(\mathbf{q}^{\epsilon,M}(u)) - f(\mathbf{q}^{\epsilon,M}(t))$ can be re-written as

$$
\begin{aligned}
\delta f(\mathbf{q}^{\epsilon,M})(u,t) &= \frac{1}{\sqrt{\epsilon}}\sum_{i=1}^{l}\int_t^u \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\tau))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\tau))d\tau \\
&= \frac{1}{\sqrt{\epsilon}}\sum_{i=1}^{l}\int_t^u \Big\{\frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(t))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(t;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(t)) \\
&+ \int_t^\tau \big[\frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\big]'_\sigma d\sigma\Big\}d\tau \\
&= \sum_{i=1}^{l}\int_t^u \Big\{\frac{1}{\sqrt{\epsilon}}\frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(t))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(t;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(t)) \\
&+ \frac{1}{\epsilon}\int_t^\tau \Big[\sum_{j=1}^{l}\Big(\frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma))\frac{\partial G_i^M}{\partial q_j}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma)) \\
&+ \frac{\partial^2 f}{\partial q_i\partial q_j}(\mathbf{q}^{\epsilon,M}(\sigma))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\Big) \\
&\times G_j^M(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma)) + \frac{\tau-\sigma}{\epsilon^a}\sum_{k=1}^{m}\frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma)) \\
&\times \frac{\partial G_i^M}{\partial y_k}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))G_k^M(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\Big]d\sigma\Big\}d\tau
\end{aligned}
$$

We shall multiply by $\Phi = \Phi(\mathbf{q}^\epsilon(s_1), \cdots, \mathbf{q}^\epsilon(s_n))$ and pass to the limit as $\epsilon \downarrow 0$. By applying the mixing Lemma A.1 to the first term on the right-hand-side we get

$$
\begin{aligned}
&\Big|\frac{1}{\sqrt{\epsilon}}E\{\int_t^u \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(t))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(t;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(t))\Phi d\tau\}\Big| \\
&\qquad \le \|\nabla f\|_\infty \|F\|_\infty \|\Phi\|_\infty \frac{1}{\sqrt{\epsilon}}\int_t^u \tilde{\phi}(\frac{\tau-t}{\epsilon})d\tau \to 0.
\end{aligned}
$$

For the other terms we introduce the notations

$$
\begin{aligned}
I_{ij}^1 &= \frac{1}{\epsilon}\int_t^u\int_t^\tau \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma))G_{i,q_j}^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))) \\
&\qquad \times G_j^M(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\,\Phi d\sigma d\tau, \\
I_{ij}^2 &= \frac{1}{\epsilon}\int_t^u\int_t^\tau \frac{\partial^2 f}{\partial q_i\partial q_j}(\mathbf{q}^{\epsilon,M}(\sigma))G_i^M(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))) \\
&\qquad \times G_j^M(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\,\Phi d\sigma d\tau,
\end{aligned}
$$

$$I^3_{ik} \quad = \quad \frac{1}{\epsilon^{1+a}} \int_t^u \int_t^\tau (\tau-\sigma) \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma)) \frac{\partial G_i^M}{\partial y_k}(\frac{\tau}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))$$
$$\times G_k^M(\frac{\sigma}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\sigma))\, \Phi d\sigma d\tau$$

Next, we compute the limit of $E\{I^1_{ij}\}$ when $\epsilon \to 0$. By introducing the functions

$$H_{ij}(\tau/\epsilon, \sigma/\epsilon, \mathbf{y}_1, \mathbf{y}_2) = F_i(\tau/\epsilon, \mathbf{y}_1)\, F_j(\sigma/\epsilon, \mathbf{y}_2),$$
$$\bar{H}_{ij} = E\{H_{ij}\} \quad \text{and} \quad \tilde{H}_{ij} = H_{ij} - \bar{H}_{ij},$$

we have that $I^1_{ij} = I^{11}_{ij} + I^{12}_{ij}$ where

$$I^{11}_{ij} = \frac{1}{\epsilon} \int_t^u \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma)) \varphi_M(\mathbf{q}^{\epsilon,M}(\sigma)) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}^{\epsilon,M}(\sigma))$$
$$\times \Big(\int_\sigma^u \bar{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}) d\tau \Big) \Phi d\sigma$$
$$I^{12}_{ij} = \frac{1}{\epsilon} \int_t^u \int_t^\tau \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\sigma)) \varphi_M(\mathbf{q}^{\epsilon,M}(\sigma)) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}^{\epsilon,M}(\sigma))$$
$$\times \tilde{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}) \Phi d\sigma d\tau.$$

Furthermore,

$$\frac{1}{\epsilon} \int_\sigma^u \bar{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\sigma;\tau)}{\epsilon^a}, \frac{\mathbf{x}^{\epsilon,M}(\sigma)}{\epsilon^a}) d\tau = \int_0^{\frac{u-\sigma}{\epsilon}} R_{ij}(s, s\epsilon^{1-a} \tilde{\mathbf{q}}^{\epsilon,M}(\sigma)) ds$$

and we get that

$$E\{I^{11}_{ij}\} \to E^M\{ \int_t^u b^{M,1}_{ij}(X(\sigma)) \frac{\partial f}{\partial q_i}(X(\sigma))\, \Phi d\sigma\}$$

where

$$b^{M,1}_{ij}(\mathbf{q}) = \varphi_M(\mathbf{q}) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}) \times \begin{cases} a_{ij}(s, s\tilde{\mathbf{q}}) ds & \text{if } a = 1, \\ a_{ij}(s, \mathbf{0}) ds & \text{if } 0 < a < 1. \end{cases}$$

Next we show that $E\{I^{12}_{ij}\} \to 0$. Let us decompose this term as $I^{12}_{ij} = J^1_{ij} + J^2_{ij}$ where

$$J^1_{ij} = \frac{1}{\epsilon} \int_t^u \int_t^\tau \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(t)) \varphi_M(\mathbf{q}^{\epsilon,M}(t)) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}^{\epsilon,M}(t))$$
$$\times \tilde{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(t;\tau)}{\epsilon^a}, \frac{\mathbf{l}^{\epsilon,M}(t;\sigma)}{\epsilon^a}) \Phi d\sigma d\tau,$$
$$J^2_{ij} = \frac{1}{\epsilon} \int_t^u \int_t^\tau \int_t^\sigma \frac{d}{d\rho} \Big\{ \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\rho)) \varphi_M(\mathbf{q}^{\epsilon,M}(\rho)) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}^{\epsilon,M}(\rho))$$
$$\times \tilde{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a}) \Big\}\, \Phi\, d\rho d\sigma d\tau.$$

Applying the second mixing Lemma A.2 with $n = 2m$,

$$U(\frac{\tau}{\epsilon}, \mathbf{y}) = F_i(\frac{\tau}{\epsilon}, \mathbf{y}_1), \quad V(\frac{\sigma}{\epsilon}, \mathbf{y}) = F_j(\frac{\sigma}{\epsilon}, \mathbf{y}_2),$$
$$Z(\frac{\rho}{\epsilon}) = \varphi_M(\mathbf{q}^{\epsilon,M}(\rho)) \frac{\partial \varphi_M}{\partial q_j}(\mathbf{q}^{\epsilon,M}(\rho)) \frac{\partial f}{\partial q_i}(\mathbf{q}^{\epsilon,M}(\rho)) \Phi,$$
$$\epsilon^a \mathbf{y}_\rho = (\mathbf{l}^{\epsilon,M}(\rho;\tau), \mathbf{l}^{\epsilon,M}(\rho;\sigma)) \quad \text{and} \quad \rho = t,$$

we get that

$$|E\{J_{ij}^1\}| \le 4\|\nabla\varphi_M\|_\infty \|F\|_\infty^2 E\{|\Phi|\}\frac{1}{\epsilon}\int_t^u \int_t^\tau \tilde{\phi}^{\frac{1}{2}}(\frac{\tau-\sigma}{\epsilon})\tilde{\phi}^{\frac{1}{2}}(\frac{\sigma-t}{\epsilon})d\sigma d\tau \to 0.$$

