Superresolution and Duality for Time-Reversal of Waves in Self-Similar Media

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We analyze the time reversal of waves in a turbulent medium using the parabolic Markovian model. We prove that the time reversal resolution can be a nonlinear function of the wavelength and independent of the aperture. We establish a duality relation between the turbulence-induced wave spread and the time-reversal resolution which can be viewed as an uncertainty inequality for random media. The inequality becomes an equality when the wave structure function is Gaussian.

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Introduction

Time reversal is the process of recording the signal from a remote source, time-reversing and back-propagating it to retrofocus around the source. Time reversal of acoustic waves has been demonstrated to hold exciting technological potentials in subwavelength focusing, dispersion compensation, communications, imaging, remote-sensing and target detection in unknown environments (see [1] and references therein). The same should hold for the electromagnetic waves as well. Time reversal of electromagnetic waves is closely related to optical phase conjugation (OPC) which used to be limited to monochromatic waves. With the advent of experimental techniques, time reversal of high frequency EM waves hold diverse potential applications including real-time adaptive optics, laser resonators, high-power laser systems, optical communication and information processing, image transmission, spatial and temporal filtering, spectroscopy etc [1].

Time reversal refocusing is the result of the time-reversal invariance of the wave equations, acoustic or electromagnetic, in time invariant media. The surprising and important fact is that the refocal spot in a richly scattering medium is typically *smaller* than that in the homogeneous medium. That is, the time reversal resolution is enhanced rather than hampered by the inhomogeneities of the medium. This sub-diffraction-limit retrofocusing is sometimes called *superresolution* and in certain regimes has been explained mathematically by using radiative transfer equations.

In the previous experimental, numerical or theoretical results the superresolution comes as a *linear* function of the wavelength but *independent* of the aperture. In this letter we show that in fractal media the resolution can be a *superlinear* (between linear and quadratic) function of the wavelength and at the same time independent of the aperture. The lowest achievable refocal spot size in this nonlinear regime is on the order of the smallest scale of the medium fluctuations. Below the inner scale the resolution is diffraction-limited while above the outer scale it is the previously reported aperture-independent enhanced resolution [1, 2].

We will focus our analysis on the widely used *parabolic Markovian* model for waves in atmospheric turbulence [6]. Neglecting the depolarization effect let us write the forward propagating wave field E at the carrier wave number k as $E(z,\mathbf{x}) = \Psi(z,\mathbf{x})e^{i(kz-\omega t)}, \mathbf{x}\in\mathbb{R}^2$ where the complex wave amplitude Ψ satisfies the Schröndinger equation in the non-dimensionalized form

$$i2k\frac{\partial\Psi}{\partial z}+\gamma\Delta_\perp\Psi+\frac{k^2}{\gamma}V(z,\mathbf{x})\circ\Psi=0 \quad (1)$$

with $\Delta_\perp$ being the Laplacian in the transverse coordinates $\mathbf{x}\in\mathbb{R}^2$ and V the fluctuation of the refractive index. Here the Fresnel number γ equals $L_z k_0^{-1} L_x^{-2}$ with k_0 being the reference wavenumber, L_z and L_x the reference scales in the longitudinal and transverse directions, respectively. The notation $\circ$ in eq. (1) means the Stratonovich product (v.s. Itô product). In the Markovian model $V(z,\cdot)$ is assumed to be a δ-correlated-in-z stationary random field such that

$$\langle V(z,\mathbf{x})V(z',\mathbf{x}')\rangle=\delta(z-z')\int\Phi(0,\mathbf{p})e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}d\mathbf{p}$$

where $\Phi(\vec{\mathbf{k}}), \vec{\mathbf{k}}=(\xi,\mathbf{p})\in\mathbb{R}^3$ is the power spectrum density of the refractive index fluctuation and, in the case of atmospheric turbulence, has a power-law behavior in the inertial range. For simplicity of presentation we assume an isotropic power-law

$$\Phi(\vec{\mathbf{k}})=\sigma_H|\vec{\mathbf{k}}|^{-1-2H}|\vec{\mathbf{k}}|^{-2}, \quad |\vec{\mathbf{k}}|\in(L_0^{-1},\ell_0^{-1}) \quad (2)$$

