CHAPTER 3

COMPLEX MANIFOLDS AND COVERS

Chap. 4 replaces the field $\mathbb{C}(z, f(z))$ generated by an algebraic function $f(z)$ over $\mathbb{C}(z)$ by a geometric object, a 1-dimensional complex manifold (*Riemann surface*) that maps to the Riemann sphere $\mathbb{P}^1_z$. To prepare for this idea requires building some manifolds, and developing intuition for basic examples. We use fundamental groups to create new 1-dimensional complex manifolds from the space $U_{\boldsymbol{z}}$ with $\boldsymbol{z}$ a finite subset of $\mathbb{P}^1_z$.

Chap. 5 collects various Riemann surfaces into families. The parameter spaces for these families — one point in the space for each member of the family — are manifolds called *moduli spaces*. Chap. 4 has a prelude, the moduli space classically called the *j-line*: $\mathbb{P}^1_j \setminus \{\infty\}$. We use it for more general families than do classical texts on Riemann surfaces. Our moduli spaces may have arbitrarily high complex dimension. Still, their construction uses covering spaces (coming from fundamental groups) of open subsets of projective spaces. This chapter builds an intuition for using group theory to construct these spaces.

1. Fiber products and relative topologies

There is so much topology and we have so little space for it despite the need for some special constructions. The treatment is expedient and not completely classical to emphasize some subtle properties of manifolds.

1.1. Set theory constructions. For X and Y sets, the *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of subsets of the set X *indexed* by the set I. The *union* of $\{X_\alpha\}_{\alpha\in I}$ is the set of $x \in X$ for which $x \in X_\alpha$ for some $\alpha \in I$. Denote this $\bigcup_{\alpha\in I} X_\alpha$. The *complement* of X_α in X, $X \setminus X_\alpha$, is $\{x \in X \mid x \notin \bigcup_{\alpha\in I} X_\alpha\}$. The *intersection* of $\{X_\alpha\}_{\alpha\in I}$ is the set of $x \in X$ with $x \in X_\alpha$ for each $\alpha \in I$. Denote this $\bigcap_{\alpha\in I} X_\alpha$.

Definition 1.1. For X_1 and X_2 sets, $Y_i \subset X_i$, $i = 1, 2$, let $f : Y_1 \to Y_2$ be a one-one onto function. The *sum* of X_1 *and* X_2 *along* f is the disjoint union of $X_1 \setminus Y_1$, Y_2, and $X_2 \setminus Y_2$. Denote this $X_1 \bigcup_f X_2$. Along with this, we have maps $f_i : X_i \to X_1 \bigcup_f X_2$, $i = 1, 2$: with $f_2(x_2) = x_2$ for $x_2 \in X_2$, $f_1(x_1) = x_1$ if $x_1 \in X_1 \setminus Y_1$, and $f_1(x_1) = f(x_1)$ for $x_1 \in Y_1$. Call f_1 and f_2 the *canonical* maps.

Example 1.2 (The set behind a non-Hausdorff space). Consider

$$X_i = \{(t, i) \in \mathbb{R}^2 \mid -1 < t < 1\},\ i = 1, 2,\ \text{with}$$

$Y_i = X_i \setminus \{(0, i)\}$, $i = 1, 2$, and $f : Y_1 \to Y_2$ by $f(t, 1) = (t, 2)$ for $(t, 1) \in Y_1$. Then, $X_1 \bigcup_f X_2$ is the disjoint union of X_2 and the point $(0, 1)$ (see Def. 1.4 and Ex. 2.4).

DEFINITION 1.3 (Set theoretic fiber products). Let $f_i : X_i \to Z$ be two functions with range Z, $i = 1, 2$. The *fiber product* $X_1 \times_Z X_2$ consists of

$$\{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}.$$

Denote the natural map back to Z by $f_1 \times_Z f_2$. Suppose $X_i \subset Z$ and $f_i : X_i \to Z$ is inclusion, $i = 1, 2$. Then, identify $X_1 \times_Z X_2$ with $X_1 \cap X_2$.

Suppose $X_1 = X_2 = Z = \mathbb{C}$, and f_1 and f_2 are polynomials. Then, $X_1 \times_Z X_2$ is the subset of $(x_1, x_2) \in \mathbb{C}^2$ defined by $f_1(x_1) = f_2(x_2)$. Define the ith *projection map*, $\text{pr}_i : X_1 \times_Z X_2 \to X_i$ by $\text{pr}_i(x_1, x_2) \mapsto x_i$, $i = 1, 2$.

The fiber product is an *implicit* set: an equation describes it.

The ball of radius r about $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the *basic open set* $\{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x} - \boldsymbol{x}_0| < r\}$. When necessary denote this $B(\boldsymbol{x}_0, r)$. *Open sets* of $\mathbb{R}^n$ are either empty or are (arbitrary) unions of basic open sets. *Closed sets* are complements (in $\mathbb{R}^n$) of open sets. *Bounded sets* are those contained in some basic open set. The collection of open sets, $\mathcal{U}$, in $\mathbb{R}^n$ therefore satisfies the axioms for a *topology*: $\mathcal{U}$ contains the empty set and the whole space, and it is closed under taking arbitrary unions and finite intersections.

DEFINITION 1.4 (Relative topology I). Let X be a subset of $\mathbb{R}^n$. Denote the collection of sets $X \cap U$ for U open subset in $\mathbb{R}^n$ by $\mathcal{U}_X$. Then $\mathcal{U}_X$ gives the *relative topology* on X. For $x_1, x_2 \in X$, two distinct points, $B(x_1, r/3) \cap X$ and $B(x_2, r/3) \cap X$ are disjoint open *neighborhoods* of the respective points x_1 and x_2 if $r = |x_1 - x_2|$. Thus, in this relative topology, X is a *Hausdorff space*.

Suppose X (resp. Y) is a topological space with open sets $\mathcal{U}_X$ (resp., $\mathcal{U}_Y$). Let $f : X \to Y$ be a function with *domain* a subset of X. Then f is *continuous* (for the *relative topology*) if for each $U \in \mathcal{U}_Y$,

$$f^{-1}(U) = \{x \text{ in the domain of } f \mid f(x) \in U\} \text{ is in } \mathcal{U}_X.$$

For U open in Y, denote *restriction* of f to $f^{-1}(U)$ by $f_U : f^{-1}(U) \to U$. If f is continuous, so is f_U.

The concept of relative topology generalizes to data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ on a set X with the following properties: $\bigcup_{\alpha \in I} X_\alpha = X$; $\varphi_\alpha : X_\alpha \to \mathbb{R}^n$ is a one-one map into $\mathbb{R}^n$; and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(X_\alpha \cap X_\beta) \to \varphi_\beta(X_\alpha \cap X_\beta)$ is a continuous function for each $\alpha, \beta \in I$. We call the functions $\{\varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha,\beta \in I}$ *transition functions*.

DEFINITION 1.5 (Relative topology II). Let X and $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be as above. Consider subsets of X that are unions of $\varphi_\alpha^{-1}(U)$ with U running over open sets of $\varphi_\alpha(X_\alpha)$, $\alpha \in I$. Denote this collection of sets by $\mathcal{U}_X$. The topology on X from $\mathcal{U}_X$ is the *relative topology on* X induced from the topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. For $x \in X$ and U an open set containing x, U is a *neighborhood* of x.

1.2. Extending topologies from $\mathbb{R}^n$. Two sets of topologizing data on X, $\{(X'_{\alpha'}, \varphi'_{\alpha'})\}_{\alpha' \in I'}$ and $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, are *equivalent* (the *same*, or *give the same topology*) if each defines the same open sets on X.

Consider X and Y, topological spaces with respective data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ and $\{(Y_\beta, \psi_\beta)\}_{\beta \in J}$. A one-one map $f : X \to Y$ is a (*topological*) *embedding* if the topologizing data from $\{(f^{-1}(Y_\beta), \psi_\beta \circ f)\}_{\beta \in J}$ is equivalent to $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Note: Ex. 2.4 has a space with no embedding in $\mathbb{R}^n$ (for any n). It isn't Hausdorff. Yet, each point has a neighborhood embeddable as an open interval in $\mathbb{R}^1$.

Associate to each subset Y of a topological space X the *closure* $\bar{Y}$ of Y in X: $\bar{Y}$ (a closed set) is the points $x \in X$ with each neighborhood of x containing at least one point of Y. If each neighborhood of x contains a point of Y distinct from x, then x is a *limit point of* Y.

Compact subsets of $\mathbb{R}^n$ are those both closed and bounded. The Heine-Borel covering theorem [**Rud76**, p. 40] characterizes these sets through the concept of an *open covering*. A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{R}^n$ is an open cover of Y if $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$. Then Y has the *finite covering property* if for *each* open cover $\mathcal{U}$ there is a *finite* collection $\{U_{\alpha_i}\}_{i=1}^t$, $\alpha_1, \dots, \alpha_t \in I$, covering Y.

THEOREM 1.6 (Heine-Borel). *The finite covering property is equivalent to compactness for subsets of* $\mathbb{R}^n$.

Thus, for any topological space X, without reference to the concept of bounded set, one says a subset Y is *compact* if it has the finite covering property.

A subset Y of a topological space X is *disconnected* if there are two nonempty open sets U_1 and U_2 of Y (in the relative topology) with $U_1 \cap U_2$ empty and $U_1 \bigcup U_2 = Y$. If Y is not disconnected call it *connected* (in X). For any $x \in X$, there is a maximal connected set U_x containing x. So, each topological space decomposes into a union of disjoint *connected components*. If $f : Y \to X$ is continuous, the image of any connected subset of Y is a connected subset of X.

2. Functions on X from functions on $\mathbb{R}^n$

There are several points to make about Def. 1.5. First it includes many topologies as our next example illustrates.

EXAMPLE 2.1. Let X be any set whose points, x_α, are indexed by $\alpha \in I$. Let $X_\alpha = \{x_\alpha\}$ and $\varphi_\alpha : \{x_\alpha\} \to \{\mathbf{0}\}$, $\alpha \in I$, where $\mathbf{0}$ is the origin of $\mathbb{R}^n$. The relative topology on X is the *discrete topology*.

By using another target space Y with a well-known topology on it (like the *p*-adic numbers $\mathbb{Z}_p$, replacing $\mathbb{R}^n$), we could include *p*-adic topologies, too. Still, it does not include all the topologies significant to modern mathematics even for spaces we consider as manifolds. Later we will extend it to *Grothendieck topologies*. It is appropriate for that example to notice we don't need a topology on X to start the process (§2.1).

Further, the point of topologizing data is to pull back functions (differentials, and other objects) from $\mathbb{R}^n$ so X has local functions (differentials, etc.) just like those of $\mathbb{R}^n$. Since $\mathbb{R}^n$ also has the notion of *real analytic*, *differentiable* and *harmonic* functions, transition functions also allow us to pull those back, to identify such functions on X. For these definitions, however, to be meaningful, they must be locally independent of which function we use for pullback. This requires the transition functions also have these respective properties (§3).

When $n = 2m$ is even, suppose the following two conditions hold.

(2.1a) We have chosen a fixed $\mathbb{R}$ linear map $L = L_n : \mathbb{R}^n \to \mathbb{C}^m$.
(2.1b) Using L, the transition functions are analytic from $\mathbb{C}^m \to \mathbb{C}^m$.

These conditions allow identifying a set of functions in a neighborhood of any point on X as analytic (§3.1.2).

Finally, there is a warning. Local function theory immediately challenges us to identify global functions and differentials on X through their local definitions. There is an immediate first problem to assure a simple property we expect from functions in $\mathbb{R}^n$. If a function f in a neighborhood of $x \in X$ has good behaviour as $x' \in X$ approaches x, then it should have a unique limit value (see §2.2 on the Hausdorff property).

2.1. Defining a topological space from its atlas. Def. 1.5 shows we don't need X to start with a topology. It inherits one from its topologizing data. So, it is reasonable to ask if we need an a priori space X at all.

2.1.1. *Equivalence relations define topological spaces.* For example, suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets in $\mathbb{R}^n$, and for some subset $(\beta, \alpha) \in I \times I$, there are invertible continuous maps $\psi_{\beta,\alpha} : V^\alpha_\beta \to V^\beta_\alpha$, with V^β_α open in U_α (resp. V^α_β open in U_α). Can we form an X so that $\{\psi_{\beta,\alpha}\}_{\alpha,\beta \in I}$ are the transition functions for its topological structure? Almost!

Let X be the disjoint union $\dot\cup_{\alpha \in I} U_\alpha$ modulo the relation R_I on this union defined by $x \in U_\alpha \sim x' \in U_\beta$ if $\psi_{\beta,\alpha}(x) = x'$. If R_I is an equivalence relation, then the equivalence classes form a set X and on it a topological structure. On this space, of course, the open sets do look like those of $\mathbb{R}^n$ (in contrast to Ex. 2.1). The following lemma keeps track of the definitions.

Lemma 2.2. *The relation R_I is an equivalence relation if and only if the following properties hold:*

(2.2a) $\psi_{\alpha,\alpha}$ *is the identity map;* $\psi_{\alpha,\beta} = \psi^{-1}_{\beta,\alpha}$*; and*
(2.2b) $\psi_{\gamma,\beta} \circ \psi_{\beta,\alpha} = \psi_{\gamma,\alpha}$ *wherever any two of the maps are defined.*

Suppose R_I is an equivalence relation. Then the inverse of the natural inclusion maps $U_\alpha \to X$ are functions φ_α giving transition functions $\varphi_\beta \circ \varphi^{-1}_\alpha = \psi_{\beta,\alpha}$.

2.1.2. *Quotient topologies.* Suppose X is a topological space with topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Let $f : X \to Y$ be any surjective map. Then, there is a topology on Y with open sets $\mathcal{U}_Y$ the images by f of all sets in $\mathcal{U}_X$. We can't, however, expect topologizing data on Y by pushing down the functions φ_α without extra conditions. It usually makes sense to write f for restriction of f to any subset $V \subset X$. The argument here, however, requires tracking the domain, and so we write f_V.

Let J be the subset of I for which $f_{X_\beta} : X_\beta \to Y$ is one-one for $\beta \in J$. Let $\mathcal{U}_{X,Y}$ be $\{X_\beta\}_{\beta \in J}$ and assume $\mathcal{U}_{X,Y}$ is a cover of X. With no loss assume the coordinate chart for X contains only sets from $\mathcal{U}_{X,Y}$. The hypothesis provides coordinate functions $\psi_\alpha : f(X_\alpha) \to \mathbb{R}^n$ by setting $\psi_\alpha = \varphi_\alpha \circ f^{-1}_{X_\alpha}$ on $f(X_\alpha)$.

From Lem. 2.2 we want an equivalence relation on $\dot\cup_{\alpha \in J} \psi_\alpha(f(X_\alpha))$ that reproduces the set Y as equivalence classes: $y \in \psi_\alpha(f(X_\alpha)) \sim y' \in \psi_\beta(f(X_\beta))$ if $\psi_{\beta,\alpha}(y) = y'$. So, the problem is to define $\psi_{\beta,\alpha}$, using that $f^{-1}_{X_\alpha}$ is different from $f^{-1}_{X_\beta}$ on $f(X_\alpha) \cap f(X_\beta)$. If $f(X_\alpha \cap X_\beta) = f(X_\alpha) \cap f(X_\beta)$, then it is consistent to define $\psi_{\beta,\alpha}$ as $\psi_\beta \circ \psi^{-1}_\alpha = \varphi_\beta \circ \varphi^{-1}_\alpha$. More generally, an additional hypothesis is essentially necessary and sufficient if we use the full set $\mathcal{U}_{X,Y}$.