For J_{ij}^2 we have the following decomposition

$$\begin{aligned}
J_{ij}^2 = \frac{1}{\epsilon^{3/2}}\int_t^u\int_t^\tau\int_t^\sigma \Big\{ & \sum_{s=1}^{l}\frac{\partial\Theta_M}{\partial q_s}(\mathbf{q}^{\epsilon,M}(\rho))G_s^M(\frac{\rho}{\epsilon},\frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a},\mathbf{q}^{\epsilon,M}(\rho)) \\
& \times \tilde{H}_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a},\frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a}) \\
& + \sum_{k=1}^{m}\Theta_M(\mathbf{q}^{\epsilon,M}(\rho))G_k^M(\frac{\rho}{\epsilon},\frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a},\mathbf{q}^{\epsilon,M}(\rho)) \\
& \times \big[(\frac{\tau-\rho}{\epsilon^a})\frac{\partial\tilde{H}_{ij}}{\partial y_k^1}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a},\frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a}) \\
+ (\frac{\sigma-\rho}{\epsilon^a})\frac{\partial\tilde{H}_{ij}}{\partial y_k^2} & (\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a},\frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a})\big]\Big\}\,\Phi\, d\rho d\sigma d\tau \\
= \sum_{s=1}^{l}K_{ijs}^1 + \sum_{k=1}^{m}(K_{ijk}^2 + K_{ijk}^3),
\end{aligned}$$

where $\Theta_M(\mathbf{q}) = \frac{\partial f}{\partial q_i}(\mathbf{q})\varphi_M(\mathbf{q})\frac{\partial\varphi_M}{\partial q_j}(\mathbf{q})$. Now applying the second mixing Lemma A.2 with

$$\begin{gathered}
U(\frac{\tau}{\epsilon},\mathbf{y}) = F_i(\frac{\tau}{\epsilon},\mathbf{y}_1), \quad V(\frac{\sigma}{\epsilon},\mathbf{y}) = F_j(\frac{\sigma}{\epsilon},\mathbf{y}_2), \\
Z(\frac{\rho}{\epsilon}) = \frac{\partial\Theta_M}{\partial q_s}(\mathbf{q}^{\epsilon,M}(\rho))G_l^M(\frac{\rho}{\epsilon},\frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a},\mathbf{q}^{\epsilon,M}(\rho))\Phi \\
\text{and} \quad \epsilon^a\mathbf{y}_\rho = (\mathbf{l}^{\epsilon,M}(\rho;\tau),\mathbf{l}^{\epsilon,M}(\rho;\sigma)),
\end{gathered}$$

for

$$\begin{aligned}
K_{ijs}^1 = \frac{1}{\epsilon^{3/2}}\int_t^u\int_t^\tau\int_t^\sigma & \frac{\partial\Theta_M}{\partial q_s}(\mathbf{q}^{\epsilon,M}(\rho))G_s^M(\frac{\rho}{\epsilon},\frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a},\mathbf{q}^{\epsilon,M}(\rho)) \\
& \times \tilde{H}_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a},\frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a})\,\Phi\, d\rho d\sigma d\tau
\end{aligned}$$

we get that

$$\begin{aligned}
|E\{K_{ijs}^1\}| &\le \|\nabla\Theta_M\|_\infty\|F\|_\infty E\{|\Phi|\}\frac{1}{\epsilon^{3/2}}\int_t^u\int_t^\tau\int_t^\sigma \tilde{\phi}^{\frac{1}{2}}(\frac{\tau-\sigma}{\epsilon})\tilde{\phi}^{\frac{1}{2}}(\frac{\sigma-\rho}{\epsilon})d\rho d\sigma d\tau \\
&\le C_1(u-t)\sqrt{\epsilon} \to 0.
\end{aligned}$$

Furthermore, applying the second mixing Lemma A.2 with

$$\begin{gathered}
U(\frac{\tau}{\epsilon},\mathbf{y}) = \frac{\partial F_i}{\partial y_k^1}(\frac{\tau}{\epsilon},\mathbf{y}_1), \quad V(\frac{\sigma}{\epsilon},\mathbf{y}) = F_j(\frac{\sigma}{\epsilon},\mathbf{y}_2), \\
Z(\frac{\rho}{\epsilon}) = (\frac{\tau-\rho}{\epsilon^a})\Theta_M(\mathbf{q}^{\epsilon,M}(\rho))G_k^M(\frac{\rho}{\epsilon},\frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a},\mathbf{q}^{\epsilon,M}(\rho))\Phi \\
\text{and} \quad \epsilon^a\mathbf{y}_\rho = (\mathbf{l}^{\epsilon,M}(\rho;\tau),\mathbf{l}^{\epsilon,M}(\rho;\sigma)),
\end{gathered}$$

for

$$K^2_{ijk} = \frac{1}{\epsilon^{3/2}} \int_t^u \int_t^\tau \int_t^\sigma \Theta_M(\mathbf{q}^{\epsilon,M}(\rho)) G^M_k(\frac{\rho}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\rho)) \\ \times (\frac{\tau-\rho}{\epsilon^a}) \frac{\partial \tilde{H}_{ij}}{\partial y^1_k}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a}) \, \Phi \, d\rho d\sigma d\tau$$

we get that

$$\begin{aligned} |E\{K^2_{ijk}\}| &\leq \|\Theta_M\|_\infty \|F\|_\infty E\{|\Phi|\} \frac{1}{\epsilon^{3/2}} \int_t^u d\tau \int_t^\tau d\sigma \tilde{\phi}^{\frac{1}{2}}(\frac{\tau-\sigma}{\epsilon}) \\ &\quad \times \int_t^\sigma (\frac{\tau-\rho}{\epsilon^a}) \tilde{\phi}^{\frac{1}{2}}(\frac{\sigma-\rho}{\epsilon}) d\rho \\ &\leq C_2(u-t)\epsilon^{3/2-a} \to 0. \end{aligned}$$

Then using

$$U(\frac{\tau}{\epsilon}, \mathbf{y}) = F_i(\frac{\tau}{\epsilon}, \mathbf{y}_1), \quad V(\frac{\sigma}{\epsilon}, \mathbf{y}) = \frac{\partial F_j}{\partial y^2_k}(\frac{\sigma}{\epsilon}, \mathbf{y}_2),$$
$$Z(\frac{\rho}{\epsilon}) = (\frac{\sigma-\rho}{\epsilon^a}) \Theta_M(\mathbf{q}^{\epsilon,M}(\rho)) G^M_k(\frac{\rho}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\rho)) \Phi$$
$$\text{and} \quad \epsilon^a \mathbf{y}_\rho = (\mathbf{l}^{\epsilon,M}(\rho;\tau), \mathbf{l}^{\epsilon,M}(\rho;\sigma)),$$

for

$$K^3_{ijk} = \frac{1}{\epsilon^{3/2}} \int_t^u \int_t^\tau \int_t^\sigma \Theta_M(\mathbf{q}^{\epsilon,M}(\rho)) G^M_k(\frac{\rho}{\epsilon}, \frac{\mathbf{x}^{\epsilon,M}(\rho)}{\epsilon^a}, \mathbf{q}^{\epsilon,M}(\rho)) \\ \times (\frac{\sigma-\rho}{\epsilon^a}) \tilde{H}_{ij,y^2_k}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\tau)}{\epsilon^a}, \frac{\mathbf{l}^{\epsilon,M}(\rho;\sigma)}{\epsilon^a}) \, \Phi \, d\rho d\sigma d\tau,$$

we get that

$$\begin{aligned} |E\{K^3_{ijk}\}| &\leq \|\Theta_M\|_\infty \|F\|_\infty E\{|\Phi|\} \frac{1}{\epsilon^{3/2}} \int_t^u d\tau \int_t^\tau d\sigma \tilde{\phi}^{\frac{1}{2}}(\frac{\tau-\sigma}{\epsilon}) \\ &\quad \times \int_t^\sigma (\frac{\sigma-\rho}{\epsilon^a}) \tilde{\phi}^{\frac{1}{2}}(\frac{\sigma-\rho}{\epsilon}) d\rho \\ &\leq C_3(u-t)\epsilon^{3/2-a} \to 0. \end{aligned}$$