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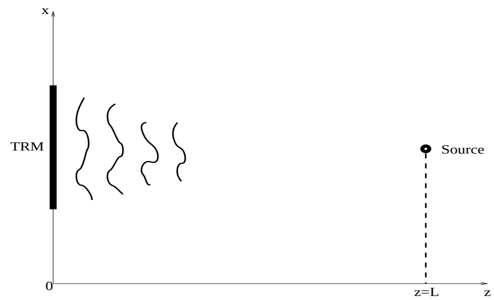

FIG. 1: The time reversal process

where L_0 and ℓ_0 are respectively the outer and inner scales of the turbulence and σ_H a constant factor. Usually H is taken to be 1/3 in the self-similar theory of turbulence. We assume that the spectrum decays sufficiently fast for $|\vec{\mathbf{k}}| \gg \ell_0^{-1}$ while staying bounded for $|\vec{\mathbf{k}}| \ll L_0^{-1}$.

For this model of propagation we will prove an uncertainty inequality where the conjugate quantities are the forward wave spread and the time-reversal resolution. The inequality becomes an *equality* when the wave structure function is Gaussian. This also helps illustrating an experimentally observed, close relation between the time reversal resolution and the correlation length of the scattered wave field prior to time reversal [1].

Time reversal process

In the time reversal procedure, a source $\Psi_0(\mathbf{x})$ located at $z = L$ emits a signal with the carrier wavenumber k toward the time reversal mirror (TRM) of aperture A located at $z = 0$ through a turbulent medium. The transmitted field is captured and time reversed at the TRM and then sent back toward the source point through the same turbulent medium, see Figure 1.

The time-reversed, back-propagated wave field at $z = L$ can be expressed as

$$\Psi_{\rm tr}(\mathbf{x}) \tag{3}$$
$$= \int G(L,\mathbf{x},\mathbf{x}_m)\overline{G(L,\mathbf{x}_s,\mathbf{x}_m)\Psi_0(\mathbf{x}_s)}\mathbb{I}_A(\mathbf{x}_m)d\mathbf{x}_m d\mathbf{x}_s$$
$$= \int e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)/\gamma} W(L,\frac{\mathbf{x}+\mathbf{x}_s}{2},\mathbf{p})\overline{\Psi_0(\mathbf{x}_s)}d\mathbf{p}d\mathbf{x}_s \tag{4}$$

where $\mathbb{I}_A$ is the indicator function of the TRM, G the propagator of the Schrödinger equation and W the Wigner distribution function

$$W(z,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}G(z,\mathbf{x}+\gamma\mathbf{y}/2,\mathbf{x}_m) \times\overline{G(z,\mathbf{x}-\gamma\mathbf{y}/2,\mathbf{x}_m)}\mathbb{I}_A(\mathbf{x}_m)d\mathbf{y}d\mathbf{x}_m.$$

Here we have used the fact that time reversing of the signal is equivalent to the phase conjugating of its spatial component.

The Wigner distribution W satisfies a closed form equation, the Wigner-Moyal equation [4], and for the Markovian model its moments also satisfy closed form equations. In particular, the mean field equation is

$$\frac{\partial\langle W\rangle}{\partial z} + \frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}\langle W\rangle = \mathcal{Q}\langle W\rangle \tag{5}$$

with the scattering operator $\mathcal{Q}$ given by

$$\mathcal{Q}f(\mathbf{x},\mathbf{p}) = \frac{k^2}{4\gamma^2}\int\Phi(0,\mathbf{q})\left[-2f(\mathbf{x},\mathbf{p}) + f(\mathbf{p}+\gamma\mathbf{q}) + f(\mathbf{x},\mathbf{p}-\gamma\mathbf{q})\right]d\mathbf{q}. \tag{6}$$