Lemma 2.3. *Suppose in addition to the above, for each pair $X_\alpha, X_\beta \in \mathcal{U}_{X,Y}$ with $f(X_\alpha) \cap f(X_\beta) \neq \emptyset$, there exists $X_{\beta'} \in \mathcal{U}_{X,Y}$ with*

(2.3) $f(X_{\beta'}) = f(X_\beta)$ *and* $f(X_\alpha \cap X_{\beta'}) = f(X_\alpha) \cap f(X_{\beta'})$.

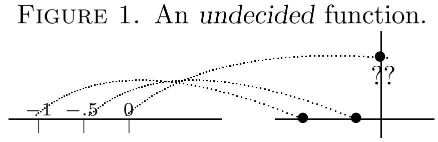
Then, the topologizing data on X provides topologizing data on Y.

Proof. Apply f to $\mathcal{U}_{X,Y}$ get $\mathcal{U}_Y$. Suppose $f(X_\alpha) \cap f(X_\beta) \neq \emptyset$. Then, choose $(X_{\beta'}, \varphi_{\beta'})$ and form $\psi_{\beta,\alpha}$ by replacing f^{-1}_β by $f^{-1}_{\beta'}$. □

2.2. $\mathbb{R}^n$-like behavior requires Hausdorffness. Here is the problem with a space that isn't Hausdorff. Suppose $f : [0,1) \to X$ is a continuous function, everything of a path except the end point. Manifolds in this book appear as extensions of open subsets of $\mathbb{R}^n$. So, the only thing that should prevent us from extending our path (continuously) to $f^* : [0,1] \to X$ is that there is no point $f^*(1) \in X$ giving a continuous f^*. If there are several possible choices $f^*(1)$ giving a continuous function f^*, these extending points would have more exotic neighborhoods than do points in $\mathbb{R}^n$. In practice, the use of Hausdorff is to assure in theorems of Chap. Chap. 4 that there is a unique manifold solution to many existence problems.

EXAMPLE 2.4 (Continuation of Ex. 1.2). As in Ex. 1.2, let $\varphi_i : X_i \to \mathbb{R}^1$ by $\varphi_i(t,i) = t$, $i = 1, 2$. The relative topology on $X_1 \bigcup_f X_2$ is not Hausdorff [9.1].

FIGURE 1. An *undecided* function.



$f(x)=(x,0)$, $\quad -1 \le x < 0$, $\quad f(0)=?$

There is a topological formulation of the possibility that we could end a path in two different points. That is, $(f,f) : [0,1) \to X \times X$ has topological closure not in the diagonal $\Delta_X = \{(x,x) \mid X] \times X\}$. That is, if $f^*(1)$ and $f^\dagger(1)$ are two different ways to extend f to a path on $[0,1]$, then $(f^*(1), f^\dagger(1))$ is in the closure of Δ_X. Conveniently, the exact property that prevents this situation is that X is Hausdorff [9.1b].

LEMMA 2.5. *X is Hausdorff if and only if Δ_X is closed in $X \times X$ [9.1d].*

Here is a classical fact. If $f : X \to Y$ is continuous and one-one and Y is Hausdorff, then the restriction of f to any compact subset of X is a homeomorphism onto its image. This uses that the image of a compact set is compact; then Hausdorff assures that the image of the compact set (and all closed subsets of it) is closed. It is, however, common to have such an f where the *inverse image* of some compact sets are not compact. For example, let $f : \mathbb{C}_z^* \to \mathbb{C}_z$ be the identity map. Then, the inverse image of the unit disk is not compact (compare with [9.1e]). Call a map $f : X \to Y$ *proper* if the inverse image of compact sets is compact.

3. Manifolds: differentiable and complex

Let X be a topological space with topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ (relative to $\mathbb{R}^n$). We add conditions to define differentiable and complex manifolds. Classical cases of the latter include the Riemann sphere, the complex torus and algebraic sets defined by $m \in \mathbb{C}[z,w]$ with nonzero gradient everywhere.

DEFINITION 3.1. Let X be a Hausdorff space with $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ as topologizing data. Assume φ_α maps U_α to an open connected subset of $\mathbb{R}^n$ for each $\alpha \in I$. Call X an *n-dimensional* (*topological*) *manifold*.

In this case, replace the open sets X_α by the notation U_α. Call $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ a *coordinate system* or *atlas*. An individual member $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a *(coordinate) chart*. Ex. 2.4 shows the Hausdorff condition isn't automatic.

3.1. Manifold structures. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on an open set U. For $\boldsymbol{x}_0 \in U$ and $\boldsymbol{v} \in \mathbb{R}^n$, the *directional derivative* of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{v}$ is the limit

$$\lim_{t\to 0}\frac{f(\boldsymbol{x}_0+t\boldsymbol{v})-f(\boldsymbol{x}_0))}{t} \stackrel{\text{def}}{=} \frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0),$$

if it exists. If $\boldsymbol{e}_i = \boldsymbol{v}$ is the vector with 1 in the ith coordinate and 0 in the other coordinates, denote the directional derivative by $\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0)$. Then

$$\nabla f(\boldsymbol{x}_0) \stackrel{\text{def}}{=} \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}_0),\dots,\frac{\partial f}{\partial x_n}(\boldsymbol{x}_0)\right)$$

is the *gradient* of f at $\boldsymbol{x}_0$.

LEMMA 3.2. [**Rud76**, p. 218] *Suppose $\frac{\partial f}{\partial x_i}$ exists and is continuous near $\boldsymbol{x}_0$ for $i = 1,\dots,n$. Then, for each vector $\boldsymbol{v}$, $\frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0)$ exists and equals $\nabla f(\boldsymbol{x}_0)\cdot\boldsymbol{v}$.*

Call a function satisfying the hypotheses of Lemma 3.2 *differentiable* at $\boldsymbol{x}_0$. A function $\boldsymbol{f} = (f_1(\boldsymbol{x}),\dots,f_m(\boldsymbol{x}))$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is differentiable at $\boldsymbol{x}_0$ if each of the coordinate functions $f_i(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}_0$. While it is not absolutely necessary, our manifolds often have transition functions with continuous partial derivatives of all orders: *smoothly* differentiable.

Assume $g : \mathbb{R}^m \to \mathbb{R}$ is a composite of $\mathbb{R}^m \xrightarrow{H} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$. Let $\boldsymbol{y}_0 \in \mathbb{R}^m$. Suppose each coordinate function from $H(\boldsymbol{y}) = (h_1(\boldsymbol{y}),\dots,h_n(\boldsymbol{y}))$ of H is differentiable at $\boldsymbol{y}_0$ and f is differentiable at $H(\boldsymbol{y}_0)$. Write $J(H)(\boldsymbol{y}_0)$ for the matrix whose ith row is $\nabla h_i(\boldsymbol{y}_0)$. As a slight generalization of Lem. 3.2, $\nabla g(\boldsymbol{y}_0)$ exists and equals

$$= \nabla f(H(\boldsymbol{y}_0))\cdot J(H)(\boldsymbol{y}_0). \tag{3.1}$$

3.1.1. *Differentiable functions.* Let X be an n-dimensional manifold. Denote an atlas for it by $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$.

DEFINITION 3.3. Call X a *differentiable manifold* if each transition function $\varphi_\beta\circ\varphi_\alpha^{-1}$ is smoothly differentiable on its domain of definition.

For any $x \in U_\alpha$ on a chart of a differentiable manifold X, define the (smoothly) differentiable functions on U_α to be $C^\infty(U_\alpha) = \{f\circ\varphi_\alpha \mid f \in \mathrm{C}^\infty(\varphi_\alpha(U_\alpha))\}$. This definition should be independent of the chart: We declare that restricting a differentiable function to an open subset of U_α still gives a differentiable function. This, however, must be compatible with the definition of differentiable using any other coordinate chart (U_β,φ_β) which also contains x.

LEMMA 3.4. *Suppose $x \in U_\alpha\cap U_\beta$, and $f\circ\varphi_\alpha$ is restriction of a differentiable function to an open neighborhood W of x in $U_\alpha\cap U_\beta$. Then, $f\circ\varphi_\alpha = g\circ\varphi_\beta$ for some differentiable function g defined on $\varphi_\beta(W)$.*

PROOF. Write $f\circ\varphi_\alpha$ as $f\circ\varphi_\alpha\circ\varphi_\beta^{-1}\circ\varphi_\beta$ and take g as $f\circ\varphi_\alpha\circ\varphi_\beta^{-1}$. This is defined on $\varphi_\beta(W)$. As the composite of two differentiable functions f and $\varphi_\alpha\circ\varphi_\beta^{-1}$, g is differentiable from (3.1). □

DEFINITION 3.5 (Global differentiable functions on X). If X is a differentiable manifold, then a function $f : X \to \mathbb{R}$ is differentiable if its restriction to each U_α in a coordinate chart is differentiable.

3.1.2. *Complex functions.* Decompose a complex number z_i into its real and complex parts as $x_i + i\,y_i$. This produces (as in (2.1)) a natural one-one map:

$$L = L_n : \mathbb{R}^{2n} \to \mathbb{C}^n \text{ by } (x_1, y_1, \dots, x_n, y_n) \mapsto (z_1, \dots, z_n).$$

Topologize $\mathbb{C}^n$ so L (and its inverse) are continuous. Identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ to consider any differentiable function: $g : \mathbb{R}^{2n} \to \mathbb{R}$ as a function $g \circ L^{-1} : \mathbb{C}^n \to \mathbb{R}$. Further, a pair u and v of differentiable functions with a common domain U from $\mathbb{R}^{2n} \to \mathbb{R}$ produces a differentiable function $f : \mathbb{C}^n \to \mathbb{C}$ on U:

$$\boldsymbol{z} \mapsto u \circ L^{-1}(\boldsymbol{z}) + iv \circ L^{-1}(\boldsymbol{z}).$$

Call $f : \mathbb{C}^n \to \mathbb{C}$ *analytic* at $\boldsymbol{z}_o = (z_{1,0}, \dots, z_{n,0})$ if each complex partial derivative

$$\frac{\partial f}{\partial z_i}(\boldsymbol{z}') = \lim_{z_i \to z_i'} \frac{(f(z_1', \dots, z_{i-1}', z_i, z_{i+1}', \dots, z_n') - f(\boldsymbol{z}'))}{z_i - z_i'}$$

exists and is continuous, $i = 1, \dots, n$, with $\boldsymbol{z}'$ near $\boldsymbol{z}_0$. We say $\boldsymbol{f} = (f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}))$ from $\mathbb{C}^n$ to $\mathbb{C}^m$ is *analytic* at $\boldsymbol{z}_0$ if each *coordinate function* $f_i(\boldsymbol{z})$ is analytic at $\boldsymbol{z}_0$. Analytic functions behave for differentiation (or integration) as if each z_i ranging over a 2-dimensional set were a single real variable. [9.4] explores how changing the particular linear identification L_n affects this definition. In the first half of the 1800's, researchers realized the geometry underlying this definition could characterize special recurring collections of integrals. A motivating problem (Chap. 4) was whether the integrals of these functions were serious new functions. By, however, defining —as in Def. 3.6 —analytic manifolds, Riemann replaced complicated sets of functions by geometric properties.

To match with previous notation, if U be an open connected subset of $\mathbb{C}^n$, denote the analytic functions on U by $\mathcal{H}(U)$. The natural quotient field $\mathcal{M}(U)$ of $\mathcal{H}(U)$ (Lem. 3.9), the field of meromorphic functions on U, consists of ratios from $\mathcal{H}(U)$ with nonzero denominators. When $n = 1$, at each point of U any meromorphic function takes a well-defined value in $\mathbb{P}^1_z$. Simple examples like $\frac{z_1}{z_2}$ at $(0, 0)$ show this is not true for $n \geq 2$ [9.11e].

DEFINITION 3.6. Let X be a $2n$-dimensional manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ where $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$. Call X an *analytic* (or *complex*) *n-dimensional manifold* if each transition function $\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is analytic on $\varphi_\alpha(U_\alpha \cap U_\beta)$. So, an analytic manifold is differentiable. A *Riemann surface* is a 1-dimensional complex manifold.

For any $x \in U_\alpha$ on a chart $\mathcal{U}$ of an analytic manifold X, define analytic (resp. meromorphic) functions on U_α to be $\mathcal{H}_\mathcal{U}(U_\alpha) = \{f \circ \varphi_\alpha \mid f \in \mathcal{H}(\varphi_\alpha(U_\alpha))\}$ (resp. $\mathcal{M}_\mathcal{U}(U_\alpha)$ where we replace f analytic by f meromorphic). Exactly as previously, Lem. 3.4 has a version for analytic or meromorphic functions. What changes if we adjust the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ in simple ways?

DEFINITION 3.7. Assume $X = X_\mathcal{U}$ is an n-dimensional analytic manifold, and $h_\alpha : \mathbb{C}^n \to \mathbb{C}^n$ is one-one, differentiable, but not necessarily analytic, on $\varphi_\alpha(U_\alpha)$ for each $\alpha \in I$. Topologies of X from $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I} = \mathcal{U}$ or $\{(U_\alpha, h_\alpha \circ \varphi_\alpha)\}_{\alpha \in I} = \mathcal{U}_{\boldsymbol{h}}$ are the same. Call $\boldsymbol{h}$ a *coordinate adjustment* and $\mathcal{U}_{\boldsymbol{h}}$ the *adjustment* of $\mathcal{U}$ by $\boldsymbol{h}$. Then, $\boldsymbol{h}$ is an *analytic adjustment* if transition functions for $\mathcal{U}_{\boldsymbol{h}}$ are analytic.

Only special coordinate adjustments are analytic. Even if $\boldsymbol{h}$ is an analytic adjustment, unless all the h_α s are analytic themselves, the functions we call analytic (or meromorphic) on an open set U_α of $X_\mathcal{U}$ are usually different from those on the same open set of $X_{\mathcal{U}_{\boldsymbol{h}}}$. For example, suppose $I = \{\alpha\}$ and $U_\alpha = D$ is an open set in

$\mathbb{C}$. Then, the functions $\mathcal{H}_{(D,h)}(D) = \{f \circ h \mid f \in \mathcal{H}(D)$ we call analytic on $\{(D,h)\}$ are the same as $\mathcal{H}(D)$ if and only if h is analyic.

If D is simply connected (and not all of $\mathbb{C}_z$), then Riemann's Mapping Theorem says $\mathcal{H}(D,h)$ is isomorphic as a ring to the convergent power series on the unit disk in $\mathbb{C}_z$. [**Ahl79**, p. 230] says this if h is the identity, though composing with h^{-1} for any diffeomorphisms is a ring isomorphism. A nontrivial case of adjustments is where all the h_α s are the same (see [9.4c]). We explore this further in Chap. 4 §7.7.1. In the next observation (see §5.2.1 for the definition of $\frac{\partial}{\partial \bar z}$) denote range variables for $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ by $z_{\alpha,1}, \dots, z_{\alpha,n}$.

LEMMA 3.8. *That $X_{\mathcal{U}_{\boldsymbol{h}}}$ is an analytic manifold is equivalent to*

(3.2) $h_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ h_\alpha^{-1}$ *is analytic on* $h_\alpha \circ \varphi_\alpha(U_\alpha \cap U_\beta)$ *for all* $(\alpha,\beta) \in I^2$*:* $\frac{\partial}{\partial \bar z_{\alpha,i}}$ *applied to each of its matrix entries is 0,* $i = 1, \dots, n$.