Finally, all these estimates lead to the conclusion that $E\{I^{12}_{ij}\} \to 0$, therefore

$$E\{I^1_{ij}\} \to E^M\{\int_t^u b^{M,1}_{ij}(X(\sigma)) \frac{\partial f}{\partial q_i}(X(\sigma)) \, \Phi \, d\sigma\} \tag{70}$$

as ϵ goes to zero. Furthermore, proceeding in a similar way one gets that

$$E\{I^2_{ij}\} \to E^M\{\int_t^u a^M_{ij}(X(\sigma)) \frac{\partial^2 f}{\partial q_i \partial q_j}(X(\sigma)) \, \Phi \, d\sigma\}$$
$$E\{I^3_{ik}\} \to E^M\{\int_t^u b^{M,2}_{ik}(X(\sigma)) \frac{\partial f}{\partial q_i}(X(\sigma)) \, \Phi \, d\sigma\}$$

as ϵ goes to zero, where

$$b_{ij}^{M,1}(X) = \varphi_M(X)\frac{\partial \varphi_M}{\partial X_j}(X) \times \begin{cases} a_{ij}(\tilde{X}), & \text{if } a = 1 \\ a_{ij}(\mathbf{0}), & \text{if } 0 < a < 1 \end{cases}$$

$$b_{ik}^{M,2}(X) = \varphi_M^2(X) \times \begin{cases} \frac{\partial a_{ik}}{\partial X_k}(\tilde{X}), & \text{if } a = 1 \\ \frac{\partial a_{ik}}{\partial X_k}(\mathbf{0}), & \text{if } 0 < a < 1 \end{cases}$$

$$a_{ij}^{M}(X) = \varphi_M^2(X) \times \begin{cases} a_{ij}(\tilde{X}), & \text{if } a = 1 \\ a_{ij}(\mathbf{0}), & \text{if } 0 < a < 1. \end{cases}$$

Note that

$$\sum_{i,j=1}^{l} a_{ij}^M(X)\frac{\partial^2 f(X)}{\partial X_i \partial X_j} + \sum_{i=1}^{l} b_i^M(X)\frac{\partial f(X)}{\partial X_i} = \mathcal{L}^M f(X)$$

where $\mathcal{L}^M$ is defined by (67) and $b_i^M(X) = \sum_{j=1}^{l} b_{ij}^{M,1}(X) + \sum_{k=1}^{m} b_{ik}^{M,2}(X)$. So, finally by adding all contributions we get that

$$I = E^M\Big\{\int_t^u ds\mathcal{L}^M f(X(s))\Phi(X(s_1),\cdots,X(s_n))\Big\}$$

which together with (69) implies the desired result (68).

A.2.4. *Removal of cutoff and weak convergence.* In this final step we remove the cutoff in M and prove the weak convergence of the measures R^ϵ. The argument is the same as in [25, step (vi), pp. 118–120], so we just highlight the main points.

Since $\mathbf{q}^{\epsilon,M}(s)$ is continuous the measure $R^{\epsilon,M}$ is actually supported in $\mathcal{C}^l$, furthermore the uniform bounds in M of a_{ij}^M and b_i^M imply that R^M is concentrated in $\mathcal{C}^l$ and that the family of measures $\{R^M\}_{M\geq|\mathbf{q}_0|}$ is tight in $\mathcal{C}^l$. Next, if M_k is a sequence for which R^{M_k} converges weakly in $\mathcal{C}^l$ to some measure R^*, as the coefficients of $\mathcal{L}^M$ converge boundedly, and uniformly on compact sets to those of $\mathcal{L}$, and the R^M martingale property (68) is valid, we get that

$$f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds$$

is a $(R^*, {\mathcal{M}^*}_0^t)$ martingale, where ${\mathcal{M}^*}_u^v$ is the trace of $\mathcal{M}_u^v$ on $\mathcal{C}^l$. By the hypothesis there is only one measure with this property, then $R^* = R_{\mathbf{q}_0}$.

Next, let S_0 be a closed subset of $C([0,T];\mathbb{R}^l)$ and $S = \{X \in \mathcal{C}^l : X|_{[0,T]} \in S_0\}$, we can carry out the same calculations presented in [25, pp.119-120] to show that $\limsup_{\epsilon\downarrow 0} R^\epsilon\{S\} \leq R_{\mathbf{q}_0}\{S\}$.

As in [25], this implies that for each bounded continuous functional Ψ on $\mathcal{C}^l$ such that $\Psi(X)$ depends on $X(\cdot)$ restricted to $[0,T]$ one has that

$$\lim_{\epsilon\to 0}\int \Psi(X)R^\epsilon\{dX\} = \int \Psi(X)R_{\mathbf{q}_0}\{dX\}.$$

And by appealing to the same argument presented there, we finally have that R^ϵ weakly converges to $R_{\mathbf{q}_0}$ in $\mathcal{C}^l$.

A.3. **Two-particle problem .** For any $0 < \epsilon \le 1$ define the stochastic process $(\mathbf{x}_1^\epsilon(s), \mathbf{q}_1^\epsilon(s), \mathbf{x}_2^\epsilon(s), \mathbf{q}_2^\epsilon(s))$ with values in $\mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^l$, for $s \ge 0$, as the solution of the system of ODEs

$$\begin{cases} \dfrac{d\mathbf{x}_1^\epsilon}{ds} = \tilde{\mathbf{q}}_1^\epsilon \\ \dfrac{d\mathbf{q}_1^\epsilon}{ds} = \dfrac{1}{\sqrt{\epsilon}} F(\dfrac{s}{\epsilon}, \dfrac{\mathbf{x}_1^\epsilon}{\epsilon^a}) \\ \dfrac{d\mathbf{x}_2^\epsilon}{ds} = \tilde{\mathbf{q}}_2^\epsilon \\ \dfrac{d\mathbf{q}_2^\epsilon}{ds} = \dfrac{1}{\sqrt{\epsilon}} F(\dfrac{s}{\epsilon}, \dfrac{\mathbf{x}_2^\epsilon}{\epsilon^a}) \\ (\mathbf{x}_1^\epsilon(0), \mathbf{q}_1^\epsilon(0), \mathbf{x}_2^\epsilon(0), \mathbf{q}_2^\epsilon(0)) = (\mathbf{x}_{10}, \mathbf{q}_{10}, \mathbf{x}_{20}, \mathbf{q}_{20}) \end{cases} \tag{71}$$

where $(\mathbf{x}_{10}, \mathbf{q}_{10}, \mathbf{x}_{20}, \mathbf{q}_{20})$ is non-random.

Theorem A.2. *Suppose that conditions C.1–C.5 (page 22) are fulfilled. Assume additionally that* $m \ge 2$ *and* $\tilde{\mathbf{q}}_{10} \ne \tilde{\mathbf{q}}_{20}$. *Then the stochastic process* $(\mathbf{x}_1^\epsilon(s), \mathbf{q}_1^\epsilon(s), \mathbf{x}_2^\epsilon(s), \mathbf{q}_2^\epsilon(s))$ *converges weakly in* $\mathcal{C}^{m+l} \times \mathcal{C}^{m+l}$ *as* $\epsilon \to 0$, *to the process* $(\mathbf{x}_1(s), \mathbf{q}_1(s), \mathbf{x}_2(s), \mathbf{q}_2(s))$ *such that*

$$\mathbf{x}_\alpha(s) = \mathbf{x}_{\alpha 0} + \int_0^s \tilde{\mathbf{q}}_\alpha(s') ds', \quad \alpha = 1, 2$$

and $\mathbf{q}_\alpha(s)$, $\alpha = 1, 2$ *are two independent* l*-dimensional diffusions starting from* $\mathbf{q}_{\alpha 0}$, $\alpha = 1, 2$, *respectively. Furthermore, the generators* $\mathcal{L}^a_{\mathbf{q}_\alpha}$, $\alpha = 1, 2$, *corresponding to these diffusion processes are defined by* (55).

The proof of the above theorem follows the same lines as in the previous section, the main different being the fact that we need to keep the trajectories separated, in order to eliminate the mixed derivatives. This difficulty is overcome in the same way as in Theorem 4.4 from [2, page 100].

We shall establish that the probability measure R_2^ϵ induced by the process $(\mathbf{q}_1^\epsilon(\cdot), \mathbf{q}_2^\epsilon(\cdot))$ on $\mathcal{C}^{2l}$ weakly converges to the measure $R_2 = R_{\mathbf{q}_{10}} \otimes R_{\mathbf{q}_{20}}$, where $R_{\mathbf{q}_{\alpha 0}}$ is the probability measure on $\mathcal{C}^l$ associated with the diffusion starting from $\mathbf{q}_{\alpha 0}$ with generator $\mathcal{L}^a_{\mathbf{q}_\alpha}$.