Eq. (5) is exactly solvable and the mean refocused field of the parabolic Markovian model can be expressed as

$$\langle\Psi_{\rm tr}\rangle(\mathbf{x}) \tag{7}$$
$$= \frac{1}{(2\pi)^2}\int d\mathbf{x}'d\mathbf{q}d\mathbf{w}\overline{\hat{\Psi}_0(\mathbf{q})}\mathbb{I}_A(\mathbf{x}')\exp\left[i\mathbf{q}\cdot\mathbf{x}\right] \times\exp\left[i(\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')-\gamma L\mathbf{w}\cdot\mathbf{q}/k-\gamma L|\mathbf{w}|^2/2k)\right] \times\exp\left[-k^2/(2\gamma^2)\int_0^L D_*(-s\gamma\mathbf{w}/k)ds\right]$$

where the structure function D_* is given by

$$D_*(\mathbf{x}) = \int\Phi(0,\mathbf{q})\left[1-e^{i\mathbf{x}\cdot\mathbf{q}}\right]d\mathbf{q}. \tag{8}$$

Here and below $\hat{f}$ denotes $\mathcal{F}f$ the Fourier transform of f. The main property of D_* we need in the subsequent analysis is the inertial range asymptotic:

$$D_*(r) \approx C_*^2 r^{2H_*}, \quad \ell_0 \ll r \ll L_0, \tag{9}$$

where the effective Hölder exponent H_* is given by

$$H_* = \begin{cases} H+1/2 & \text{for } H\in(0,1/2) \\ 1 & \text{for } H\in(1/2,1] \end{cases} \tag{10}$$

and the structure parameter C_* is proportional to σ_H. Outside of the inertial range we have instead $D_*(r) \sim r^2, r \ll \ell_0$ and $D_*(r) \to D_*(\infty)$ for $r \to \infty$ where $D_*(\infty) > 0$ is a constant.

Let us consider a point source located at $(L,\mathbf{x}_0)$ by substituting the Dirac-delta function $\delta(\mathbf{x}-\mathbf{x}_0)$ for Ψ_0 in (7). We then obtain the point-spread function for time reversal $\mathcal{P}_{tr} = \mathcal{P}_0 T_{tr}$ with

$$\mathcal{P}_0(\mathbf{x}-\mathbf{x}_0) \equiv \left(\frac{k}{\gamma L}\right)^2 e^{i\frac{k}{2\gamma L}(|\mathbf{x}|^2-|\mathbf{x}_0|^2)}\hat{\mathbb{I}}_A\left(\frac{k}{\gamma L}(\mathbf{x}-\mathbf{x}_0)\right)$$
$$T_{tr}(\mathbf{x}-\mathbf{x}_0) \equiv \exp\left[-k^2/(2\gamma^2)L\int_0^1 D_*(-s(\mathbf{x}-\mathbf{x}_0))ds\right]. \tag{11}$$

In the absence of random inhomogeneity the function $T_{\rm tr}$ is unity and the resolution scale ρ_0 is determined solely by $\mathcal{P}_0$:

$$\rho_0 \sim \gamma\frac{\lambda L}{A}, \quad \lambda=\frac{2\pi}{k}. \tag{12}$$

This is the classical (Rayleigh) resolution formula where the retrofocal spot size is proportional to λ and the distance to the TRM, and inversely proportional to the aperture A.

Anomalous Focal Spot-size

First we consider the situation where there may be an inertial range behavior. This requires

$$k^2\gamma^{-2}D_*(\infty)L \gg 1 \tag{13}$$

where $D_*(\infty)=\lim_{r\to\infty}D_*(r)$.