If the $\{h_\alpha\}_{\alpha \in I}$ *are all analytic, then* $\mathcal{H}_{\mathcal{U}}(U_\alpha) = \mathcal{H}_{\mathcal{U}_{\boldsymbol{h}}}(U_\alpha)$ *for all* $\alpha \in I$.

Suppose $X_{\mathcal{U}}$ and $X_{\mathcal{U}_{\boldsymbol{h}}}$ are both analytic manifolds. Lem. 3.8 shows the local analytic functions change unless $\boldsymbol{h}$ consists of analytic functions. We regard the complex structures as the same if and only if both $X_{\mathcal{U}}$ and $X_{\mathcal{U}_{\boldsymbol{h}}}$ have the same analytic functions in a neighborhood of each point. A special case appears often in the theory of complex manifolds. It is when all the functions h_α are complex conjugation (Chap. 4 Lem. 7.15). Notice: Complex conjugation reverses orientation in $\mathbb{C}$ by mapping clockwise paths around the origin to counterclockwise paths.

3.1.3. *A tentative definition of algebraic manifold.* For complex manifolds, a coordinate chart allows us to define global meromorphic functions as a collection $g_\alpha \in \mathcal{M}(U_\alpha)$ for which $g_\alpha = g_\beta$ on any points of $U_\alpha \cap U_\beta$ where both make sense. Our major study treats families of compact Riemann surfaces. Often each family member appears explicitly with a finite set of points removed, using Riemann's Existence Theorem to produce such surfaces as covers of $U_{\boldsymbol{z}}$. Meromorphic functions mean for us functions meromorphic on some compactification of this manifold. This includes that the functions are ratios of holomorphic functions at those points that might not be included in the initial presentation. For example, global meromorphic functions on $U_{\boldsymbol{z}}$ refer to elements of $\mathbb{C}(z)$. They are among the ratios of algebraic functions on $U_{\boldsymbol{z}}$, so they have no essential singularities as we approach $\boldsymbol{z}$.

Understanding manifolds which have a coordinate description is important to the goals of this book. When we deal with compact complex manifolds, global coordinate functions live inside the field of global meromorphic functions. Our first tentative definition of algebraic excludes some manifolds that everyone considers algebraic. Still, it is simple, close to the general meaning of algebraic and it leads naturally to that definition.

LEMMA 3.9. *Suppose $X_{\mathcal{U}}$ is a connected topological space and an analytic manifold. Then, the (global) meromorphic functions on $X = X_{\mathcal{U}}$ form a field, $\mathbb{C}(X)$.*

PROOF. Add (resp. multiply) functions of form $f_1(\varphi_\alpha)$ and $f_2(\varphi_\alpha)$ by computing the value at $x \in U_\alpha$ as $f_1(\varphi_\alpha(x)) + f_2(\varphi_\alpha(x))$ (resp. $f_1(\varphi_\alpha(x))f_2(\varphi_\alpha(x))$). Quotients, too, are obvious for they will also be ratios of holomorphic functions at each point. We need only to see that $\mathbb{C}(X)$ is an integral domain. If, however, $f_1(\varphi_\alpha(x))f_2(\varphi_\alpha(x)) = 0$ for $x \in U_\alpha$, then $f_1(\boldsymbol{z})f_2(\boldsymbol{z}) = 0$ for $\boldsymbol{z}$ on the open set $\varphi_\alpha(U_\alpha)$. Chap. 2 [9.8a] shows either $f_1(\varphi_\alpha)$ or $f_2(\varphi_\alpha)$ is 0 on U_α. □

A goal for compact Riemann surfaces is to understand adjustments well enough to be able to list the isomorphism classes of fields $\mathbb{C}(X_{\mathcal{U}_{\boldsymbol{h}}})$, the *function field* of

$X_{\mathcal{U}_{\boldsymbol{h}}}$, as $\boldsymbol{h}$ varies. How can we describe the complete set of function fields up to isomorphism? This book shows how to apply various answers to many seemingly unrelated problems.

Suppose $x_1, x_2 \in X$ and $f \in \mathbb{C}(X_{\mathcal{U}})$ are holomorphic in a neighborhood of x_1 and x_2 and takes different values there. We say f *separates* x_1, x_2. If for each pair of distinct points $x_1, x_2 \in X$ there is an $f \in \mathbb{C}(X_{\mathcal{U}})$ separating them, we say $\mathbb{C}(X_{\mathcal{U}})$ separates points. Suppose $X_{\mathcal{U}}$ has complex dimension n, $x \in X_{\mathcal{U}}$ is in a coordinate chart $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ and there are n functions $f_1, \dots, f_n \in \mathbb{C}(X_{\mathcal{U}})$ all holomorphic in a neighborhood of x. If the Jacobian of $f_1, \dots, f_n$ —determinant of the matrix with (i,j)-entry of $\frac{\partial f_i \circ \varphi_\alpha^{-1}}{\partial z_j}$, $i = 1, \dots, n$, $j = 1, \dots, n$ —is nonzero at $\varphi_\alpha(x)$, we say $f_1, \dots, f_n$ *separate tangents* at x.

DEFINITION 3.10. An n-dimensional compact complex manifold X (with topologizing data $\mathcal{U}$) is $\mathbb{P}^1$-*algebraic* if there is a collection $f_1, \dots, f_N \in \mathbb{C}(X_{\mathcal{U}})$ so the following conditions hold.

(3.3a) For each $x \in X_{\mathcal{U}}$, there is a collection $\epsilon_1, \dots, \epsilon_N \in \{\pm 1\}$ (dependent on x) so that $f_1^{\epsilon_1}, \dots, f_N^{\epsilon_N}$ are all holomorphic at x.
(3.3b) Among $f_1^{\epsilon_1}, \dots, f_N^{\epsilon_N}$ there are n that separate tangents at x.
(3.3c) Given distinct $x_1, x_2 \in X$, one from $f_1, \dots, f_N$ separates x_1 and x_2.

Note: In (3.3c), if f_i is holomorphic at x, and $f_i(x) = 0$, we include ∞ as the value of $1/f_i(x)$. Algebraic manifolds are the analytic manifolds $X_{\mathcal{U}}$ most significant to us ($\mathbb{P}^1$-algebraic manifolds are a special case; see §4.1.2). There are 2-dimensional analytic manifolds with function fields consisting only of constant functions. Our examples will be complex torii. The phrase *abelian variety* (Chap. 4§6.9; usually with a extra structure called a *polarization*) is the name for a complex torus that is algebraic. Chap. 4 analyzes all analytic structures on a dimension one complex torus by corresponding them precisely to the isomorphism class of their function fields. This topic starts in § 3.2.2.

There are two distinct generalizations: To compact Riemann surfaces and to abelian varieties. The former are $\mathbb{P}^1$-algebraic while the latter are not in general.

3.2. Classical examples. We discuss two natural first cases of compact complex manifolds.

3.2.1. *The Riemann sphere* $\mathbb{P}^1_z$. Let X be the disjoint union of the complex plane $\mathbb{C}$ and a point labeled ∞. Here is a coordinate chart:

$$\begin{aligned} U_1 = \mathbb{C}, &\quad \varphi_1 : U_1 \to \mathbb{C} \text{ by } \varphi_1(z) = z; \text{ and} \\ U_2 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}, &\quad \varphi_2 : U_2 \to \mathbb{C} \text{ by } \varphi_2(\infty) = 0 \text{ and} \\ &\quad \varphi_2(z) = \tfrac{1}{z} \text{ for } z \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

Chap. 2 used the Riemann sphere. It embeds in $\mathbb{R}^3$. So it is Hausdorff. Then, X is a complex manifold: $\varphi_2 \circ \varphi_1^{-1}(z) = \varphi_1 \circ \varphi_2^{-1}(z) = \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$ are analytic.

If a complex manifold is compact, some atlas for it contains only finitely many elements. The Riemann sphere required only two (one wouldn't do, would it?).

3.2.2. *Complex torus.* An atlas for our next example will require four open sets. Let ω_1 and ω_2 be two nonzero complex numbers satisfying the *lattice condition*: $\frac{\omega_2}{\omega_1}$ is not real. Consider the *lattice* ω_1 and ω_2 generate:

$$L(\omega_1, \omega_2) = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}. \tag{3.4}$$

The lattice condition guarantees the natural quotient map $\mathbb{C} \to \mathbb{C}/L(\omega_1, \omega_2)$ has open sets that are like open sets in $\mathbb{C}$ [9.6c]. According to Lem. 2.3, the manifold structure on $\mathbb{C}$ automatically gives the manifold structure on $\mathbb{C}/L(\omega_1, \omega_2)$. Use the chart $\{(U'_i, \varphi'_i)\}_{i\in\{0,1,2,3\}}$ of Fig. 3 with φ'_i the inclusion of U'_i in $\mathbb{C}$. This assures satisfying the Lem. 2.3 condition: Each $z \in \mathbb{C}$ has an $i = i_z$ for which $z \in U'_i$ and the natural map $\mathbb{C} \to \mathbb{C}/L(\omega_1, \omega_2)$ is one-one on U_i.

The resulting complex manifold $\mathbb{C}/L(\omega_1, \omega_2)$ depends only on $L(\omega_1, \omega_2)$. Among the many choices we can make of ω_1, ω_2 generating this lattice, it is traditional to choose them satisfying special conditions. Elements of the group $\mathrm{SL}_2(\mathbb{Z})$ act on ω_1, ω_2 to give all pairs of basis elements Chap. 2 [9.15c]. Further, for $a \in \mathbb{C}^*$ the scaling $\mathbb{C} \to \mathbb{C}$ by $z \mapsto az$ induces a homomorphism $\mathbb{C}/L(\omega_1, \omega_2) \to \mathbb{C}/L(a\omega_1, a\omega_2)$ of abelian groups. At the level of coordinate charts, the same scaling gives the map. So, it induces an analytic isomorphism (for precision use Def. 4.1). With no loss take $a = 1/\omega_1$, to change the basis of the lattice to $1, \omega_2/\omega_1$. The ratio $\omega_2/\omega_1 = \tau$ aptly indicates the shape of the parallelogram (3.6). This starts a typical normalization for the complex structure. If we could uniquely indicate the complex structure by τ, that would be an excellent way to parametrize them. The problem is that the complex structure depends only on the lattice $L(1, \tau)$ generated by 1 and τ. Many values of τ giving the same $L(1, \tau)$. For example, here are three obvious changes:

(3.5a) If necessary, replace $\{1, \tau\}$ by $\{1, -\tau\}$ to assume $\Im(\tau)$ is in the *upper half plane* $\mathbb{H} \stackrel{\text{def}}{=} \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$; or

(3.5b) replace $\{1, \tau\}$ by $\{1, \tau + n\}$ for some integer n to assume $0 \le \Re(\tau) < 1$; or

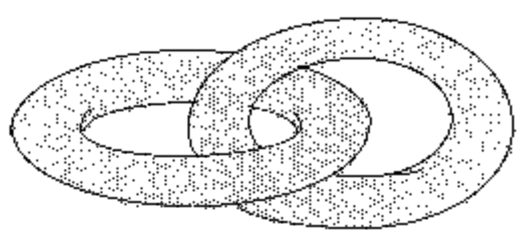
(3.5c) scale by $-1/\tau$ to replace $\{1, \tau\}$ by $\{1, -1/\tau\}$.

Changes from (3.5) generate a group, $\mathrm{PSL}_2(\mathbb{Z})$ ($< \mathrm{PSL}_2(\mathbb{R})$; §8.2), acting on $\tau \in \mathbb{H}$.

LEMMA 3.11. *Together,* (3.5) *permits restricting a* τ *representing a given complex torus (up to isomorphism) to the narrow strip in* $\mathbb{H}$ *over the closed interval* $[0, 1) \subset \mathbb{R}$ *lying within the closed unit circle around the origin.*

Transition functions restrict on each connected component of an intersection of charts to be translation in the complex plane. Topologically this is the same as a *torus* in $\mathbb{R}^3$. Topologists deal with torii, too, though they concentrate especially on the topological space in which the torii sit (see [9.5] for the point of Fig. 2). We care most about this additional complex structure, while they rarely distinguish between one complex torus and another. See §7.2.3 for additional comments on attempts to draw pictures in $\mathbb{R}^3$.

FIGURE 2. These two torii could unknot in $\mathbb{R}^4$.

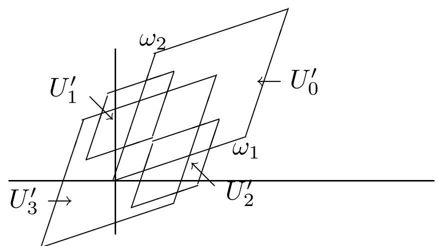


Here is the set behind the manifold:

$$X = \{t_1\omega_1 + t_2\omega_2 \mid 0 \le t_i < 1, i = 1, 2\}. \tag{3.6}$$

Standard open parallelograms in $\mathbb{C}$ represent each of four coordinate charts in Fig. 3, U_i, $i = 0$, 1, 2, 3, that do lie in X.

FIGURE 3. Four open sets sort of covering a torus



Let $U_0 = \{t_1\omega_1 + t_2\omega_2 \mid 0 < t_i < 1, i = 1, 2\}$, with $\varphi_0 : U_0 \to \mathbb{C}$ the identity map. The corresponding U_0' is equal to U_0 in Fig. 3. On the other hand, consider

$$U_1 = \{t_1\omega_1 + t_2\omega_2 \mid \frac{1}{3} < t_2 < \frac{2}{3} \text{ and either } 0 \le t_1 < \frac{1}{3} \text{ or } \frac{2}{3} < t_1 < 1\},$$

and $\varphi_1 : U_1 \to \mathbb{C}$ by

$$\varphi_1(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_1 < \frac{1}{3} \\ (t_1-1)\omega_1 + t_2\omega_2 & \text{for } \frac{2}{3} < t_1 < 1. \end{cases}.$$

Form the corresponding U_1' by translating a pieces of the range of φ_1.

The remaining charts are similar (though slightly more complicated):

$$\begin{aligned} U_2 = \ & \{t_1\omega_1 + t_2\omega_2 \mid \tfrac{1}{3} < t_1 < \tfrac{2}{3} \text{ and either } 0 \le t_2 < \tfrac{1}{3} \text{ or } \tfrac{2}{3} < t_1 < 1\}, \\ & \varphi_2(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_2 < \frac{1}{3} \\ t_1\omega_1 + (t_2-1)\omega_2 & \text{for } \frac{2}{3} < t_2 < 1. \end{cases} \end{aligned}$$

$$\begin{aligned} U_3 = \ & \{t_1\omega_1 + t_2\omega_2 \mid 0 \le t_1 < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < t_1 < 1,\ 0 \le t_2 < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < t_2 < 1\}, \text{ and} \\ & \varphi_3(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_1, t_2 < \frac{1}{2}, \\ (t_1-1)\omega_1 + t_2\omega_2 & \text{for } \frac{1}{2} < t_1 < 1, 0 \le t_2 < \frac{1}{2}, \\ t_1\omega_1 + (t_2-1)\omega_2 & \text{for } 0 \le t_1 < \frac{1}{2}, \frac{1}{2} < t_2 < 1, \\ (t_1-1)\omega_1 + (t_2-1)\omega_2 & \text{for } \frac{1}{2} < t_1, t_2 < 1. \end{cases} \end{aligned}$$

To see X is a 1-dimensional complex manifold check the transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$. For each i and j, $\varphi_i(U_i \cap U_j)$ is the union of a finite number of connected open sets. For example,

$$\varphi_0(U_0 \cap U_1) = U_1' \setminus \Big\{t_2\omega_2 \mid \frac{1}{3} < t_2 < \frac{2}{3}\Big\}.$$

On each connected component of $\varphi_i(U_i \cap U_j)$, $\varphi_j \circ \varphi_i^{-1}$ is translation by one of the complex numbers $\delta_1\omega_1 + \delta_2\omega_2$ where δ_k is 0 or ± 1, $k = 1, 2$.