We further consider the case where $\mathbf{x}_{10} = \mathbf{x}_{20} = \mathbf{0}$, remarking that in case this condition is not satisfied we can proceed in a similar way.

A.3.1. *Truncated process and tightness.* For the two-particles problem the truncated process is constructed in order to keep the trajectories bounded and separated.

Let $M > |\mathbf{q}_{j0}|$, $j = 1, 2$ and consider a cutoff function $\varphi_M : \mathbb{R}^l \to [0, 1]$, $\varphi_M \in C^\infty(\mathbb{R}^l)$ such that

$$\varphi_M(\mathbf{q}) = \begin{cases} 0, & |\mathbf{q}| \ge 2M \\ 1, & |\mathbf{q}| \le M. \end{cases} \tag{72}$$

Let $N > 0$ and define $\tilde{Q}_N := \{(\tilde{\mathbf{q}}_1', \tilde{\mathbf{q}}_2') \in \mathbb{R}^{2m} : |\tilde{\mathbf{q}}_j' - \tilde{\mathbf{q}}_{j0}| \le \frac{2}{N+1}, j = 1, 2\}$. Choose N such that $\gamma_N := \inf\{|\tilde{\mathbf{q}}_1' - \tilde{\mathbf{q}}_2'| : (\tilde{\mathbf{q}}_1', \tilde{\mathbf{q}}_2') \in \tilde{Q}_N\} > 0$. We then have that the cones in $\mathbb{R}^{m+1}$ with vertex in the origin and basis $B_\alpha = \{(\mathbf{q}, 1) : |\mathbf{q} - \mathbf{q}_{\alpha 0}| \le \frac{1}{N+1}\}$, $\alpha = 1, 2$ are separated and consequently $\lambda_N > 0$, where

$$\lambda_N := \inf\{|\mathbf{q}_1' - \varrho \mathbf{q}_2'| \wedge |\mathbf{q}_2' - \varrho \mathbf{q}_1'| : \mathbf{q}_\alpha' = (\tilde{\mathbf{q}}_\alpha', 1),\ \alpha = 1, 2, \quad (\tilde{\mathbf{q}}_1', \tilde{\mathbf{q}}_2') \in \tilde{Q}_N\}.$$

Let $p_1 := 2^{q+2}pM$, $t^{(p)} = 1/p$ and $t_k^{(p_1)} = k/p_1$, $k = 1, 2, \ldots$. We introduce the smooth function $\psi_N : \mathbb{R}^m \to [0,1]$, with uniformly in N bounded gradient and such that

$$\psi_N(\tilde{\mathbf{q}}) = \begin{cases} 0, & \text{if } |\tilde{\mathbf{q}}| \geq \frac{2}{N+1} \\ 1, & \text{if } |\tilde{\mathbf{q}}| \leq \frac{1}{N+1} \end{cases} \tag{73}$$

and the cutoff functions $\Psi_\alpha : [0,\infty) \times \mathbb{R}^m \to [0,1]$, $\alpha = 1, 2$:

$$\Psi_\alpha(t, \tilde{\mathbf{q}}) = \begin{cases} 1, & \text{if } t \geq t^{(p)} \\ \psi_N(\tilde{\mathbf{q}} - \tilde{\mathbf{q}}_{\alpha 0}), & \text{if } 0 \leq t < t^{(p)}. \end{cases} \tag{74}$$

Further, we consider the functions $\xi_k : \mathbb{R}^m \times \mathcal{D}^m \to [0,1]$ smooth when the path $\tilde{X}(\cdot) \in \mathcal{D}^m$ is fixed and such that

$$\xi_k(\mathbf{y}; \tilde{X}(\cdot)) = \begin{cases} 1, & \text{if } |\mathbf{y} - \int_0^{t_k^{(p_1)}} \tilde{X}(s)ds| \geq \frac{2}{q} \\ 0, & \text{if } |\mathbf{y} - \int_0^{t_k^{(p_1)}} \tilde{X}(s)ds| \leq \frac{1}{q} \end{cases} \tag{75}$$

and introduce the cutoff function $\Xi : [0,\infty) \times \mathbb{R}^m \times \mathcal{D}^m \to [0,1]$:

$$\Xi(t, \mathbf{y}; \tilde{X}(\cdot)) = \begin{cases} 1, & \text{if } 0 \leq t < t^{(p)} \\ \xi_k(\mathbf{y}; \tilde{X}(\cdot)), & \text{if } t_k^{(p_1)} \leq t < t_{k+1}^{(p_1)} \text{ where } t_k^{(p_1)} \geq t^{(p)}. \end{cases} \tag{76}$$

We finally sum up the effect of all these cutoff functions by defining $\Theta_\alpha : [0,\infty) \times \mathbb{R}^m \times \mathbb{R}^l \times \mathcal{D}^m \to [0,1]$, $\alpha = 1, 2$:

$$\Theta_\alpha(t, \mathbf{y}, \mathbf{q}; \tilde{X}(\cdot)) = \varphi_M(\mathbf{q}) \Psi_\alpha(t, \tilde{\mathbf{q}}) \Xi(t, \mathbf{y}; \tilde{X}(\cdot)). \tag{77}$$

Set

$$G_\alpha^{\epsilon,M,N,p,q}(t, \mathbf{y}, \mathbf{q}; \tilde{X}(\cdot)) = \Theta_\alpha(\epsilon t, \epsilon \mathbf{y}, \mathbf{q}; \tilde{X}(\cdot)) F(t, \mathbf{y})$$

and define the truncated system of ODEs:

$$\begin{cases} \dfrac{d\bar{\mathbf{x}}_1^\epsilon}{ds} = \tilde{\bar{\mathbf{q}}}_1^\epsilon \\ \dfrac{d\bar{\mathbf{q}}_1^\epsilon}{ds} = \dfrac{1}{\sqrt{\epsilon}} G_1^{\epsilon,\tilde{M}}(\dfrac{s}{\epsilon}, \dfrac{\bar{\mathbf{x}}_1^\epsilon}{\epsilon^a}, \bar{\mathbf{q}}_1^\epsilon) \\ \dfrac{d\bar{\mathbf{x}}_2^\epsilon}{ds} = \tilde{\bar{\mathbf{q}}}_2^\epsilon \\ \dfrac{d\bar{\mathbf{q}}_2^\epsilon}{ds} = \dfrac{1}{\sqrt{\epsilon}} G_2^{\epsilon,\tilde{M}}(\dfrac{s}{\epsilon}, \dfrac{\bar{\mathbf{x}}_2^\epsilon}{\epsilon^a}, \bar{\mathbf{q}}_2^\epsilon) \end{cases} \tag{78}$$

with initial conditions $(\mathbf{x}_{10}, \mathbf{q}_{10}, \mathbf{x}_{20}, \mathbf{q}_{20})$, where

$$G_1^{\epsilon,\tilde{M}}(t, \bar{\mathbf{y}}, \mathbf{q}) = G_1^{\epsilon,M,N,p,q}(t, \mathbf{y}, \mathbf{q}; \tilde{\bar{\mathbf{q}}}_2^\epsilon(\cdot)),$$
$$G_2^{\epsilon,\tilde{M}}(t, \bar{\mathbf{y}}, \mathbf{q}) = G_2^{\epsilon,M,N,p,q}(t, \mathbf{y}, \mathbf{q}; \tilde{\bar{\mathbf{q}}}_1^\epsilon(\cdot)),$$

Note the simplifying notation $\tilde{M} = (M, N, p, q)$.

We remark that the main differences between the cutoff function defined in Section A.2 and the function Θ_α defined above are that the latter depends on time t, both variables $\mathbf{y}$, $\mathbf{q}$ and also, parametrically, on a path from $\mathcal{D}^m$; while the former only depends on $\mathbf{q}$. Furthermore, we notice that when $0 \leq t < t^{(p)}$ or $t_k^{(p_1)} \leq t < t_{k+1}^{(p_1)}$ and the path remain fixed Θ_α is independent of t and smooth on the other arguments.