In the presence of random inhomogeneities the retrofocal spot size is determined by $\mathcal{P}_0$ or $T_{\rm tr}$ depending on which has a smaller support. For the power-law spectrum we have the inertial range asymptotic

$$T_{\rm tr}(\mathbf{x}) \sim \exp\left[-C_*^2k^2L|\mathbf{x}-\mathbf{x}_0|^{2H_*}\gamma^{-2}(4H_*+2)^{-1}\right]$$

for $\ell_0 \ll |\mathbf{x}-\mathbf{x}_0| \ll L_0$. Under the following condition

$$(C_*k\gamma^{-1}\sqrt{L})^{1/H_*} \gg k\gamma^{-1}L^{-1}A \sim \rho_0^{-1} \tag{14}$$

the function $T_{\rm tr}$ is much more sharply localized around $\mathbf{x}_0$ than $\mathcal{P}_0$. We define the turbulence-induced time-reversal resolution as

$$\rho_{\rm tr}=\sqrt{\int|\mathbf{x}-\mathbf{x}_0|^2T^2_{\rm tr}(\mathbf{x}-\mathbf{x}_0)d\mathbf{x}/\int T^2_{\rm tr}(\mathbf{x})d\mathbf{x}}$$

which then has the inertial range asymptotic

$$\rho_{\rm tr}\sim\left(\frac{\gamma\lambda}{C_*\sqrt{L}}\right)^{1/H_*}, \quad \ell_0\ll\rho_{\rm tr}\ll L_0. \tag{15}$$

The nonlinear law (15) is valid only down to the inner scale ℓ_0 below which the linear law prevails $\rho_{\rm tr}\sim\gamma\lambda L^{-1/2}$. We see that under (13)-(14) $\rho_{\rm tr}$ is independent of the aperture, has a superlinear dependence on the wavelength in the inertial range. Moreover, the resolution is enhanced as the distance L and random inhomogeneities (C_*) increase. This effect can be explained by the notion of *turbulence-induced aperture* which enlarges as L and C_* increase because the TRM is now able to capture signals initially propagating in the more oblique directions (see more on this below).

To recover the linear law previously reported in [2], let us consider the situation where $\rho_{\rm tr}=O(\gamma)$ and take the limit of vanishing Fresnel number $\gamma\to 0$ in eq. (8) by setting $\mathbf{x}=\gamma\mathbf{y}$. Then we have

$$\begin{aligned}\lim_{\gamma\to0}D_*(\gamma\mathbf{y}) &= D_0|\mathbf{y}|^2 \\ D_0 &= \frac{1}{2}\int\Phi(0,\mathbf{q})|\mathbf{q}|^2d\mathbf{q}.\end{aligned} \tag{16}$$

The resulting mean retrofocused field $\langle\Psi_{\rm tr}(\mathbf{y})\rangle$ is Gaussian in the offset variable $\mathbf{y}$ and the refocal spot size on the original scale is given by

$$\rho_{\rm tr}\sim\gamma\lambda(D_0L)^{-1/2}.$$

Hence the linear law prevails in the sub-inertial range.

Turbulence-induced aperture and duality

Intuitively speaking, the turbulence-induced aperture is closely related to how a wave is spread in the course of propagation through the turbulent medium. A quantitative estimation can be given by analyzing the spread of wave energy.

Let us calculate the mean energy density in $\left\langle|\Psi(z,\mathbf{x})|^2\right\rangle$ with the Gaussian initial wave amplitude

$$\Psi(0,\mathbf{x})=\exp\left[-|\mathbf{x}|^2/(2\alpha^2)\right].$$

We obtain

$$\begin{aligned}\left\langle|\Psi(L,\mathbf{x})|^2\right\rangle &= \left(\frac{\alpha}{2\sqrt{\pi}}\right)^d\int e^{-|\mathbf{w}|^2[\alpha^2/4+\gamma^2L^2/(4k^2\alpha^2)]} \\ &\times\exp\left[-k^2/(2\gamma^2)L\int_0^1D_*(\gamma L\mathbf{w}s/k)ds\right]e^{i\mathbf{w}\cdot\mathbf{x}}\,d\mathbf{w}.\end{aligned}$$

Hence the turbulence-induced broadening can be identified as convolution with the kernel which is the inverse Fourier transform $\mathcal{F}^{-1}T$ of the transfer function

$$T(\mathbf{w})=\exp\left[-k^2/(2\gamma^2)L\int_0^1D_*(\gamma L\mathbf{w}s/k)ds\right].$$

In view of (11), we obtain that

$$\mathcal{F}^{-1}T(\mathbf{x})=\frac{k^2}{\gamma^2L^2}\mathcal{F}^{-1}T_{\rm tr}(\frac{k\mathbf{x}}{\gamma L}).$$