With this manifold structure, X is the *complex torus with periods* ω_1 *and* ω_2.

3.3. Manifolds from algebraic functions. Let $m \in \mathbb{C}[z, w]$ be an irreducible polynomial. Denote the branch points of m by $\boldsymbol{z}$ with $z_0 \in U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \boldsymbol{z}$ as in Chap. 2 Def. 6.3. Assume $f(z)$ is analytic in a neighborhood of z_0 and it satisfies $m(z, f(z)) \equiv 0$. Chap. 2 started with two definitions of algebraic functions Def. 1.1 and Def. 1.2. They characterize the same set of functions (Chap. 2 Prop. 7.3).

Riemann's Existence Theorem starts by attaching to each algebraic function a unique (up to analytic isomorphism) compact complex manifold of dimension 1. The next two examples are the first step in that construction, producing an open subset of the final manifold. We introduce some algebraic geometry using as an excuse showing how to construct explicit manifold compactifications in special cases. We expect coordinates for the abstract compactification of a general Riemann surface to be somewhat mysterious.

3.3.1. *An unramified cover of* $U_{\boldsymbol{z}}$. Consider first the set

$$X^{[0]} = X_f = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid z \notin \boldsymbol{z},\ m(z,w) = 0\}.$$

PROPOSITION 3.12. *The projection map* $\mathrm{pr}_z : X^{[0]} \to U_{\boldsymbol{z}}$ *by* $(z,w) \mapsto z$ *produces a natural atlas on* $X^{[0]}$ *making it a connected complex manifold. For* $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0, z_1)$ *(*Chap. 2 §*1.1), naturally identify the manifolds* X_f *and* X_{f_λ}.

PROOF. To simplify the construction, assume $\infty \in \boldsymbol{z}$. As usual, apply an element of $\mathrm{PGL}_2(\mathbb{C})$ to $\boldsymbol{z}$ to arrange that situation (Chap. 2 §5.2.1; see Lem. 4.3).

Use the implicit function theorem (Chap. 2 §6.2) as follows. For $(z',w') \in X^{[0]}$, let $\Delta_{z'}$ be the open disk centered at z' of radius the minimum distance from z' to a point of $\boldsymbol{z}$. Then, for some one-one analytic function $f_{z',w'}(z)$ the following holds.

(3.7) The points $(z, f_{z',w'}(z))$ are on $X^{[0]}$ and $f_{z',w'}(z') = w'$.

For each (z',w') let $U_{z',w'}$ be the range of $z \mapsto F_{z'}(z) \stackrel{\text{def}}{=} (z, f_{z',w'}(z))$ on $\Delta_{z'}$. The inverse of $F_{z'}$ is pr_z, projection of a pair (z,w) onto its z-coordinate. Compatible with the definition of manifold, here denote pr_z by $\varphi_{z',w'}$. Then, $F_{z'}$ *parametrizes* the neighborhood $U_{z',w'}$ of (z',w') and $\varphi_{z',w'}$ maps it into $\mathbb{C}_z$. If $V = U_{z',w'} \cap U_{z'',w''}$ is nonempty, then $\varphi_{z'',w''} \circ \varphi^{-1}_{z',w'}$ is the identity map on the overlap of $\Delta_{z'} \cap \Delta_{z''}$.

That gives an atlas. As it is a subspace of the Hausdorff space $\mathbb{C} \times \mathbb{C}$, $X^{[0]}$ is Hausdorff. So, it is a connected (from Chap. 2 §6.4) complex manifold. Let λ be a path as in the statement of the proposition. The point set of X_f consists of pairs $(z',x') \in \mathbb{C} \times \mathbb{C}$ of the form $(z', f_\gamma(z'))$ with $\gamma : [a,b] \to U_{\boldsymbol{z}}$ with $\gamma(a) = z_0$ and $\gamma(b) = z'$. As X_{f_λ} is connected, we can write any point on it as the endpoint of $(z, f_{\lambda\cdot\gamma})$ for some λ. So, X_{f_λ} is the same subset of points in $\mathbb{C} \times \mathbb{C}$. □

Note: Each $z' \in U_{\boldsymbol{z}}$ has a neighborhood $\Delta_{z'}$ with this property.

(3.8) pr_z restricted to each connected component $U_{z',w'}$ of $\mathrm{pr}_z^{-1}(\Delta_{z'})$ is a homeomorphism with $\Delta_{z'}$.

This is a stronger property than pr_z being an immersion. It means $\mathrm{pr}_z : X^{[0]} \to U_{\boldsymbol{z}}$ is an (unramified) cover according to Def. 7.12. The inverse image by pr_z of small closed disks around z' are closed disks around points lying over z'. That is, the preimage of a compact set is compact, and pr_z is a *proper* map [9.1d].

REMARK 3.13 (Finite atlas). The atlas of Prop. 3.12 contains an infinite number of elements. For a manifold that adds one complication §3.2.1 and §3.2.2 don't have. This came about to include a deleted neighborhood of (z_i, w') with $z_i \in \boldsymbol{z}$ and w' a solution of $m(z_i, w')$. That's because we chose disks on $\mathbb{C}_z$ as the domain for the $F_{z'}$ parametrization. To remedy this choose other simply connected sets, including traditional slit disks given by scaling, translating and rotating

$$\{z \in \mathbb{C} \mid |z| < 1\} \setminus \{0 \le \Re(z) < 1\}.$$

Chap. 4 §2.4 has further justification for these charts.

3.3.2. *Further compactification and use of equations.* Chap. 4 Thm. 2.6 shows there is a unique compact complex manifold, up to analytic isomorphism (Def. 4.1), extending $X^{[0]}$ (and also the analytic map to $\mathbb{P}^1_z$). To get it we must compatibly add points and analytic disk neighborhoods to match with the analytic structure on $X^{[0]}$. Using the equation $m(z, f(z)) \equiv 0$ often allows adding further points (z_i, w') to $X^{[0]}$ and their local analytic functions to extend the complex manifold structure. The simplest such extension includes those points (z_i, w') where, even though z_i is a branch point, $\frac{\partial m}{\partial w}(z_i, w') \neq 0$. That is, consider

$$X^{[1]} = \Big\{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0,\ \frac{\partial m}{\partial w}(z, w) \neq 0\Big\}.$$

The variable for a local chart around w' is w. Prop. 3.15 gives the details.

EXAMPLE 3.14. Suppose $h \in \mathbb{C}[w]$ of degree $n > 1$ produces $h : \mathbb{P}^1_w \to \mathbb{P}^1_z$. Let z_i be a branch point of $m(z, w) = h(w) - z$ and let $g_{z_i} \in S_n$ be a representative of the conjugacy class attached to z_i (Chap. 2 Lem. 7.9). Then, there is a one-one correspondence between the following sets. Chap. 2 [9.4]:

(3.9a) Points (z_i, w') over z_i for which $z \mapsto (z, f_{z_i,w'}(z))$ (3.7) parametrizes a neighborhood of (z_i, w').
(3.9b) Disjoint cycles of length 1 in g_{z_i}.

Example: Consider $h_1(w) = w(w-1)(w-2)$. Use notation from Chap. 2 Lem. 7.9. The group attached to an algebraic $f_1(z)$ satisfying $h_1(f_1(z)) - z \equiv 0$ is S_3.

Branch cycles g_{z_1} and g_{z_2} at the two branch points z_1, z_2 have the shape (1)(2) (§7.1.1): disjoint cycles of length 1 and 2. So each branch point has two points above it. Then, for each z_i there are two solutions $w_{i,1}$ and $w_{i,2}$ of $h(w) - z_i$. Select $w_{i,1}$ so that $\frac{dh}{dw}(w_{i,1}) \neq 0$ and $\frac{dh}{dw}(w_{i,2}) = 0$, $i = 1, 2$. Adding $(z_i, w_{i,1})$ to $X^{[0]}$ produces an open set on which pr_z maps one-one to $\mathbb{P}^1_z$. This does not hold for the point $(z_i, w_{i,2})$. So, $X^{[1]}$ has exactly one point on it over each of z_1 and z_2.

For any $h(w)$ in Ex. 3.14, $X^{[1]}$ will have missing points in that the map pr_z is not proper over some points $z_i \in \boldsymbol{z}$ (§2.2). For f analytic in several variables $z_1, \dots, z_n$ in a neighborhood of a point $\boldsymbol{z}_0$, we call

$$\nabla f(\boldsymbol{z}_0) \stackrel{\text{def}}{=} \Big(\frac{\partial f}{\partial z_1}(\boldsymbol{z}_0), \dots, \frac{\partial f}{\partial z_n}(\boldsymbol{z}_0)\Big)$$

the *complex gradient* of f at $\boldsymbol{z}_0$. Now consider a set (usually) larger than $X^{[1]}$:

$$X^{[2]} = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0,\ \nabla(m)(z, w) \neq 0\}.$$

PROPOSITION 3.15. *A natural atlas makes $X^{[2]}$ into a complex manifold.*

PROOF. Since $X^{[2]}$ is a subspace of $\mathbb{C} \times \mathbb{C}$ it is Hausdorff. From Prop. 3.12 we have only to add (z_i, w') lying over $z_i \in \boldsymbol{z}$ sitting in $X^{[2]}$ to their neighborhoods in $X^{[1]}$. Change the w' coordinate by an element of $\mathrm{PGL}_2(\mathbb{C})$ to assume none of the finitely many w' s is ∞.

By assumption $\nabla(m)(z_i, w') \neq 0$, though by definition $\frac{\partial m}{\partial w}(z_i, w') = 0$. Therefore, $\frac{\partial m}{\partial z}(z_i, w') \neq 0$. Apply the implicit function theorem to find a disk $\Delta_{w'} \subset \mathbb{C}_w$ and $h_{z_i,w'}(w)$ analytic on $\Delta_{w'}$ with the following properties.

(3.10a) The points $(h_{z_i,w'}(w), w)$ are on $X^{[1]}$.
(3.10b) The radius of $\Delta_{w'}$ is the minimum distance from w' to any branch point of $m^*(w, z) \stackrel{\text{def}}{=} m(z, w)$ (switch the variables z and w).

Similar to the proof of Prop. 3.12, let $V_{z_i,w'}$ be the range of $w \mapsto (h_{z_i,w'}(w), w)$ on $\Delta_{w'}$. Then the coordinate map at (z_i, w') is pr_w by $(z, w) \mapsto w$.

The essence of producing the manifold structure is to check the transition functions. The key check occurs when the intersection a neighborhood of (z_i, w') meets a neighborhood of (z'', w'') with $z'' \notin \boldsymbol{z}$. For example:

$$\mathrm{pr}_w \circ \mathrm{pr}_z^{-1} : z \mapsto (z, f_{z'',w''}(z)) \mapsto f_{z'',w''}(z)$$

is analytic. Similarly, so is

$$\mathrm{pr}_z \circ \mathrm{pr}_w^{-1} : w \mapsto (h_{z_i,w'}(w), w) \mapsto h_{z_i,w'}(w).$$

That concludes the proof of the lemma. □

4. Coordinates and meromorphic functions

Here we define analytic maps between complex manifolds. In many areas of mathematics, being able to compare all objects of study with a core of special cases can help. For example, it is helpful to know that all finite groups have a Jordan-Hölder series of finite simple groups and that this collection of finite simple groups (including their multiplicities) is an invariant of the group. Still, even an expert on the classification of finite simple groups can't be confident of a complete understanding of the finite group from knowing its Jordan-Hölder series.

For certain compact complex manifolds, knowing how to use their meromorphic functions can help decide how such a manifold fits among all related manifolds. That is a rough statement of how we use coordinates on compact complex manifolds. This subsection uses explicit (though only partial) compactification of Riemann surfaces of algebraic functions to illustrate how coordinates give defining equations.

4.1. Comparing analytic spaces. We define maps between analytic spaces, and then emphasize the significance of such maps to $\mathbb{P}^1$.

4.1.1. *Maps between spaces.* Let X_i be a differentiable (resp., complex) manifold of dimension n_i with topologizing data $\{(U_{\alpha_i}, \varphi_{\alpha_i})\}_{\alpha_i \in I_i}$. Consider a function $f : X_1 \to X_2$ and the functions

$$\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1} : \varphi_{\alpha_1}(U_{\alpha_1} \cap f^{-1}(U_{\alpha_2})) \to \varphi_{\alpha_2}(f(U_{\alpha_1}) \cap U_{\alpha_2}) \tag{4.1}$$

for $(\alpha_1, \alpha_2) \in I_1 \times I_2$.

Definition 4.1 (Analytic map). Call f differentiable (resp. analytic) if the functions of (4.1) are differentiable (resp. analytic) on their domains. For $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^m$, this is equivalent to f being differentiable as usual. If f is one-one and onto, call f a differentiable (resp. analytic) isomorphism between X_1 and X_2.

The phrase *isomorphism* in Def. 4.1 implies there is a differentiable (resp. analytic) $g : X_2 \to X_1$ inverse to f. That is the gist of our next statement.

Lemma 4.2. *Let X and Y be differentiable manifolds. Assume $f : Y \to X$ is a differentiable map, and in a neighborhood U_y of some point $y \in Y$, one-one. Then, there exists differentiable $g : f(U_y) \to U_y$ that is an inverse to f. So, if f is one-one and onto, it has differentiable inverse. If we replace the word differentiable by analytic, there is an analogous result.*

Proof. Both statements are consequences of the *inverse function theorem*. This says that a local inverse exists and is differentiable. There is an inverse function to a one-one onto map (§2.2), so the differentiability is all we need. The definition of differentiable (or analytic) function reverts this result to one about $f : \mathbb{R}^n \to \mathbb{R}^n$ (or for $f : \mathbb{C}^n \to \mathbb{C}^n$) for some integer n.

Chap. 2 §6.1 discusses the inverse function theorem for one complex variable. The full inverse function theorem is an inductive procedure for several complex variables. See [**C89**, p. 72] or [**Rud76**, p. 224] for the general case. For differentiable functions, equation (3.1) says that an inverse g to $f : \mathbb{R}^n \to \mathbb{R}^n$ would have Jacobian matrix $J(g)(\boldsymbol{y}) = J(f(\boldsymbol{x}))^{-1}$ at $\boldsymbol{y} = f(\boldsymbol{x})$. This is a differential equation for $g = (g_1(\boldsymbol{y}), \dots, g_n(\boldsymbol{y}))$, given f. The case when f is real analytic is much more likely for our use, and that has easier proofs in the literature. □

4.1.2. $\mathbb{P}^1$-*algebraic spaces.* Let $\varphi : X \to Y$ be an analytic map of complex manifolds. If U is an open subset of Y, denote the *restriction* of φ over U by $\varphi_U : \varphi^{-1}(U) \to U$. Then, composing holomorphic functions on an open set $U \subset Y$ with φ produces a map $\varphi^* : \mathcal{H}(U) \to \mathcal{H}(f^{-1}(U))$. In particular, if both spaces are connected, and φ is onto, this induces an injection $\varphi^* : \mathbb{C}(X) \to \mathbb{C}(Y)$, an embedding of the function field of Y into that of X.