For each fixed $\tilde{M} = (M, N, p, q)$, the tightness of the family of measures $R_2^{\epsilon,\tilde{M}}$, $0 < \epsilon \le 1$, induced by the truncated process $(\bar{\mathbf{q}}_1^\epsilon(s), \bar{\mathbf{q}}_2^\epsilon(s))$ in $\mathcal{D}^{2l} = D([0,\infty); \mathbb{R}^{2l})$ follows after minor changes of the computations performed in Section A.2, since we only need to establish inequality (66) for $0 \le t < u \le t^{(p)}$ or $t_k^{(p_1)} \le t < u \le t_{k+1}^{(p_1)}$ with $t^{(p)} \le t_k^{(p_1)}$.

A.3.2. *Limit identification.* As in the case of one particle, our next step consists in the identification of the limit points of the family of measures $R_2^{\epsilon,M,N,p,q}$, as $\epsilon \to 0$. We represent by $X(t) = (X_1(t), X_2(t))$ the t-coordinate function in $\mathcal{D}^{2l}$ and let

$$Y_\alpha(t) = \int_0^t \tilde{X}_\alpha(s) ds, \quad \alpha = 1, 2.$$

For any $f \in C_0^\infty(\mathbb{R}^{2l})$ define the operator

$$(79) \quad (\mathcal{L}^{\tilde{M}} f)(\mathbf{q}_1, \mathbf{q}_2; \tilde{X}(\cdot)) = \sum_{\alpha=1,2} \Big\{ \frac{1}{2} \sum_{i,j=1}^{l} A_{\alpha ij}^{\tilde{M}}(s, \mathbf{q}_\alpha; \tilde{X}(\cdot)) \frac{\partial^2 f}{\partial q_i^\alpha \partial q_j^\alpha} + \sum_{i=1}^{l} b_{\alpha i}^{\tilde{M}}(s, \mathbf{q}_\alpha; \tilde{X}(\cdot)) \frac{\partial f}{\partial q_j^\alpha} \Big\}$$

with the coefficients defined as follows

$$A_{\alpha ij}^{\tilde{M}}(s, \mathbf{q}; \tilde{X}(\cdot)) = \Lambda_\alpha^2(s, \mathbf{q}; \tilde{X}(\cdot)) \times \begin{cases} A_{ij}(\tilde{\mathbf{q}}), & \text{if } a = 1 \\ A_{ij}(\mathbf{0}), & \text{if } 0 < a < 1, \end{cases}$$

$$b_{\alpha i}^{\tilde{M}}(s, \mathbf{q}; \tilde{X}(\cdot)) = \Lambda_\alpha(s, \mathbf{q}; \tilde{X}(\cdot)) \times \sum_{j=1}^{l} \begin{cases} \frac{\partial}{\partial q_j}\big(\Lambda_\alpha(s, \mathbf{q}; \tilde{X}(\cdot)) a_{ij}(\tilde{\mathbf{q}})\big), & \text{if } a = 1 \\ \frac{\partial}{\partial q_j}\Lambda_\alpha(s, \mathbf{q}; \tilde{X}(\cdot)) a_{ij}(\mathbf{0}) + \Lambda_\alpha(s, \mathbf{q}; \tilde{X}(\cdot)) \frac{\partial a_{ij}}{\partial q_j}(\mathbf{0}), & \text{if } 0 < a < 1 \end{cases}$$

where

$$\Lambda_\alpha(s, \mathbf{q}; \tilde{X}(\cdot)) \Theta_\alpha(s, \mathbf{q}, Y_\alpha(s); \tilde{X}_{\hat{\alpha}}(\cdot)) \quad \text{and} \quad \hat{\alpha} = \begin{cases} 1, & \text{if } \alpha = 2 \\ 2, & \text{if } \alpha = 1. \end{cases}$$

We next establish a martingale property for any limit measure $R_2^{\tilde{M}}$. We prove that for any integer $n > 0$, bounded continuous function $\Phi : (\mathbb{R}^{2l})^n \to \mathbb{R}$ and $s_1 \le \cdots \le s_n \le t < u$

$$(80) \quad E_2^{\tilde{M}}\{(f(X(u)) - f(X(t)))\Phi(X(s_1), \cdots, X(s_n))\} = E_2^{\tilde{M}}\{\int_t^u (\mathcal{L}^{\tilde{M}} f)(X(s); \tilde{X}(\cdot)) \Phi(X(s_1), \cdots, X(s_n))\}$$

where $E_2^{\tilde{M}}\{\cdot\}$ represents expectation with respect to $R_2^{\tilde{M}}$. As has been remarked in [26] it is sufficient to consider the case where $0 \le t < u \le t^{(p)}$ or $t_k^{(p_1)} \le t < u \le t_{k+1}^{(p_1)}$ with $t_k^{(p_1)} \ge t^{(p)}$.

The calculations for obtaining the martingale property are very similar to those performed in Section A.2, therefore we will not present the details.

We proceed to determine the limit

$$
\begin{aligned}
I &= E_2^{\tilde M}\{(f(X(u)) - f(X(t)))\Phi(X(s_1),\cdots,X(s_n))\} \\
&= \lim_{n\to\infty} E\{(f(\bar{\mathbf{q}}^{\epsilon_n}(u)) - f(\bar{\mathbf{q}}^{\epsilon_n}(t)))\Phi(\bar{\mathbf{q}}^{\epsilon_n}(s_1),\cdots,\bar{\mathbf{q}}^{\epsilon_n}(s_n))\},
\end{aligned} \tag{81}
$$

where the sequence $\{\epsilon_n\}$ is such that $R_2^{\epsilon,\tilde M}$ weakly converges to $R_2^{\tilde M}$ and $\epsilon_n \to 0$, as $n \to +\infty$. To simplify notations, we drop the index n and let ϵ go to zero through the sequence $\{\epsilon_n\}$.

We have that $\delta f(\bar{\mathbf{q}}^\epsilon)(u,t) = f(\bar{\mathbf{q}}^\epsilon(u)) - f(\bar{\mathbf{q}}^\epsilon(t))$ can be re-written as

$$
\begin{aligned}
\delta f(\bar{\mathbf{q}}^\epsilon)(u,t) &= \frac{1}{\sqrt{\epsilon}} \sum_{\substack{i=1\\ \alpha=1,2}}^{l} \int_t^u \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\tau)) G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{x}}^\epsilon_\alpha(\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\tau)) d\tau \\
&= \frac{1}{\sqrt{\epsilon}} \sum_{\substack{i=1\\ \alpha=1,2}}^{l} \int_t^u \Big\{ \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(t)) G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(t;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(t)) \\
&+ \int_t^\tau \frac{d}{d\sigma}\big[\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma)) G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma))\big] d\sigma \Big\} d\tau \\
&= \sum_{\substack{i=1\\ \alpha=1,2}}^{l} \int_t^u \Big\{ \frac{1}{\sqrt{\epsilon}} \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(t)) G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(t;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(t)) \\
&+ \frac{1}{\epsilon} \int_t^\tau \Big[\sum_{j=1}^{l} \Big(\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma)) \frac{\partial G^\epsilon_{\alpha i}}{\partial q_j}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) \\
&\times G^\epsilon_{\alpha j}(\frac{\sigma}{\epsilon}, \frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) + \sum_{\beta=1,2} \frac{\partial^2 f}{\partial q_i^\alpha \partial q_j^\beta}(\bar{\mathbf{q}}^\epsilon(\sigma)) \\
&\times G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) G^\epsilon_{\beta j}(\frac{\sigma}{\epsilon}, \frac{\bar{\mathbf{x}}^\epsilon_\beta(\sigma)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\beta(\sigma)) \Big) \\
&+ \frac{\tau-\sigma}{\epsilon^a} \sum_{k=1}^{m} \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) \frac{\partial G^\epsilon_{\alpha i}}{\partial y_k}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) \\
&\times G^\epsilon_{\alpha k}(\frac{\sigma}{\epsilon}, \frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma)) \Big] d\sigma \Big\} d\tau.
\end{aligned}
$$