We define the turbulence-induced forward spread σ_* as

$$\sigma_*=\sqrt{\int|\mathbf{x}|^2T^2(\mathbf{x})d\mathbf{x}/\int T^2(\mathbf{x})d\mathbf{x}}$$

which together with $\rho_{\rm tr}$ then satisfies the uncertainty inequality:

$$\sigma_*\rho_{\rm tr}\ge\frac{\gamma L}{k}. \tag{17}$$

The equality holds when $T_{\rm tr}$ is Gaussian, i.e. when $\rho_{\rm tr} \leq \ell_0$ or $\ell_0 \ll \rho_{\rm tr} \ll L_0$ with $H_* = 1$. This strongly suggests the definition of the turbulence-induced aperture as $A_* = \gamma\,\lambda L/\rho_{\rm tr}$ which is entirely analogous to (12).

Because the coherence length of the wave field is closely related to the spread, it is not surprising then to find that the turbulence-induced (de)coherence length δ_* associated with $\left\langle \overline{\Psi(L,\mathbf{x})}\Psi(L,\mathbf{y})\right\rangle$ is directly related to $\rho_{\rm tr}$. Indeed, one can show that $\delta_* \approx \rho_{\rm tr}$ when the effect of the turbulent medium is dominant over diffraction [5].

Discussion

In summary, we have shown for the parabolic Markovian model that, first, the time reversal resolution can be aperture independent and depend on the wavelength in a nonlinear way. This is due to the self-similar nature of the media. Second, we prove an uncertainty inequality for random media where the conjugate variables are the forward wave spread and the time-reversal resolution. The equality is attained when the wave structure function $T_{\rm tr}$ is Gaussian.

The preceding analysis has been carried out for a narrow-band signal. Because of the linearity of the equation a wide-band signal $u_0(t,\mathbf{x})$ can be decomposed into frequency components each of which can be analyzed as above and then resynthesized. The mean retrofocused signal can be calculated as

$$\begin{aligned}
&\left\langle u_{\rm tr}\right\rangle(\tau,\mathbf{x}) \qquad (18)\\
&= \frac{1}{2\pi\gamma^2L^2}\int\overline{u_0(t,\mathbf{y})}\int\hat{\mathbb{I}}_A\left(\frac{k(\mathbf{x}+\mathbf{y})}{\gamma L}\right)e^{-ik(t+\tau)}\\
&\quad\times e^{ik|\mathbf{x}|^2/(2\gamma L)}e^{-ik|\mathbf{y}|^2/(2\gamma L)}k^2T_{\rm tr}(\mathbf{x}-\mathbf{y})dkdt
\end{aligned}$$

from which it follows that the turbulence-induced spread in time is given by convolution with a *Gaussian* kernel because $T_{\rm tr}$ is Gaussian in k, see (11). The Gaussian kernel has a offset-**x**-depending variance $\sigma^2_{\rm tr}(\mathbf{x}) = L\int_0^1 D_*(s\mathbf{x})ds/\gamma^2$ which grows rapidly with the offset if $L \gg 1, \gamma \ll 1$. It is precisely this rapid change of temporal dispersion rate with the offset that produces the sharp spatial retrofocusing of the time-reversed pulse.

Our results above have been limited to the mean value of the time-reversed retrofocused field. Its second or higher moments can be determined from those of the Wigner distribution which are not exactly solvable. However, the mean field is sufficient for determining all the higher moments in case of self-averaging. Self-averaging occurs, for example, when the narrow-band beam width in the transverse directions is large compared to the correlation length of the random medium or when the signal is wide-band [1]. The former case has been analyzed extensively in the literature (see [4] and references therein) and there arise several canonical radiative transfer equations as the self-averaging scaling limits. The latter case of wide-band signals has only been studied for the one-dimensional medium where the issue of spatial focusing does not arise, see [3]. In the near-self-averaging regime the second moment of the Wigner distribution can be calculated and will be reported elsewhere.

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