Chap. 2 Def. 4.13 includes the definition of analytic maps from a domain on $\mathbb{P}^1_w$ to $\mathbb{P}^1_z$, a special case of Def. 4.1. More generally, for any complex manifold X, a nonconstant analytic map $\varphi : X \to \mathbb{P}^1_z$ is a meromorphic function on X (represented by z). Chap. 2 Lem. 2.1 guarantees a nonconstant map of compact Riemann surfaces is surjective. This also applies to φ, even if X (compact) has larger dimension, for again these functions come locally from power series expressions and so give an open map. Further, if X is a compact Riemann surface, Chap. 4 Thm. 2.6 shows any meromorphic function on X extends to give an analytic map from X to projective 1-space. Chap. 4 Lem. 2.1 shows the following points. If X is compact (and φ is nonconstant), then φ has a degree, $|\varphi^{-1}(z')|$ for $z' \in \mathbb{P}^1_z$ not in a finite set of values where this cardinality is a smaller number. Further, if we count points in $\varphi^{-1}(z')$ with *appropriate multiplicity* for their appearance in the fiber, the degree is independent of $z' \in \mathbb{P}^1_z$.

Many compact complex manifolds of dimension at least 2 (example: $\mathbb{P}^n$, $n \geq 2$, [9.11e]), have the following property. Though they have many nonconstant meromorphic functions, none are represented by an analytic map to $\mathbb{P}^1_z$. The compact complex manifolds that are $\mathbb{P}^1$-algebraic are exactly those that embed in $(\mathbb{P}^1)^N$ for some integer N. That is, they have sufficiently many functions represented by an analytic map to $\mathbb{P}^1$, the gist of condition (3.3a).

A virtue of the definition $\mathbb{P}^1$-algebraic is its simplicity, this use of special elements of the function field giving maps to $\mathbb{P}^1$. Still, Chap. 4 §5.2 extends this, as is traditional, to say a manifold is *algebraic* if it embeds in $\mathbb{P}^N$ for some N. The effect of that is to show why a set of basic principles forces extending $\mathbb{P}^1$-algebraic manifolds to include $\mathbb{P}^N$ as algebraic. We hope this adds historical perspective on what was less than a century ago a complicated issue. Witness this [**Mu66**, p. 15] quote on going directly from affine space to projective space:

> Among others, Poncelet realized that an immense simplication could be introduced in many questions by by considering "projective" algebraic sets (cf. Felix Klein, *Die Enwicklund der Mathematik*, Part I, p. 80–82). Even to this day, ... projective algebraic sets play a central role in algebro-geometric questions: therfore we shall define them as soon as possible.

Mumford's quote, and the total acceptance of it in [**Har77**], shouldn't deny the natural way that $\mathbb{P}^1$-algebraic spaces and fiber products illuminate special meromorphic functions arise in providing coordinates.

In practice, on many intensely studied algebraic manifolds, you can choose a finite set, $f_1, \dots, f_m$, of global meromorphic functions to construct the manifold, whose points we can then *see* as given by the values of $f_1, \dots, f_m$ at the given point. From these, it is theoretically possible to construct anything else you would expect attached to the manifold from $f_1, \dots, f_m$. Still, much classical algebraic geometry spends great time on using coordinates (embeddings in projective space) of special types to make these constructions. For many applications, however, this is a too-detailed reliance on specific use of coordinates. We hope discussions in this chapter help the reader see why coordinates are necessary, though one shouldn't insist on seeing them explicitly at all stages.

We especially study families of compact Riemann surfaces with each family member appearing with an attached equivalence class of maps to $\mathbb{P}^1_z$. What, however, is the analogy, so important to individual measurements, for comparing different function fields (Lem. 3.9) associated to different complex manifolds? Where would we expect such comparisons to arise? Comparing Riemann surfaces is possible if there is an efficient labeling of function field generators. The easiest event is if all these Riemann surfaces embed naturally in a space with global coordinates that restrict to give coordinates on the individual surfaces. §4.2 gives examples of how coordinates can help compactify some Riemann surfaces.

An easy way to get new analytic maps from old appears if $\varphi : X \to \mathbb{P}^1_z$ is a meromorphic function. Let $\alpha \in \mathrm{PGL}_2(\mathbb{C})$. Then $\alpha \circ \varphi : X \to \mathbb{P}^1_z$ is a new meromorphic function.

For Ex. 3.14, Prop. 3.15, produces $X^{[2]}$ analytically isomorphic to $\mathbb{C}_w$. We already knew this was a manifold. The proof of Props. 3.12 and 3.15 simplifies because $\infty \in \boldsymbol{z}$. The following lemma removes that assumption [9.1b].

Lemma 4.3. *Let $U_i \subset \mathbb{P}^1_z$, $i = 1, 2$ be domains. Let $\varphi : X \to U_1 \cup U_2$ denote projection of a manifold for an algebraic function onto the z coordinate. With $\alpha_i \in \mathrm{PGL}_2(\mathbb{C})$, $i = 1, 2$, assume $\alpha_i \circ \varphi_{U_i} : \varphi^{-1}(U_i) \to \alpha_i^{-1}(U_i)$ is a manifold from the construction of Prop. 3.15, $i = 1, 2$. Then, X is a complex manifold extending the manifold structure on $\varphi^{-1}(U_i)$.*

Assume X is a manifold from Prop. 3.15. Let $\varphi : X \to U \subset \mathbb{P}^1_z$ be the algebraic function giving projection onto the z coordinate. Riemann's Existence Theorem (Chap. 4) produces a unique compact complex manifold $\bar{X}$ containing X as an open subset. We do this by extending φ to an analytic map $\bar{\varphi} : \bar{X} \to \mathbb{P}^1_z$. This is an abstract approach to *compactification*. It will help to see preliminary examples that relate compactifications and coordinates. In §4.2 we give these.

4.2. Compactifications and fiber products. Continue the notation for m and its branch points $\boldsymbol{z}$ from §3.3. Denote

$$\{w' \in \mathbb{P}^1_w \mid (z', w') \in X^{[0]},\ z' \notin \boldsymbol{z}\} \text{ by } U_{\mathrm{pr}_z^{-1}(\boldsymbol{z})}.$$

To further compactify we might embed the subset $X^{[0]}$ of $U_{\boldsymbol{z}} \times U_{\mathrm{pr}_z^{-1}(\boldsymbol{z})}$ into a compact space Z; then take the closure X of $X^{[0]}$ in Z. (Or apply to the already extended spaces $X^{[1]}$ or $X^{[2]}$.) As a closed subspace of compact space, X is compact.

4.2.1. *Local holomorphic functions from equations.* We note especially that equations give more than an (*implicit*) description of a point set. Using the implicit function theorem, they often give local parametrizing functions. In this section we use spaces Z to compactify that give natural local equations around points of the closure of $X^{[0]}$. Such equations help decide which points of the closure have extensions to the analytic structure on $X^{[0]}$ (or just manifold structure). This is an aspect of saying such Z provide *global coordinates*.

We need a notation for holomorphic functions compatible with §1.3 for the Laurent field $\mathcal{L}_{z'}$. We use $\mathcal{L}^h_{z'}$ for the ring of functions, with each holomorphic in some disk (dependent on the function) about z': power series $\sum_{n=0}^{\infty} a_n(z-z')^n$, convergent in some neighborhood of z'. For a general space X and point $x \in X$, the notation would be $\mathcal{L}^h_{X,x}$. For the holomorphic elements of $\mathcal{P}_{z',e}$ use $\mathcal{P}^h_{z',e}$.

We've been giving examples of point sets $\{(z,w) \mid m(z,w)=0\}$ in $\mathbb{C}\times\mathbb{C}$ using just one equation. Defining algebraic functions $f(z_1,\dots,z_n)$ in several variables is easy: Consider $X_m = \{(z_1,\dots,z_n,w) \mid m(z_1,\dots,z_n,w)=0\}$, and we say m algebraically defines $f(z_1,\dots,z_n)$, holomorphic in the variables $z_1,\dots,z_n$, if $m(z_1,\dots,z_n,f(z_1,\dots,z_n)) \equiv 0$. Also, the notation above extends to consider $\mathcal{L}^h_{z'_1,\dots,z'_n}$. Suppose $m(z'_1,\dots,z'_n,w')=0$, for $(z'_1,\dots,z'_n,w') \in \mathbb{C}^{n+1}$. Assume also that m defines $f(z_1,\dots,z_n)$ algebraically, and $f(z'_1,\dots,z'_n)=w'$. Then, we say the local holomorphic (or analytic) functions around $(z'_1,\dots,z'_n,w')$ consists of elements of the ring $\mathcal{L}^h_{z'_1,\dots,z'_n}$. This ring is invariant under analytic change of variables.

The next definition extends this to consider local holomorphic functions even with no a priori algebraic function f satisfying m. Recall the residue class map $\mathrm{rc}_{z'_1,\dots,z'_n,w'} : \mathbb{C}[z_1,\dots,z_n,w] \to \mathbb{C}$ by $(z_1,\dots,z_n,w) \mapsto (z'_1,\dots,z'_n,w')$. This is a ring homomorphism, and we record this in the form of the following. The completion of the ring $\mathbb{C}[z_1,\dots,z_n,w]/(m)$ at $(z'_1,\dots,z'_n)$ is

$$\mathcal{L}^h_{z'_1,\dots,z'_n}[z_1,\dots,z_n,w]/(m(z_1,\dots,z_n,w)) \stackrel{\text{def}}{=} \mathcal{L}^h_{X_m,z'_1,\dots,z'_n}.$$

DEFINITION 4.4. Analytic functions on X_m around $(z'_1,\dots,z'_n,w')$ are elements of the localization of $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n}$ at $w=w'$:

$$\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'} \stackrel{\text{def}}{=} \{u/v \mid u \in \mathcal{L}^h_{X_m,z'_1,\dots,z'_n},\quad v \in \mathbb{C}[z_1,\dots,z_n,w], \\ \text{with } u(z'_1,\dots,z'_n,w') \neq 0.$$

LEMMA 4.5. *If $z'_1,\dots,z'_n,w'$ is on X_m, then $\mathrm{rc}_{z'_1,\dots,z'_n,w'}$ factors naturally through $\mathbb{C}[z_1,\dots,z_n,w]/(m)$ and even through $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}$. This defines the value of $s \in \mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}$ at $(z'_1,\dots,z'_n,w')$ as $\mathrm{rc}_(s)$. Suppose the leading coefficient of $m(z,w)$ is invertible in $\mathcal{L}^h_{z'_1,\dots,z'_n}$ and W_m is the set of distinct solutions $w'=w$ of $m(z'_1,\dots,z'_n,w)=0$ (the case of multiple zeros being our emphasis). Then, there is a natural injective homomorphism*

$$\mathcal{L}^h_{X_m,z'_1,\dots,z'_n} \to \oplus_{w'\in W_m} \mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}.$$

DEFINITION 4.6 (Local holomorphic functions). Suppose $m(z'_1,\dots,z'_n,w') = 0$, for $(z'_1,\dots,z'_n,w') \in \mathbb{C}^{n+1}$ and there are but finitely many solutions w to $m(z'_1,\dots,z'_n,w')=0$. Then, the local holomorphic (or analytic) functions that m defines consist of elements of $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}[z_1,\dots,z_n,w]/(m(z_1,\dots,z_n,w)) = R$. We say this defines a manifold neighborhood if R is isomorphic to the convergent power series around a point of $\mathbb{C}^n$.

It is appropriate to say R is the *restriction* of local holomorphic functions on $\mathbb{C}^{n+1}$ to the set X_m around $(z'_1,\dots,z'_n,w')$. Further, the definition works as well if several equations, $m_1,\dots,m_u$, instead of just one, define the set.

4.2.2. $\mathbb{P}^1_z \times \mathbb{P}^1_w$ *compactification.* Since $Z = \mathbb{P}^1_z \times \mathbb{P}^1_w$ is a product of compact spaces, it is compact. Further, the compactification of $X^{[0]}$, if it is a manifold, suits the definition for $\mathbb{P}^1$-algebraic in (3.3).

The natural manifold structure on Z has four open sets in its atlas following Ex. 3.2.1. Label these $U_{i,z} \times U_{j,w}$, $1 \le i,j \le 2$: $U_{1,z} = \mathbb{C}_z$ and $U_{2,z} = \mathbb{C}^*_z \cup \{\infty\}$, etc. The atlas gives an isomorphism of each of the four opens sets $U_{i,z} \times U_{j,w}$ with $\mathbb{C}\times\mathbb{C}$, by a map we call $\varphi_{i,j}$. Let $\bar{X}$ be the closure of $X^{[0]}$ in Z. We describe the part of $\bar{X}$ lying inside $U_{i,z} \times U_{j,w}$ by an algebraic equation. Then a previous procedure allows checking points at which X has a manifold structure.

Start with $\bar{X} \cap U_{2,z} \times U_{2,w}$, and leave the other open sets as analogous. On the open subset $\mathbb{C}^* \times \mathbb{C}^* \subset U_{2,z} \times U_{2,w}$, $\varphi_{2,2}$ acts as

$$(z,w) \mapsto (1/z, 1/w) = (z', w').$$

An equation in (z',w') describes $\varphi_{2,2}$ applied to $X \cap (U_{2,z} \times U_{2,w}) = X_{2,2}$: $\varphi_{2,2}(X_{2,2})$ is the closure of $\{(z',w') \mid m(1/z',1/w') = 0\}$ in $\mathbb{C}_{z'} \times \mathbb{C}_{w'}$. Get the closure points by allowing z' or w' to go to 0. To include those limit values, multiply $m(1/z',1/w')$ by the minimal powers of z' and w' to clear the denominators.

EXAMPLE 4.7 (Continuation of Ex. 3.14). Continue with $m(z,w) = h(w) - z$ and $\deg(h) = n$. The set $\varphi_{2,2}(X_{2,2})$ is $\{(z',w') \mid z'h^*(w') - (w')^n = 0\}$ where $h^*(w') = h(1/w')(w')^n$. Check that $X_{1,2}$ and $X_{2,1}$ have no new points beyond those already in $X_{1,1}$. Still, $X_{2,2}$ has a new point, corresponding to $(z',w') = (0,0)$. The gradient of $z'h^*(w') - (w')^n$ at zero is $(h^*(0),0) \ne (0,0)$. So, there is a manifold neighborhood of this point [9.10a].

4.2.3. *Tensor products and fiber products of* $\mathbb{P}^1$ *covers.* We combine two cases of Ex. 4.7. Suppose $m(z,w) = h(w) - g(z)$, a *variables separated* equation. Rename z to a variable w', and use z for the value $h(w)$. Rewrite $m(z,w)$ as $m(w',w)$.

Consider $(w',w) \in \mathbb{C}_{w'} \times \mathbb{C}_w$ satisfying $m(w',w) = 0$. Call this X_m. Denote the Riemann surface for a function $w'(z)$ (resp. $w(z)$, as in Ex. 4.7) of z satisfying $h(w'(z)) \equiv z$ (resp. $g(w) = z$) by $X_{w'}$ (resp. X_w). There is a map $\varphi_{w'} : X_{w'} \to \mathbb{P}^1_z$ by $w' \mapsto h(w') = z$. Similarly for a map φ_w.