By multiplying with Φ, we get the decomposition

$$
\delta f(\bar{\mathbf{q}}^\epsilon)(u,t)\Phi = \sum_{\substack{i=1\\ \alpha=1,2}}^{l} [I^0_{\alpha i} + \sum_{j=1}^{l}(I^1_{\alpha ij} + \sum_{\beta=1,2} I^2_{\alpha\beta ij}) + \sum_{k=1}^{m} I^3_{\alpha ik}],
$$

where

$$
\begin{aligned}
I^0_{\alpha i} &= \frac{1}{\sqrt{\epsilon}} \int_t^u \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(t)) G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(t;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(t); \bar{\mathbf{q}}_{\hat\alpha}(\cdot)) \Phi d\tau, \\
I^1_{\alpha ij} &= \frac{1}{\epsilon} \int_t^u \int_t^\tau \frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma)) \frac{\partial G^\epsilon_{\alpha i}}{\partial q_j}(\frac{\tau}{\epsilon}, \frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma); \bar{\mathbf{q}}_{\hat\alpha}(\cdot))) \\
&\times G^\epsilon_{\alpha j}(\frac{\sigma}{\epsilon}, \frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a}, \bar{\mathbf{q}}^\epsilon_\alpha(\sigma); \bar{\mathbf{q}}_{\hat\alpha}(\cdot))\, \Phi d\sigma d\tau,
\end{aligned}
$$

$$I^2_{\alpha\beta ij} = \frac{1}{\epsilon}\int_t^u\int_t^\tau \frac{\partial^2 f}{\partial q_i^\alpha \partial q_j^\beta}(\bar{\mathbf{q}}^\epsilon(\sigma))G^\epsilon_{\alpha i}(\frac{\tau}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}_{\hat\alpha}(\cdot)))$$
$$\times G^\epsilon_{\beta j}(\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{x}}^\epsilon_\beta(\sigma)}{\epsilon^a},\bar{\mathbf{q}}^\epsilon_\beta(\sigma);\bar{\mathbf{q}}_{\hat\beta}(\cdot))\,\Phi d\sigma d\tau,$$
$$I^3_{\alpha ik} = \frac{1}{\epsilon^{1+a}}\int_t^u\int_t^\tau(\tau-\sigma)\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma))\frac{\partial G^\epsilon_{\alpha i}}{\partial y_k}(\frac{\tau}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}_{\hat\alpha}(\cdot))$$
$$\times G^\epsilon_{\alpha k}(\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a},\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}_{\hat\alpha}(\cdot))\,\Phi d\sigma d\tau.$$

By applying mixing Lemma A.1, similarly as in Section A.2, we get

$$|I^0_{\alpha i}| \le \|\nabla f|_\infty\|F\|_\infty\|\Phi\|_\infty\frac{1}{\sqrt{\epsilon}}\int_t^u\tilde\phi(\frac{\tau-t}{\epsilon})d\tau = C_1\sqrt{\epsilon}\to 0.$$

Next, we compute $\lim_{\epsilon\to 0} E\{I^1_{\alpha ij}\}$. By introducing the functions

$$H_{ij}(\tau/\epsilon,\sigma/\epsilon,\mathbf{y}_1,\mathbf{y}_2) = F_i(\tau/\epsilon,\mathbf{y}_1)\,F_j(\sigma/\epsilon,\mathbf{y}_2),$$
$$\bar H_{ij} = E\{H_{ij}\}\quad\text{and}\quad \tilde H_{ij} = H_{ij}-\bar H_{ij},$$

we arrive to the decomposition $I^1_{\alpha ij} = I^{11}_{\alpha ij}+I^{12}_{\alpha ij}$ where

$$I^{11}_{\alpha ij} = \frac{1}{\epsilon}\int_t^u\int_t^\tau\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma))\Theta_\alpha(\sigma,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))$$
$$\times\frac{\partial\Theta_\alpha}{\partial q_j}(\tau,\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))\bar H_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a})\Phi d\sigma d\tau$$
$$I^{12}_{\alpha ij} = \frac{1}{\epsilon}\int_t^u\int_t^\tau\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma))\Theta_\alpha(\sigma,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))$$
$$\times\frac{\partial\Theta_\alpha}{\partial q_j}(\tau,\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))\tilde H_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a})\Phi d\sigma d\tau$$

Moreover, since $|\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)-\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)| = (\tau-\sigma)|\tilde{\bar{\mathbf{q}}}^\epsilon_\alpha(\sigma)|\le 2M(\tau-\sigma)$ we get that

$$|\delta I^{11}_{\alpha ij}| = \big|I^{11}_{\alpha ij}-\frac{1}{\epsilon}\int_t^u\int_t^\tau\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma))\Theta_\alpha(\sigma,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))$$
$$\times\frac{\partial\Theta_\alpha}{\partial q_j}(\tau,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))\bar H_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\frac{\bar{\mathbf{x}}^\epsilon_\alpha(\sigma)}{\epsilon^a})\Phi d\sigma d\tau\big|$$
$$\le\frac{C_2}{\epsilon}\int_t^u\int_t^\tau(\tau-\sigma)|R_{ij}(\frac{\tau-\sigma}{\epsilon},\frac{\tau-\sigma}{\epsilon^a}\tilde{\bar{\mathbf{q}}}^\epsilon_\alpha(\sigma))|d\sigma d\tau$$
$$\le C_2\epsilon\int_t^u\int_0^{\frac{u-\sigma}{\epsilon}}s|R_{ij}(s,s\epsilon^{1-a}\tilde{\bar{\mathbf{q}}}^\epsilon_\alpha(\sigma))|dsd\sigma\to 0.$$

Therefore,

$$I^{11}_{\alpha ij} = \int_t^u\frac{\partial f}{\partial q_i^\alpha}(\bar{\mathbf{q}}^\epsilon(\sigma))\Theta_\alpha(\sigma,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))\frac{\partial\Theta_\alpha}{\partial q_j}(\sigma,\bar{\mathbf{x}}^\epsilon_\alpha(\sigma),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot))$$
$$\times\big(\int_0^{\frac{u-\sigma}{\epsilon}}R_{ij}(s,s\epsilon^{1-a}\tilde{\bar{\mathbf{q}}}^\epsilon_\alpha(\sigma))ds\big)\Phi d\sigma+O(\epsilon)$$

uniformly in Ω, and consequently we get that

$$E\{I^{11}_{\alpha ij}\}\to E_2^{\tilde M}\{\int_t^u b^{\tilde M,1}_{\alpha ij}(\sigma,X_\alpha(\sigma);\tilde X(\cdot))\frac{\partial f}{\partial q_i^\alpha}(X(\sigma))\,\Phi d\sigma\}$$

where

$$b^{\tilde M,1}_{\alpha ij}(\sigma,\mathbf{q};\tilde X(\cdot)) = \Theta_\alpha(\sigma,\mathbf{q},Y_\alpha(\sigma);\tilde X_{\hat\alpha}(\cdot))\frac{\partial\Theta_\alpha}{\partial q_j}(\sigma,\mathbf{q},Y_\alpha(\sigma);\tilde X_{\hat\alpha}(\cdot)) \times \begin{cases}\int_0^\infty R_{ij}(s,s\tilde{\mathbf{q}})ds, & \text{if } a=1\\ \int_0^\infty R_{ij}(s,\mathbf{0})ds, & \text{if } 0<a<1.\end{cases}$$

Proceeding in a similar way as in Section A.2 we can prove that $E\{I^{12}_{\alpha ij}\}\to 0$ and finally, get that

$$E\{I^1_{\alpha ij}\}\to E_2^{\tilde M}\{\int_t^u b^{\tilde M,1}_{\alpha ij}(\sigma,X_\alpha(\sigma);\tilde X(\cdot))\frac{\partial f}{\partial q_i^\alpha}(X(\sigma))\,\Phi\,d\sigma\}.$$