Compare with Def. 1.3: X_m as a set is the same as the fiber product of these two maps. Now apply the $\mathbb{P}^1_{w'} \times \mathbb{P}^1_w$ compactification to $m(w',w)$. The resulting set is $\bar{X}_{w'} \times_{\mathbb{P}^1_z} \bar{X}_w = \bar{X}_m$. (In our example, $\bar{X}_{w'} = \mathbb{P}^1_{w'}$ and $\bar{X}_w = \mathbb{P}^1_w$.) This is the fiber product (over $\mathbb{P}^1_z$) of the compactifications of $X_{w'}$ and X_w from Ex. 4.7.

Now consider points of $\bar{X}_m$ to decide what are the natural local analytic functions in a neighborhood within one of the four charts for $\mathbb{P}^1 \times \mathbb{P}^1$:

$$X_{i,j} = U_{i,z} \times U_{j,w},\ 1 \le i,j \le 2. \tag{4.2}$$

For $(w'_0, w_0) \in \bar{X}_m$. Let $e_{w'_0}$ (resp. e_{w_0}) be the ramification index (Chap. 2 Def. 7.6) of w'_0 over $h(w'_0) = z_0$ (resp. w_0 over $g(w_0) = z_0$). New cases are with $e_{w'_0} = e' > 1$ *and* $e_{w_0} = e > 1$.

Local holomorphic functions in a neighborhood of (w'_0, w_0) that come from the coordinates w' and w are analytic in the solutions w' of $h(w') = z$ expanded about w'_0 and in the solutions w of $g(w) = z$ expanded about w_0. As usual, use ζ_d for the complex number $e^{2\pi i/d}$. Assume R is a ring, and S_1 and S_2 are two R algebras.

Then the tensor product $S_1 \otimes_R S_2$ is the natural direct sum of R algebras. That is, it is an R algebra T with R algebra homomorphisms $\psi_i : S_i \to T$, $i = 1, 2$ ($\psi_1 : s_1 \in S_1 \mapsto s_1 \otimes 1$, etc.) and any such homomorphism will naturally factor through the map to $S_1 \otimes_R S_2$. As in Chap. 2 Cor. 7.5: $[e_1, e_2]$ is the least common multiple of e_1 and e_2; $u(z) = (z - z')^{1/[e_1,e_2]}$ is a choice of $[e_1, e_2]$th root of $z - z'$ (a generator of $\mathcal{P}_{z',[e_1,e_2]}$); and $\zeta_d = e^{2\pi i/d}$. Our first lemma is a famous consequence of the Euclidean algorithm.

LEMMA 4.8. *Assume K is a characteristic 0 field and $f \in K[x]$ is $\prod_{i=1}^{u} g_i(x)^{r_i}$ with $g_1, \dots, g_u$ irreducible and distinct monic polynomials over K. Then the natural map $\mu : K[x]/(f(x)) \to \oplus_{i=1}^{u} K[x]/(g_i^{e_i})$ by $h(x) \mapsto (h \mod (g_i^{e_1}), \dots, h \mod (g_i^{e_1})$ is an isomorphism.*

PROOF. Check that the kernel of μ trivial. So this linear vector space map, injects a space of dimension $\deg(f)$ into one of the same dimension $\sum_{i=1}^{u} e_i \deg(g_i)$. Conclude: μ is onto. □

PROPOSITION 4.9. *Suppose U is an open subset of $\mathbb{P}^1_z$, and $\varphi : X \to U$ is an analytic map of Riemann surfaces. For x' over $\varphi(x') = z'$ with ramification index $e_{x'/z'} = e$, $\mathcal{L}^h_{X,x'}$ is a natural $\mathcal{L}^h_{\mathbb{P}^1_z,z'}$ algebra that identifies with $\mathcal{P}^h_{z',e}$.*

Let $\varphi_i : X_i \to U$ be two such maps, with $x'_i \in X_i$ over z' having ramification index e_i, $i = 1, 2$. Let $d = (e_1, e_2)$. Then, the ring of local holomorphic functions about (x_1, x_2) on $X_1 \times_{\mathbb{P}^1_z} X_2 = Y$ is $\mathcal{L}^h_{X_1,x'_1} \otimes_{\mathcal{L}^h_{\mathbb{P}^1_z} z'} \mathcal{L}^h_{X_2,x'_2}$. So $u^{e_1/d} = u_2$ (resp. $u^{e_2/d} = u_1$) is an e_2th (e_1th) root of $(z - z')$. Then, $\mathcal{L}^h_{Y,(x_1,x_2)}$ naturally identifies with $\mathcal{L}^h_{\mathbb{P}^1_z,z'}[u_1 \otimes 1, 1 \otimes u_2] = R$ (with $(u_1 \otimes 1)^{e_1} = z \otimes 1 = (1 \otimes u_2)^{e_2}$ according to the rules of tensoring over $\mathcal{L}^h_{\mathbb{P}^1_z} z'$). This ring has a single maximal ideal. There is an injective homomorphism

$$\mu : \mathcal{L}^h_{\mathbb{P}^1_z,z'}[u_1 \otimes 1, 1 \otimes u_2] \to \oplus_{j+1}^{d} \mathcal{L}^h_{\mathbb{P}^1_z,z'}[x, y]/(x^{e_1/d} - \zeta_d^j y^{e_2/d}, z = y^{e_2})$$

by $u_1 \otimes 1 \mapsto x$ and $1 \otimes u_2 \mapsto y$ in each coordinate. Each summand on the right of (4.9) *is an integral domain whose quotient field naturally identifies with $\mathcal{P}_{z',[e_1,e_2]}$.*

Then, R is an integral domain if and only if $d = 1$, and the image of μ in each summand is a proper subring of the summand unless one of e_i/d is 1. Conclude: Restricting local holomorphic functions on $X_1 \times X_2$ defines an analytic manifold structure around (x'_1, x'_2) if and only if one of the e_i s is 1. Yet, the image of μ generates the quotient field of each summand.

PROOF. According to Def. 4.1, by rewriting φ using local analytic coordinates $z_{x'}$ and $z_{z'}$ around x' and z', we get a very simple normal form. A local analytic change of variables identifies $z_{x'}$ with one of the solutions of $u^e = z_{z'}$. Chap. 2 Cor. 7.5 shows this when φ is given by an algebraic function. Chap. 4 (proof of Lem. 2.1) shows it is not dependent on a priori knowing φ is algebraic. That gives the first paragraph in the lemma.

Now consider φ_i, $i = 1, 2$, in the statement of the lemma. From above, identify an analytic coordinate around $x_i(z)$ around x'_i with $(z - z')^{1/e_i}$ and the map φ_i with the e_ith power map, $i = 1, 2$. The only relations among $u_1 \otimes 1$ and $1 \otimes u_2$ are generated by $(u_1 \otimes 1)^{e_1} = z \otimes 1 = (1 \otimes u_2)^{e_2}$ and the kernel of the map μ is in the ideal generated by this relation.

If $d > 1$, then $(u_1 \otimes 1)^{e_1/d} - \zeta_d^j (1 \otimes u_2)^{e_2/d}$ divides $(u_1 \otimes 1)^{e_1} - (1 \otimes u_2)^{e_2} = z$.

Replace $\mathcal{L}^h_{\mathbb{P}^1_z,z'}$ by $\mathcal{L}_{z'}(y) = K$, a field (leaving x as a variable). Then applying Lem. 4.8 to μ actually gives an isomorphism. The corresponding summands on the right side of (4.9) would be fields identified with the quotient fields of the summands on the right side of the actual (4.9). So, to finish the result we have only to show the quotient field of the summand $\mathcal{L}^h_{\mathbb{P}^1_z,z'}[x,y]/(x^{e_1/d} - \zeta_d^j y^{e_2/d}, z = y^{e_2})$ identifies with $\mathcal{P}_{z',[e_1,e_2]}$, though the summand itself is a proper subring of the locally holomorphic functions in $(z-z')^{1/[e_1,e_2]}$ [9.11b]. □

Now apply Prop. 4.9 to (4.2).

COROLLARY 4.10. *Restricting local holomorphic functions on* $\mathbb{P}^1_{w'} \times \mathbb{P}^1_w$ *to the fiber product* $\mathbb{P}^1_{w'} \times_{\mathbb{P}^1_z} \mathbb{P}^1_w$ *compactification gives an analytic manifold structure around* (w'_0, w_0) *if and only if* $(e'_{w_0}, e_{w_0}) = 1$.

REMARK 4.11 (simplifying the use of Prop. 4.9). Riemann's Existence Theorem gives a unique compact manifold by completing a cover of $U_{\boldsymbol{z}}$. In so doing, it computes precisely what to expect when you take the fiber product of two ramified covers of $\mathbb{P}^1_z$ (over of any other Riemann surface). Chap. 4 §3.4 shows the combinatorial result of getting d distinct points on the correctly compactified fiber product (ramified of order $[e_1, e_2]$ over z') over the pair (x'_1, x'_2) is built transparently into the use of branch cycles. Since, however, fiber products (and tensor products) are so important, Prop. 4.9 gives a relatively simple example readers may return to for help with other examples.

4.3. $\mathbb{P}^n$ compactifications. Denote the origin in $\mathbb{C}^{n+1}$ by 0. There is an action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Given a nonzero vector $\boldsymbol{v} = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$ and $\alpha \in \mathbb{C}^*$ form the result of scalar multiplication $\alpha \cdot \boldsymbol{v} = (\alpha v_0, \dots, \alpha v_n)$. *Projective n-space* is a quotient definition like that of a complex torus: $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$. Mapping $\boldsymbol{v}$ to the set equivalent to $\boldsymbol{v}$ gives $\Gamma_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

4.3.1. *An atlas on* $\mathbb{P}^n$. In this form, it can be convenient (though cumbersome) to label $\mathbb{P}^n$ as either $\mathbb{P}^n_{v_1/v_0,\dots,v_n/v_0}$ (*inhomogeneous* coordinates) or $\mathbb{P}^n_{v_0,\dots,v_n}$ (*homogenous* coordinates). The extra notation means we have added data for a standard set of coordinate functions for $\mathbb{P}^n$. Algebraic geometry texts might refer to a manifold analytically isomorphic to this manifold as $\mathbb{P}^n$. Still, there is a significance to adding specific coordinates as Chap. 5 does. To practice this distinction try [9.11e]. Taking $n = 1$ and $v_1/v_0 = z$ gives the notation for $\mathbb{P}^1_z$ from Chap. 2.

Standard coordinates on $\mathbb{P}^n$ produce standard transition functions for its manifold structure. Typical of forming an object by an equivalence relation, each point of $\mathbb{P}^n$ is a set in $\mathbb{C}^{n+1}$. As some coordinate is not 0, such a point has a *representative* with some coordinate equal 1. If you tell which coordinate that is, the representative will be unique.

Let U_i be the points with representative having 1 in the ith position. Each point of $\mathbb{P}^n$ has a representative in U_i for some i. Projecting U_i onto coordinates different from the ith gives a coordinate chart $\varphi_i : U_i \to \mathbb{C}^n$, $i = 0, \dots, n$. If $\boldsymbol{v}$ is any other representative of a point in U_i, first scale it by $1/v_i$ before this projection.

LEMMA 4.12. *The atlas* $\{U_i, \varphi_i\}_{i=1}^n$ *makes* $\mathbb{P}^n$ *a compact dimension n complex manifold. The map* $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ *is a map of analytic manifolds.*

PROOF. An explicit computation of the transition function

$$\varphi_j \circ \varphi_i^{-1} : \mathbb{C}^n_{v_0,v_1,\dots,v_{i-1},v_{i+1},\dots,v_n} \to \mathbb{C}^n_{v_0,v_1,\dots,v_{j-1},v_{j+1},\dots,v_n}$$

is easy. If $i = j$ it is the indentity. Otherwise, it maps $(v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ to $1/v_j(v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n)$ (with $v_i = 1$). It is analytic on $\varphi_i(U_i \cap U_j)$.

To see $\mathbb{P}^n$ is compact, use the standard absolute value $|v|$ on $\mathbb{C}$. Let $\mathbb{C}_c^{n+1}$ be the vectors $\boldsymbol{v}$ with $\max_{i=0}^n(|v_i|) \le 1$. This is a closed bounded subset of $\mathbb{C}^{n+1}$. So, by the Heine-Borel compactness theorem, it is compact. Every point of $\mathbb{P}^n$ has a representative in $\mathbb{C}_c^{n+1}$: Scale it by the largest nonzero entry. Now use that the image of a compact set under a continuous map is compact. An alternate could use this characterization of compactness: Infinite sequences of points in a separable metric space have convergent subsequences [9.10b].

The diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ is the image of a compact subset of the diagonal in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. Though the image is compact, until we know $\mathbb{P}^n$ is Hausdorff we can't invoke Lem. 2.5 to see the image is closed. Here, however, a direct argument can establish that $\mathbb{P}^n$ is Hausdorff. Suppose two points are in one of the U_i s, a copy of $\mathbb{C}^n$. As this is Hausdorff, separate the two points by open sets. So, given any two points it suffices to change coordinates to assure, in the new coordinates, these are both in one of the U_i s. Do that choosing a linear combination $L_{\boldsymbol{a}} = \sum_{i=0}^n a_i v_i$ so neither point lies on the zero set of $L_{\boldsymbol{a}}$. Use $L_{\boldsymbol{a}}$ in place of v_j as one of the new coordinates for any j for which $a_j \neq 0$.

Use $\Gamma_n^{-1}(U_i) = V_i \subset \mathbb{C}^{n+1}$ and the same transition functions for a coordinate chart on $\mathbb{C}^{n+1}$. This shows Γ_n is a map of complex manifolds. □

4.3.2. $\mathbb{P}^2_{z,w,u}$ *compactifications.* As in §4.2.2, let $Z' = \mathbb{P}^2_{z,w,u}$. Embed $\mathbb{C}_z \times \mathbb{C}_w$ in this by $\varphi_u^{-1} : (z, w) \mapsto (z, w, 1) \mod \mathbb{C}^* \in Z'$. Call the image U_u. Similarly, let U_w be points of $\mathbb{P}^2_{z,w,u}$ with a representative of form $(z, 1, u)$ and U_z points with a representative of form $(1, w, u)$. Take X' to be the closure of $\{(z, w) \mid m(z, w) = 0\}$ in the compact space Z'. To check points of X' for a manifold neighborhood requires an equation around each point of X'. It suffices to *define* this equation for points of $X' \cap U_z$ and $X' \cap U_w$. We do the former; the latter is similar.

Since φ_z identifies U_z with $\mathbb{C}_w \times \mathbb{C}_u$, it suffices to define the image of $X' \cap U_z$ under φ_z. With n' the total degree of m, it is

$$X'_z = \{(w, u) \mid u^{n'} m(1/u, w/u).$$

4.3.3. *Hyperelliptic curves.* Suppose $\varphi : X \to \mathbb{P}^1_z$ is a degree 2 map of compact Riemann surfaces. Let $\boldsymbol{z}$ be the finite set of branch points (as in Chap. 4 Lem. 2.1). The theme of Chap. 2 §8 is that we already know, from branches of log, what are the abelian covers of $U_{\boldsymbol{z}} = U$ (see Chap. 4 Prop. 2.10). That is, $\pi_U : X_U \to U_{\boldsymbol{z}}$ is equivalent to the cover defined by a branch of square root of $h(z) \in \mathbb{C}(z)$. Also, h has multiplicity one zeros and poles contained in $\boldsymbol{z}$ (Chap. 2 (8.1)): φ is a cover from a branch of solutions $f(z)$ of $m(z, w) = w^2 - h(z)$ with $h(z) = \frac{\prod_{i=1}^t (z - z_i)}{\prod_{j=t+1}^r (z - z_j)}$.

Suppose the z_i s are distinct, and all different from 0 or ∞ ($r = 2t$ so the degrees of the numerator and denominator are the same). Then, according to Prop. 4.9, this is an if and only if condition that for a manifold compactification given by the fiber product embedding in $\mathrm{pr}^1_z \times \mathbb{P}^1_w$. This is good, yet the standard normalization of hyperelliptic curves changes the variables so that h is a polynomial. Do this by multiplying both sides by the square of the denominator, then change the variable w to $w \prod_{j=t+1}^r (z - z_j)$. For simplicity we keep the name of the variables the same. So, now consider the equation $wr = h(z)$ where $h = \prod_{i=1}^r (z - z_i)$. Here r is even, and we assume it is at least 4. Another common normalization is make the changes

$z \mapsto z_1 + 1/z$ and $w \mapsto w/z$, thereby replacing h by a polynomial having odd degree $r \geq 3$. As it stands let us consider the $\mathbb{P}^2_{z,w,u}$ compactification.

Then, $X'_u = \{(z,w) \mid w^2 - h(z) = 0\}$ has a manifold neighborhood around each point: $\nabla(m) = 0$ implies $w = 0$ and $\frac{dh}{dz} = 0$ (z is a repeated root of h). From above,

$$
\begin{aligned}
X'_w =& \quad \{(z,u) \mid u^{n-2} - u^n h(z/u) = m^{(w)}(z,u) = 0\} \text{ and} \\
X'_z =& \quad \{(w,u) \mid u^{n-2}w^2 - u^n h(1/u) = m^{(z)}(w,u) = 0\}.
\end{aligned} \tag{4.3}
$$

On X'_w new points (not already represented on X'_u) have $u = 0$ and $z = 0$. For $r > 3$, $\nabla(m^{(w)})(0,0) = 0$. So, it has no manifold neighborhood. Note this contrasts with the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification of m, in which all points have manifold neighborhoods when you use the right algebraic change of coordinates [9.11c]. For $r = 3$, however, the point $(0,0)$ has a manifold neighborhood in $\mathbb{P}^2$. There are no new points on X'_z; $u = 0$ gives no solution in w to $m^{(z)}(w,u) = 0$.

4.3.4. *Coordinates give meromorphic functions.* Let $\bar{X}$ be the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification (§4.2.2) of $X = \{(z,w) \mid m(z,w) = 0\}$ with $m \in \mathbb{C}[z,w]$. Assume every point of $\bar{X}$ has a manifold neighborhood in this compactification. Then, every point of $\bar{X}$ has the form $(z,w) \in \mathbb{P}^1_z \times \mathbb{P}^1_w$. Thus, projection of (z,w) onto z (or onto w) provides a meromorphic function on $\bar{X}$.

Similarly, suppose $\bar{X}$ is the $\mathbb{P}^2_{z,w,u}$ compactification (§4.3.2) of X and every point of $\bar{X}$ has manifold neighborhood. Then, many meromorphic functions come from this compactification. A linear form in (z,w,u) is a nonzero linear combination of z, w, u (like $L_{\boldsymbol{a}}$, used in the proof of Lem. 4.12). Assume $\bar{X}$ is not in the zero set of any linear form. For example, suppose $m(z,w)$ is irreducible and has total degree $n > 1$.

Proposition 4.13. *Let L_1 and L_2 be linear forms in (z,w,u), not multiples of one another. Let (z_0,w_0,u_0) represent the unique point of intersection of the zero sets of L_1 and L_2. Then, with $z' = L_1(z,w,u)/L_2(z,w,u)$, there is a natural (nonconstant) meromorphic function $\bar{\varphi} : \bar{X} \to \mathbb{P}^1_{z'}$. The degree of $\bar{\varphi}$ is n if $(z_0,w_0,u_0) \notin \bar{X}$ and $n-1$ otherwise.*

Proof. Give the map by $(z,w,u) \in \bar{X} \mapsto L_1(z,w,u)/L_2(z,w,u)$. We verify this map is well-defined. If $(z_0,w_0,u_0) \notin \bar{X}$, then meaningfully assign a value $z' \in \mathbb{C} \cup \{\infty\}$ to the evaluation of L_1/L_2 at any point of $\bar{X}$. Let $H_{z'_0}$ be the line in $\mathbb{P}^2$ given as the zero set of $L_{z'_0} = L_1 - z'_0 L_2$. To see the degree, check the number of points in the intersection of $H_{z'_0}$ and $\bar{X}$ if z'_0 is suitably general. This is n. These are exactly the points that go to z'_0.

On the other hand, suppose $(z_0,w_0,u_0) \in \bar{X}$. Then each $H_{z'_0}$ goes through (z_0,w_0,u_0). If z'_0 is general, $L_1(z,w,u)/L_2(z,w,u)$ has a clear ratio value at the $n-1$ points other than (z_0,w_0,u_0). So, this gives a map of degree $n-1$ of $\bar{X} \to \mathbb{P}^1_{z'}$. Check: For only one value z'_0 is $H_{z'_0}$ tangent to $\bar{X}$ at (z_0,w_0,u_0) because we assumed $\bar{X}$ is nonsingular [9.11f]. Interpret such a z'_0 as having (z_0,w_0,u_0) above it. □

5. Paths, vectors and forms

Notation for paths started in Chap. 2 §2.2. Let X be a topological space. A path in X is a continuous $\gamma : [a,b] \to X$ for some choice of a and b with $a < b$. The points $\gamma(a)$ and $\gamma(b)$ are, respectively, the *initial* and *end* points of the path. The path γ is *closed* if $\gamma(a) = \gamma(b)$.

The idea a path being piecewise differentiable (simplicial) works if X is an n-dimensional differentiable manifold (or, more generally, a finite union of differentiable manifolds), with topologizing data $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$. Then, γ is differentiable if $\frac{d}{dt}(\varphi_\alpha \circ \gamma(t)) = \boldsymbol{v}_\alpha(t)$ exists for each $t \in [a, b]$ (use one-sided limits at the endpoints) and each $\alpha \in I$ with $\gamma(t) \in U_\alpha$. The vector $\boldsymbol{v}_\alpha(t)$ is the *tangent vector* to γ at t with respect to $(U_\alpha, \varphi_\alpha)$. It depends only on γ close to t.

As in Chap. 2, simplicial paths support applications to integration, and to forming convenient analytic continuations of functions. Still, it is awkward to analyze *homotopy classes* of paths without allowing paths that are only continuous in the homotopy (see Prop. 6.10).

5.1. Tangent vectors. The above formulation presents a tangent vector as something attached to a path. We recognize a tangent vector at a point x_0 without having a path through the point. Let $\mathcal{C}^\infty_{x_0} = \mathcal{C}_{x_0,X}$ be functions, differentiable and complex valued, defined in some neighborhood of x_0.

DEFINITION 5.1. A (complex valued) tangent vector to a differentiable manifold X at a point x_0 is a linear map $\boldsymbol{v} : C^\infty_{x_0} \to C^\infty_{x_0}$ satisfying Leiznitz's rule:

$$\boldsymbol{v}(f_1 f_2)(x_0) = \boldsymbol{v}(f_1)(x_0)f_2(x_0) + (f_1)(x_0)\boldsymbol{v}(f_2)(x_0). \tag{5.1}$$

That is, $\boldsymbol{v}$ is a derivation of $\mathcal{C}_{x_0}$ defined at x_0.

5.1.1. *Tangent vectors and paths.* To relate to tangent vectors attached to a path, assume $x_0 \in U_\alpha$. A function f in a neighborhood of x_0 defines a function $f \circ \varphi_\alpha^{-1}$ on a neighborhood of $\varphi_\alpha(x_0) \in \mathbb{R}^n$. Denote the variables of $\mathbb{R}^n$ here by $\boldsymbol{y} = (y_1, \dots, y_n)$. Consider $F : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{y} \mapsto (F_1(\boldsymbol{y}), \dots, F_n(\boldsymbol{y}))$. Suppose each coordinate function $F_i(\boldsymbol{y})$ has continuous partial derivatives. The *Jacobian matrix* $J(F)$ of F is the $n \times n$ matrix with (i, j)-entry $\frac{\partial F_i}{\partial y_j}$ at the point $\boldsymbol{y}$.

LEMMA 5.2. [**Rud76**, p. 214] *Identify derivations of functions* $f \in \mathcal{C}_{\varphi_\alpha(x_0),\mathbb{R}^n}$ *with linear combinations* $T_{\boldsymbol{v}} = \sum_{i=1}^n v_i \frac{\partial}{\partial y_i}$, $v_1, \dots, v_n \in \mathcal{C}_{\varphi_\alpha(x_0),\mathbb{R}^n}$.

So, $T_{\boldsymbol{v}}(f)(\varphi_\alpha(x_0))$ *is the directional derivative of* f *in the direction* $\boldsymbol{v}(\varphi_\alpha(x_0))$.

For $\gamma(t) \in U_\alpha \cap U_\beta$, *the chain rule relates* $\boldsymbol{v}_\alpha(t)$ *and* $\boldsymbol{v}_\beta(t)$:

$$\big(J(\varphi_\beta \circ \varphi_\alpha^{-1})_{|(\varphi_\alpha\circ\gamma)(t)}\big)(\boldsymbol{v}_\alpha(t)) = \boldsymbol{v}_\beta(t). \tag{5.2}$$

So, $\boldsymbol{v}_\alpha(t)$ *is nonzero if and only if* $\boldsymbol{v}_\beta(t)$ *is nonzero. To check if* γ *has a nonzero tangent vector doesn't depend on the choice of* $(U_\alpha, \varphi_\alpha)$.

5.1.2. *Vector fields.* A *vector field* T_U on an open set U in a (differentiable) manifold X is a differentiable assignment of derivations at each point of U. A formal definition shows the effect of transition functions from an atlas. Sometimes it is confusing to use $\boldsymbol{y}$ for variables of *all* copies of $\mathbb{R}^n$. So, we use $\boldsymbol{y}_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,n})$ for variables in the range of φ_α.

DEFINITION 5.3. Assume $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ is an atlas for the differentiable manifold X. Then, T_U consists of giving $T_\alpha = \sum_{i=1}^n f_{\alpha,i} \frac{\partial}{\partial y_{\alpha,i}}$ with the $f_{\alpha,i}$s differentiable functions on $V_\alpha = \varphi_\alpha(U_\alpha)$, for each $\alpha \in I$, subject to the following rule. Assume $U_\alpha \cap U_\beta$ is nonempty. Consider any differentiable function $f : U_\alpha \to \mathbb{R}^n$. Use the same notation T_α for the restriction of T_α to $\varphi_\alpha(U_\alpha \cap U_\beta)$. Here is a relation between T_α and T_β on $U_\alpha \cap U_\beta$:

$$T_\alpha(f \circ \varphi_\alpha^{-1}(y_{\alpha,1}, \dots, y_{\alpha,n})) = T_\beta(f \circ \varphi_\beta^{-1}(y_{\beta,1}, \dots, y_{\beta,n})). \tag{5.3}$$

Apply $(\frac{\partial}{\partial y_{\beta,1}},\dots,\frac{\partial}{\partial y_{\beta,n}})$ to $f\circ\varphi_\alpha^{-1}(\boldsymbol{y}_\beta)=f_\alpha(\boldsymbol{y}_\alpha)$ to get a *gradient* vector of $(f_{\alpha 1},\dots,f_{\alpha n})(\boldsymbol{y}_\alpha)$ functions. A traditional expression rewrites (5.3) as

$$J(\psi_{\boldsymbol{y}_\beta,\boldsymbol{y}_\alpha})^{-1}\big(\frac{\partial}{\partial y_{\alpha,1}},\dots,\frac{\partial}{\partial y_{\alpha,n}}\big)=\big(\frac{\partial}{\partial y_{\beta,1}},\dots,\frac{\partial}{\partial y_{\beta,n}}\big) \tag{5.4}$$

applied to $f(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$ [9.14c]. Thus, (5.3) translates to a linear relation between $(f_{\alpha,1},\dots,f_{\alpha,n})(\boldsymbol{y}_\alpha))$ and $(f_{\beta,i},\dots,f_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$ [9.14].

So, a chart produces a preferred basis for vector fields and a preferred basis for differential 1-forms from the coordinate functions for the chart.

Definition 5.4. As in Chap. 2 Def. 2.1, $\gamma:[a,b]\to X$ is *simplicial* if there is an integer n and $t_0=a<t_1<\cdots<t_{n-1}<t_n=b$ with $\gamma_{|_{[t_i,t_{i+1}]}}$ differentiable, $i=0,\dots,n-1$. Also, γ is *special simplicial* if either $\frac{d}{dt}(\gamma(t))$ is identically zero for $t\in(t_i,t_{i+1})$ or it is nonzero for *each* $t\in(t_i,t_{i+1})$, $i=0,\dots,n-1$. A space X is *simplicially connected* if, for each pair $x_0,x_1\in X$, there is a simplicial path $\gamma:[a,b]\to X$ with $\gamma(a)=x_0,\gamma(b)=x_1$.

Lemma 5.5 (Integrating vector fields). *Let T_U be a vector field on the open set U of the differentiable manifold X. For each $u_0\in U$ there exists $\epsilon>0$ and a unique differentiable path $\gamma:[-\epsilon,\epsilon]\to U$, with $\gamma(0)=u_0$, so the following holds. The derivation $T_{U,\gamma(t)}$ at $\gamma(t)$ is the directional derivative of γ at $t\in[-\epsilon,\epsilon]$.*

Proof. With no loss, assume u_0 is in an atlas element U_α. We summarize the meaning of the lemma using the previous notation $\varphi_\alpha:U_\alpha\to\mathbb{C}^n$.

Let $\boldsymbol{y}$ be coordinates on $\mathbb{R}^n\supset\varphi_\alpha(U_\alpha)$. Use the path $t\mapsto\varphi_\alpha\circ\gamma(t)=\gamma^*(t)$. By definition, T_{U_α} is an expression $\sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial y_i}$. The lemma says there is $\gamma^*(t)$ so $\frac{d\gamma_i^*}{dt}(t)=f_{\alpha,i}(\gamma^*(t))$, $i=1,\dots,n$.

Many books quote this result ([**Hi65**, p. 12], for example) by referring to the existence and uniqueness of solutions to ordinary differential equations. The path in U_α is then $\varphi_\alpha^{-1}(\gamma^*(t))$. All general proofs we've seen use fixed point arguments and involve considerable detail, as in the exercises of [**Rud76**, p. 118, #25–29, p. 170, #25-26] giving uniquess and existence under all conditions that would come up for us. Analytic dependence of the solutions on u_0 is considered more difficult (see [**Bo86**, p. 171-174, Thm. 4.1]). □

Suppose T_U is a vector field on U and $\gamma:[a,b]\to U$ is a differentiable path. Then, call γ an *integral* curve of T_U. With some assumptions there is a useful converse producing T_U from a path. [9.13].

5.2. Holomorphic vector fields and differential forms. Analogs of differentiable vector fields reflect the complex structure on a manifold X. The main example from Def. 5.3 has V_α as $T_\alpha=\sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial z_{\alpha,i}}$ with the $f_{\alpha,i}$ s holomorphic in the complex coordinates $z_{\alpha,i}$, $i=1,\dots,n$. Though T_α initially only applies to functions analytic in $(z_{\alpha,1},\dots,z_{\alpha,n})$, we may extend it to all differentiable functions taking complex values.