Furthermore, a straightforward modification of the corresponding calculations of Section A.2 allows us to show that

$$E\{I^2_{\alpha\alpha ij}\}\to E_2^{\tilde M}\{\int_t^u a^{\tilde M}_{\alpha ij}(\sigma,X_\alpha(\sigma);\tilde X(\cdot))\frac{\partial f}{\partial q_i^\alpha\partial q_j^\alpha}(X(\sigma))\,\Phi\,d\sigma\}$$
$$E\{I^3_{\alpha ik}\}\to E_2^{\tilde M}\{\int_t^u b^{\tilde M,2}_{\alpha ik}(\sigma,X_\alpha(\sigma);\tilde X(\cdot))\frac{\partial f}{\partial q_i^\alpha}(X(\sigma))\,\Phi\,d\sigma\}$$

as ϵ goes to zero, where

$$a^{\tilde M}_{\alpha ij}(\sigma,X_\alpha(\sigma);\tilde X(\cdot)) = \Theta^2_\alpha(\sigma,\mathbf{q},Y_\alpha(\sigma);\tilde X_{\hat\alpha}(\cdot))\times\begin{cases}\int_0^\infty R_{ij}(s,s\tilde{\mathbf{q}})ds, & \text{if } a=1,\\ \int_0^\infty R_{ij}(s,\mathbf{0})ds, & \text{if } 0<a<1,\end{cases}$$
$$b^{\tilde M,2}_{\alpha ik}(\sigma,\mathbf{q};\tilde X(\cdot)) = \Theta^2_\alpha(\sigma,\mathbf{q},Y_\alpha(\sigma);\tilde X_{\hat\alpha}(\cdot))\times\begin{cases}\int_0^\infty s\frac{\partial R_{ik}}{\partial y_k}(s,s\tilde{\mathbf{q}})ds, & \text{if } a=1,\\ \int_0^\infty s\frac{\partial R_{ik}}{\partial y_k}(s,\mathbf{0})ds, & \text{if } 0<a<1.\end{cases}$$

Next, we obtain that for $\alpha\neq\beta$, $E\{I^2_{\alpha\beta ij}\}\to 0$. It is enough to prove that in this case $E\{I^{21}_{\alpha\beta ij}\}\to 0$ where

$$I^{21}_{\alpha\beta ij} = \frac{1}{\epsilon}\int_t^u\int_t^\tau \frac{\partial^2 f}{\partial q_i^\alpha\partial q_j^\beta}(\bar{\mathbf{q}}^\epsilon(\sigma))\Theta_\alpha(\tau,\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau),\bar{\mathbf{q}}^\epsilon_\alpha(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\alpha}(\cdot)) \times\Theta_\beta(\sigma,\bar{\mathbf{x}}^\epsilon_\beta(\sigma),\bar{\mathbf{q}}^\epsilon_\beta(\sigma);\bar{\mathbf{q}}^\epsilon_{\hat\beta}(\cdot))\bar H_{ij}(\frac{\tau}{\epsilon},\frac{\sigma}{\epsilon},\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\epsilon^a},\frac{\bar{\mathbf{x}}^\epsilon_\beta(\sigma)}{\epsilon^a})\Phi d\sigma d\tau. \tag{82}$$

First consider that $0\le\sigma\le\tau<t^{(p)}$. Since for $0\le s<t^{(p)}$ we have that $|\tilde{\bar{\mathbf{q}}}^\epsilon_\gamma(s)-\tilde{\bar{\mathbf{q}}}_{\gamma 0}|\le\frac{2}{N+1}$, $\gamma=1,2$, it follows that $(\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)}{\tau},\frac{\bar{\mathbf{x}}^\epsilon_\beta(\sigma)}{\sigma})\in\tilde Q_N$ and then $\sqrt{(\tau-\sigma)^2+|\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)-\bar{\mathbf{x}}^\epsilon_\beta(\sigma)|^2}\ge\lambda_N\tau$. Consequently, we have the estimate

$$\sqrt{(\frac{\tau-\sigma}{\epsilon})^2+|\frac{\bar{\mathbf{l}}^\epsilon_\alpha(\sigma;\tau)-\bar{\mathbf{x}}^\epsilon_\beta(\sigma)}{\epsilon^a}|^2}\ge\frac{\lambda_N\tau}{\epsilon^a}.$$

Finally, by considering the condition (53) we get

$$|E\{I^{21}_{\alpha\beta ij}\}|\le\frac{C_2}{\epsilon}\int_t^u\int_t^\tau e^{-C_1\frac{\lambda_N\tau}{\epsilon^a}}d\sigma d\tau = C_2\frac{e^{-C_1\frac{\lambda_N t}{\epsilon^a}}}{\epsilon^{1-2a}}\int_0^{\frac{u-t}{\epsilon}} se^{-C_1\lambda_N s}ds\to 0.$$

On the other hand, when $t_k^{(p_1)} \leq \sigma \leq \tau < t_{k+1}^{(p_1)}$ with $t_k^{(p_1)} \geq t^{(p)}$ we have the following estimate

$$\begin{aligned} &\big|\Theta_\alpha(\tau, \bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau), \bar{\mathbf{q}}_\alpha^\epsilon(\sigma); \bar{\mathbf{q}}_{\hat{\alpha}}^\epsilon(\cdot))\Theta_\beta(\sigma, \bar{\mathbf{x}}_\beta^\epsilon(\sigma), \bar{\mathbf{q}}_\beta^\epsilon(\sigma); \bar{\mathbf{q}}_{\hat{\beta}}^\epsilon(\cdot)) \\ &\qquad \times \bar{H}_{ij}(\frac{\tau}{\epsilon}, \frac{\sigma}{\epsilon}, \frac{\bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau)}{\epsilon^a}, \frac{\bar{\mathbf{x}}_\beta^\epsilon(\sigma)}{\epsilon^a})\big| \leq C_3 e^{-C_1\sqrt{(\frac{\tau-\sigma}{\epsilon})^2 + \frac{1}{4\epsilon^{2a}q^2}}}. \end{aligned} \tag{83}$$

Indeed, either $\Theta_\alpha(\tau, \bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau), \bar{\mathbf{q}}_\alpha^\epsilon(\sigma); \bar{\mathbf{q}}_{\hat{\alpha}}^\epsilon(\cdot)) = 0$, and then the estimate holds, or $\xi_k(\bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau); \bar{\mathbf{q}}_\beta^\epsilon(\cdot)) \neq 0$. In the last case, we have that

$$|\bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau) - \bar{\mathbf{x}}_\beta^\epsilon(t_k^{(p_1)})| \geq \frac{1}{q}.$$

Furthermore, $|\bar{\mathbf{q}}_\beta^\epsilon(s)| \leq 2M$ implies that

$$|\bar{\mathbf{x}}_\beta^\epsilon(\sigma) - \bar{\mathbf{x}}_\beta^\epsilon(t_k^{(p_1)})| \leq 2M(\sigma - t_k^{(p_1)}) \leq \frac{2M}{p_1}$$

and consequently one has

$$|\bar{\mathbf{l}}_\alpha^\epsilon(\sigma;\tau) - \bar{\mathbf{x}}_\beta^\epsilon(\sigma)| \geq \frac{1}{q} - \frac{2M}{p_1} \geq \frac{1}{2q}$$

and by considering condition (53) the estimate (83) follows. Furthermore, from that estimate we obtain that

$$\begin{aligned} |E\{I_{\alpha\beta ij}^{21}\}| &\leq \frac{C_3'}{\epsilon}\int_t^u \int_t^\tau e^{-C_1\sqrt{(\frac{\tau-\sigma}{\epsilon})^2 + \frac{1}{4\epsilon^{2a}q^2}}} d\sigma d\tau \\ &\leq C_3' e^{-\frac{C_1'}{\epsilon^a}} \int_t^u d\sigma \int_0^{\frac{u-\sigma}{\epsilon}} ds e^{-C_1'' s} \to 0 \end{aligned} \tag{84}$$

where $C_1' = \frac{C_1}{2\sqrt{2}q}$ and $C_1'' = \frac{C_1}{\sqrt{2}}$.

Finally, by adding all contributions we get that

$$I = E_2^{\tilde{M}}\Big\{\int_t^u ds \mathcal{L}^{\tilde{M}} f(X(s))\Phi(X(s_1), \cdots, X(s_n))\Big\}$$

where $\mathcal{L}^{\tilde{M}}$ is given by (79) which together with (81) imply the desired result (80).

A.3.3. *Removal of cutoff and weak convergence.* In this final step, we shall remove the cutoff and establish the weak convergence of $(\mathbf{q}_1^\epsilon(\cdot), \mathbf{q}_2^\epsilon(\cdot))$. From now on, we consider that all measures are supported in the corresponding space of continuous functions and that the convergence also take place in this space. This property can be established in the same manner as we did in Section A.2.

Next, for proving weak convergence we will follow the same strategy as presented in [2, 26]. We first define a stopping time $U(\cdot; M, N, p, q)$ with the property that the dynamics of the truncated and original systems coincide up to this time. Moreover, this property allows us to identify any limit measure $R_2^{\tilde{M}}$ with R_2 on the σ-algebra corresponding to the stopping time. Finally, by choosing sufficiently large M, N we get that $U(\cdot; M, N, p, q)$ converges to infinity as $q \to \infty$ and $p \to \infty$; from which the weak convergence follows by the same calculations as in [2].

Let $X(\cdot) = (X_1(\cdot), X_2(\cdot)) \in \mathcal{C}^{2l}$ and $Y_\alpha(t) = \int_0^t \tilde{X}_\alpha(s)ds$. For such $X(\cdot)$ we define the following stopping times

$$S(N,p) := \lim_{n\to\infty} S_n(N,p)$$
$$T(M) := \lim_{n\to\infty} T_n(M)$$
$$V(p,q) := \lim_{n\to\infty} V_n(p,q)$$

where

$$S_n(N,p) = \inf\{t : 0 \le t < t^{(p)}, |\tilde{X}_\alpha(t) - \tilde{\mathbf{q}}_{\alpha 0}| > \frac{1}{N+1} - \frac{1}{n} \quad \text{for } \alpha = 1 \text{ or } 2\},$$
$$T_n(M) = \inf\{t \ge 0 : |X_\alpha(t)| > M - \frac{1}{n} \quad \text{for } \alpha = 1 \text{ or } 2\},$$
$$V_n(p,q) = \inf\{t \ge t^{(p)}, |Y_1(t) - Y_2(t)| < \frac{9}{4q} + \frac{1}{n}\}.$$

We also adopt the convention that if the corresponding set of times is empty then the stopping time is infinite.

Let $T_0 > 0$ and set

$$U(N,M,p,q) := \{S(N,p) \wedge T(M) \wedge V(p,q)\}$$

and

$$B(N,M,p,q) := \{S(N,p) \wedge V(p,q) \le T(M) \wedge T_0\},$$

note that $B \in \mathcal{M}_0^U$ (the σ-algebra generated by the stopping time U). Moreover, we have that

$$T_0 < U((\bar{\mathbf{q}}_1^\epsilon(\cdot), \bar{\mathbf{q}}_2^\epsilon(\cdot)); N, M, p, q)$$

implies that $(\bar{\mathbf{q}}_1^\epsilon(s), \bar{\mathbf{q}}_2^\epsilon(s)) = (\mathbf{q}_1^\epsilon(s), \mathbf{q}_2^\epsilon(s))$ for $s \in [0, T_0]$.

The following results will be useful for establishing the weak convergence. We begin with two simple lemmas:

Lemma A.3. *We have that* $\lim_{M\to\infty} T(M) = +\infty$*, a.s.* R_2*.*

Lemma A.4. *We have that* $\lim_{p\to\infty} S(N,p) = +\infty$*,* $\forall N$ *a.s.* R_2*.*

These lemmas are direct consequences of the continuity of the paths of the limiting diffusion process.

The next lemma shows that $S(N,p) \wedge V(p,q)$ becomes infinity as $q \to +\infty, p \to +\infty$ (in this order).

Lemma A.5. *For* N *sufficiently large and* $T_1, \eta > 0$ *arbitrary, one can find* p_0 *such that* $R_2\{S(\cdot; N,p) \wedge V(\cdot; p,q) \le T_1\} \le \eta$ *for any* $p \ge p_0$ *and* $q \ge q_0(p)$*.*

Proof. Thanks to Lemma A.4, we can take p_0 such that $R_2\{S(\cdot; N,p) \ne +\infty\} \le \frac{\eta}{3}$ for any $p \ge p_0$. Moreover, because of the continuity of the paths we also can take $M_1 = M_1(T_1, \eta)$ such that $R_2\{\sup_{0\le t\le T_1} |X_\alpha(t)| > M_1, \quad \text{for } \alpha = 1 \text{ or } 2\} \le \frac{\eta}{3}$. Now consider the event

$$B_1 = \{S(\cdot; N,p) = +\infty \quad \text{and} \quad \sup_{0\le t\le T_1} |X_\alpha(t)| \le M_1, \quad \alpha = 1, 2\}.$$

We will establish that for any fixed $p \ge p_0$,

$$\lim_{q\to\infty} R_2\{V(\cdot; p,q) \le T_1; B_1\} = 0, \tag{85}$$

and from this relation the lemma immediatly follows. □

The proof of equation (85) is a consequence of the following result. Define the diffusion process $\Gamma(\cdot) = (Z(\cdot), X(\cdot)) \in \mathbb{R}^{m+2l}$ where $Z(t) = \int_0^t \big(\tilde{X}_1(s) - \tilde{X}_2(s)\big) ds$ and $X(\cdot)$ is a diffusion associated with R. The process $\Gamma(\cdot)$ has the infinitesimal generator $\mathcal{L}_\Gamma = \mathcal{L}_{X_1} + \mathcal{L}_{X_2} + (\tilde{X}_1 - \tilde{X}_2) \cdot \nabla_Z$.

Lemma A.6. *If condition C.5 is valid (i.e. the coefficients $A_{ij}, b_i \in C^\infty(\mathbb{R}^m)$, the sub-matrix $\tilde{C}(\tilde{\mathbf{q}})$ is non-singular) and $m \geq 2$, then the process $\Gamma(\cdot)$ is a hypoelliptic diffusion.*

Proof. We need to establish that the differential operator $-\partial_t + \mathcal{L}_\Gamma^*$ is hypoelliptic. The proof of this fact follows the same line of the corresponding result in [26, pp. 59–60]. □

Now we are ready to prove equation (85). The hypoellipticity implies the following estimate for the probability distribution of $\Gamma(\cdot)$. For every $0 < t_0 \leq T$, $\lambda > 0$ there exists $K = K(t_0, T, \lambda) < \infty$ such that

$$\mathbb{P}\{|Z(s+t) - Z_0| \leq \varrho \,|\, X(s) = X_0, Z(s) = Z_1\} \leq K \varrho^m, \tag{86}$$

where $\varrho \leq 1$, $0 \leq s \leq T$, $t_0 \leq t \leq T$ and $|X_0|, |Z_0|, |Z_1| \leq \lambda$. Therefore we have that

$$\begin{aligned} R_2\{V(\cdot; p, q) \leq T_1; B_1\} &\leq R_2\{\inf_{t^{(p)} \leq t \leq T_1} |Z(t)| \leq \frac{9}{4q}; B_1\} \\ &\leq R_2\{\inf_{t^{(p)} \leq j\delta \leq T_1} |Z(j\delta)| \leq \frac{9}{2q}; B_1\} \\ &\leq (\frac{T_1 + 1}{\delta}) K' \frac{1}{q^m} = K'' \frac{1}{q^{m-1}} \end{aligned}$$

where $\delta = \frac{9}{8M_1 q}$ and K'' is a constant that depends on T_1, M_1 and p. Thus $R_2\{V(\cdot; p, q) \leq T_1; B_1\} \to 0$ as $q \to \infty$ when $m \geq 2$. This concludes the proof of equation (85) and the Lemma A.5.

A straightforward and useful consequence of the lemmas above is given by the next corollary.

Corollary A.7. *For any $\eta > 0$, there exist sufficiently large M, N, p and q such that*

$$R_2\{B(\cdot; M, N, p, q)\} \leq \eta.$$

Now with the aid of the lemmas above we can establish as in [2] that for any $T_0 > 0$ and a continuous and bounded functional H on $\mathcal{C}^{2l}$ that is $\mathcal{M}_0^{T_0}$-measurable, we have that

$$\limsup_{\epsilon \to 0} E\{H(\mathbf{q}_1^\epsilon(\cdot), \mathbf{q}_2^\epsilon(\cdot))\} \leq \int H(X) R_2\{dX\}.$$

The proof of this fact is just a repetition of the calculations in [2, pp. 126–127]. Finally, as in the cited reference, using the same argument presented in [25] (see Section A.2) the proof of the weak convergence of R_2^ϵ to $R_2 = R_{\mathbf{q}_{10}} \otimes R_{\mathbf{q}_{20}}$ is completed.

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