5.2.1. *Extend T differentiably.* Let $\boldsymbol{z}=(z_1,\dots,z_n)$ be the coordinate functions on $\mathbb{C}^n$. Write $z_j=x_j+iy_j$ and $\bar z_j=x_j-iy_j$, breaking the coordinates into their real and imaginary parts. Then, $x_j=\frac{1}{2}z_j+\bar z_j$ and $y_j=\frac{1}{2i}z_j-\bar z_j$. Define $\frac{\partial}{\partial z_j}$ on holomorphic functions $f(z_1,\dots,z_n)$ as the jth *partial* derivative with respect to the variables $z_1, \dots, z_n$. The partials $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ act on any differentiable functions of the variables $x_1, \dots, x_n, y_1, \dots, y_n$ (see Chap. 2 Lem. 2.6).

LEMMA 5.6. *The operator* $\frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ *maps* z_j *to 1,* z_k *to 0 for* $k \neq j$*. Further, it maps* $\bar{z}_l$ *to 0 for all* l*. So, it extends* $\frac{\partial}{\partial z_j}$ *to act as previously on holomorphic functions, and to kill anti-holomorphic functions. Similarly,* $\frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$ *extends* $\frac{\partial}{\partial \bar{z}_j}$ *from anti-holomorphic functions to all differentiable functions.*

5.2.2. *Vector fields in local coordinates.* Suppose T_α and T_β are the expressions for a holomorphic vector field on two coordinate charts. Interpret the relation between the $f_{\alpha,i}$ s and $f_{\beta,j}$ s given by the complex version of the Jacobian of the transition functions. So, for X a 1-dimensional complex manifold, the equation relating $f_\alpha(z_\alpha)\frac{\partial}{\partial z_\alpha}$ and $f_\beta(z_\beta)\frac{\partial}{\partial z_\beta}$ comes from expecting the same value upon application of both to $z_\beta = \psi_{\beta,\alpha}(z_\alpha)$:

$$f_\beta(\psi_{\beta,\alpha}(z_\alpha)) = f_\alpha(z_\alpha)\frac{\partial \psi_{\beta,\alpha}}{\partial z_\alpha}. \tag{5.5}$$

5.2.3. *Differential 1-forms.* Now consider the collection of differential 1-forms Ω_U defined on an open set U in a differentiable manifold X. Use notation of §5.1.2 analogous to that for vector fields. As in §Chap. 2 2.3 our motivation is to form integrals of $\omega_U \in \Omega_U$ along any piecewise differentiable path in U.

DEFINITION 5.7. Such an ω_U comes by giving $\omega_\alpha = \sum_{i=1}^n g_{\alpha,i}\, dy_{\alpha,i}$ with the $g_{\alpha,i}$ s differentiable functions on $V_\alpha = \varphi_\alpha(U_\alpha \cap U)$, for each $\alpha \in I$, subject to the following rule. If $V_\alpha \cap V_\beta$ is nonempty, denote restriction of ω_α to $\varphi_\alpha(V_\alpha \cap V_\beta)$ also by ω_α and let $\gamma : [a,b] \to V_\alpha \cap V_\beta$ be a differentiable path. Then,

$$\int_{\varphi_\alpha \circ \gamma} \omega_\alpha = \int_{\varphi_\beta \circ \gamma} \omega_\beta. \tag{5.6}$$

Equation (5.6) translates to a linear relation between $(g_{\alpha,1}, \dots, g_{\alpha,n})(\boldsymbol{y}_\alpha)$ and $(g_{\beta,i}, \dots, g_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$. This formula applies with $\gamma_{[t,t+\epsilon]}$ (restriction of γ to $[t, t+\epsilon]$) replacing γ for any value of $t \in [a,b]$ and $\epsilon > 0$. So, it gives equality of the integrands as a function of t.

DEFINITION 5.8 (Contraction). Suppose T_U is a vector field defined on U. Use the previous notation for expressing T_U on V_α: $T_\alpha = \sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial y_{\alpha,i}}$. The *contraction* of T_α and ω_α is the function $\sum_{i=1}^n f_{\alpha,i}g_{\alpha,i}$. Denote it by $\langle T_\alpha, \omega_\alpha\rangle$. More generally, the *contraction* $\langle T_U, \omega_U\rangle$ of T_U and ω_U is $F \in C_U^\infty$ with this property.

(5.7) $F \circ \varphi_\alpha^{-1}(\boldsymbol{y}_\alpha) = \langle T_\alpha, \omega_\alpha\rangle$ on $\varphi_\alpha(V_\alpha)$, for each $\alpha \in I$.

LEMMA 5.9. *As above,* $F \circ \varphi_\alpha^{-1}$ *at* $\varphi_\alpha(x)$ *does not depend on* α *and the contraction* $\langle T_U, \omega_U\rangle$ *is a differentiable function on* U*. Further, the vector of differentials* $(dy_{\beta,1}, \dots, dy_{\beta,n})$ *evaluated at* $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$ *is* $J(\psi_{\beta,\alpha})(dy_{\alpha,1}, \dots, dy_{\alpha,1})$*.*

PROOF. By explicit computation using Lemma 5.2, $f \circ \varphi_\alpha^{-1}$ is the integrand of the left of (5.6). The comment following (5.6) shows this equals the contraction for β evaluated at $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$. To conclude the proof use the vector field formula [9.14c]. Contract each side with the differentials $dy_{\beta,j}$ to see the transformation formula for differentials is inverse to that for vector fields. □

5.2.4. *Tensors.* Suppose $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ is an atlas for a differentiable manifold X. On each U_α let $\mathbb{T}^0_{U_\alpha}$ (resp. $\mathbb{D}^0_{U_\alpha}$) be the tensor algebra over $C^\infty(U_\alpha)$ generated by tangent vectors (resp. differential 1-forms) on U_α. By definition that means elements of $\mathbb{T}^0_{U_\alpha}$ are finite sums of terms $gT_1\otimes T_2\otimes\cdots\otimes T_k$ with k any nonnegative integer, $g\in C^\infty(U_\alpha)$ and $T_1,\dots,T_k$ tangent vectors on U_α. If $k=0$, the element is just the function g.

Suppose $h_1,h_2\in C^\infty$ and $T_i^{(1)}$ and $T_i^{(2)}$ are tangent vectors on U_α. Further, interpret the tensor sign $\otimes$ to be a formal symbol modulo the following relations. Replacing T_i by $h_1T_i^{(1)}+h_2T_i^{(2)}$ replaces $gT_1\otimes\cdots\otimes T_i\otimes\cdots\otimes T_k$ by the sum

$$gh_1T_1\otimes\cdots\otimes T_i^{(1)}\otimes\cdots\otimes T_k + gh_2T_1\otimes\cdots\otimes T_i^{(2)}\otimes\cdots\otimes T_k.$$

There are two things to note:

(5.8a) Unless it follows from these allowed relations, we do not expect $T_1\otimes T_2$ to equal $T_2\otimes T_1$.

(5.8b) Declaring $T_1\otimes\cdots\otimes T_k$ times $T_1'\otimes\cdots\otimes T_{k'}'$ (in that order) to be $T_1\otimes\cdots\otimes T_k\otimes T_1'\otimes\cdots\otimes T_{k'}'$ generates an associative ring multiplication on $\mathbb{T}^0_{U_\alpha}$.

Similarly for $\mathbb{D}^0(U_\alpha)$. Both have $C^\infty(U_\alpha)$ as a subring acting by multiplication on each element of $\mathbb{T}^0_{U_\alpha}$ (or $\mathbb{D}^0_{U_\alpha}$): These are *associate algebras* over $C^\infty(U_\alpha)$. We may even tensor together elements of $\mathbb{T}^0_{U_\alpha}$ and $\mathbb{D}^0_{U_\alpha}$ for a bigger algebra $\mathbb{T}^0_{U_\alpha}\otimes\mathbb{D}^0_{U_\alpha}$. In this convention, however, we can distinguish between tangent vectors and differential forms, and typically we pass all the tangent vectors to the left.

A subtlety occurs in comparing elements $\omega_\alpha\in\mathbb{T}^0_{U_\alpha}\otimes\mathbb{D}^0_{U_\alpha}$ and $\omega_\beta\in\mathbb{T}^0_{U_\alpha}\otimes\mathbb{D}^0_{U_\alpha}$ on the intersection $U_\alpha\cap U_\beta$. Use the transition function $\varphi_\beta\circ\varphi_\alpha^{-1}$ to reexpress ω_β in the variables $y_{\alpha,1},\dots,y_{\alpha,n}$ for $(U_\alpha,\varphi_\alpha)$ as previously for 1-forms (and vectors). Then, using the formal rules for $\otimes$, compare ω_α and ω_β upon their restriction to $U_\alpha\cap U_\beta$. Suppose the restriction of ω_α and ω_β (using the variables $y_{\alpha,1},\dots,y_{\alpha,n}$) are the same on $U_\alpha\cap U_\beta$. Then, we declare them together as forming a general element ω of the tensor algebra on $U_\alpha\cup U_\beta$. The subtlety is that ω likely will not be in $\mathbb{T}^0_{U_\alpha\cup U_\beta}\otimes\mathbb{D}^0_{U_\alpha\cup U_\beta}$. Drop the 0 superscript for a more general algebra.

Definition 5.10. The (mixed) tensor algebra $\mathbb{T}_X\otimes\mathbb{D}_X$ on X consists of collections $\omega_{\alpha_i}\in\mathbb{T}^0_{U_{\alpha_i}}\otimes\mathbb{D}^0_{U_{\alpha_i}}$, $i=1,\dots,t$, with $\cup_{i=1}^t U_{\alpha_i}=X$ and ω_{α_i} and ω_{α_j} restricting to equal elements in $\mathbb{T}^0_{U_{\alpha_i}\cap U_{\alpha_j}}\otimes\mathbb{D}^0_{U_{\alpha_i}\cap U_{\alpha_j}}$ for all allowed i and j.

Elements of $\mathbb{D}_X$ are *covariant* tensors. If everywhere locally $\omega\in\mathbb{D}_X$ is a sum of terms with each a tensor of exactly k differential 1-forms, then it is a k-covariant tensor. Generalize contraction (Def. 5.8) to define ω paired with k ordered tangent vectors $(T_1,\dots,T_k)$. Notice how this requires local expressions of ω as a sum of terms like $g\omega_1\otimes\cdots\otimes\omega_k$, with each ω_i a local differential 1-form. This contraction, $\langle(T_1,\dots,T_k),\omega\rangle$, is a global C^∞ function on X. For $\omega=g\omega_1\otimes\cdots\otimes\omega_k$ write it as as $g\prod_{i=1}^k\langle T_i,\omega_i\rangle$. Such an ω is *symmetric* if $\langle(T_1,\dots,T_k),\omega\rangle=\langle(T_{(1)\pi},\dots,T_{(k)\pi}),\omega\rangle$ for any permutation $\pi\in S_k$. It is *alternating* (or a *differential k-form*) if

$$\langle(T_1,\dots,T_k),\omega\rangle=\mathrm{Det}(\pi)\langle(T_{(1)\pi},\dots,T_{(k)\pi}),\omega\rangle\ \pi\in S_k\ (\S7.1.4).$$

5.2.5. *Orientation of a differentiable manifold.* A traditional and fuller treatment of the tensor algebra appears in texts on Riemannian geometry like [**Hi65**, Chap. 4]. Riemannian geometry starts with a differentiable manifold and a given symmetric 2-tensor furnished for measuring distances and angles [9.19]. From that tensor appear others for measuring other quantities on the manifold. For example, if on a differentiable 2-manifold we can measure distances along parametrized paths, then we should also be able to define the area of an open subset. The problem here is that you aren't likely to find a single parametrization by $\mathbb{R}^2$ of the whole area, and you must parametrize it in pieces, then add up the resulting areas. This forces the notion of *orientation*. The only 2-manifolds that have a well-defined area are orientable, which does include all Riemann surfaces Chap. 4 [10.9].

An orientation on a 2-dimensional differentiable manifold X consists of a rule for continuously assigning a left and right direction at the transversal meeting of two paths on the manifold. Precisely: Suppose given $\gamma^i : [-1,1] \to X$, $i = 1,2$, differentiable paths for which $x\gamma_i(0) = x \in X$, $i = 1,2$, and $(U_\alpha, \varphi_\alpha)$ is a coordinate chart containing x. So, we start with oriented 1-dimensional differential manifolds meeting at a point. Assume also that $\frac{\varphi_i \circ \gamma_i}{dt}(0) = \boldsymbol{v}_i$, $i = 1,2$, are distinct nonzero vectors. View a *traveler* as moving along $\varphi_\alpha \circ \gamma_1(t)$, facing at time $t = 0$ the direction $\boldsymbol{v}_1$ in $\mathbb{R}^2$ regarded as the (x,y) plane in $\mathbb{R}^3$. Then, the parametric line $L_0 = \{\varphi_\alpha \circ \gamma_1(0) + s\boldsymbol{v}_1 \mid s \in \mathbb{R}^1\}$ cuts the plane so that $\boldsymbol{v}_2$ points in the direction of the left half or the right half.

DEFINITION 5.11. Suppose there is a new $\{(V_\beta, \psi_\beta)\}_{\beta \in J}$ on X, compatible with the original atlas (usually taken as a subcollection of its coordinate charts) with this property. Independently of the choice of a coordinate chart in the new atlas containing x, the vector $\boldsymbol{v}_2$ lies consistently in the same half plane (left or right) defined by the corresponding L_0. Then, we say the new atlas defines an orientation at x. The atlas defines an orientation on X if it gives an orientation at each $x \in X$. Riemann surfaces are examples of oriented manifolds.

A generalizing definition inductively allows discussing an orientation of X defined by the oriented meeting of an oriented $n-1$ dimensional manifold meeting an oriented 1-dimensional manifold Chap. 4 [10.5c].

5.3. Meromorphic vector fields and differentials. The definition of vector fields and differential forms is formal. So for each chart, $(U_\alpha, \varphi_\alpha)$, it extends to objects of form $T_\alpha = \sum_{i=1}^n f_{\alpha,i} \frac{\partial}{\partial z_{\alpha,i}}$ or $\omega_\alpha = \sum_{i=1}^n f_{\alpha,i}\, dz_{\alpha,i}$ with the $f_{\alpha,i}$ s meromorphic in the complex coordinates $z_{\alpha,i}$, $i = 1, \dots, n$. Then, since the jacobian of transition functions (and its inverse) have holomorphic function entries, this assures it maps a vector of meromorphic functions to a vector of meromorphic functions.

EXAMPLE 5.12 (Differential of a meromorphic function). Suppose X is a Riemann surface (not necessarily compact) and $\psi : X \to \mathbb{P}^1_z$ is a (nonconstant) meromorphic function on X. We produce a meromorphic differential from ψ and an atlas $\mathcal{U}_X = \{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ for X. Define $d\psi_\alpha$ to be $\frac{d\psi \circ \varphi_\alpha^{-1}}{dz_\alpha}\, dz_\alpha$. Check: This is a differential form satisfying transformation formula (5.6).

Finally, let ω be a meromorphic differential 1-form on the Riemann surface X. Let $x_0 \in X$ lie in U_α where ω has the expression $f_\alpha(z_\alpha)\, dz_\alpha$. Suppose $\varphi_\alpha(x_0) = 0$. Then, the *order* m_{x_0} of ω at x_0 is the order of f_α at 0. Transition functions have neither zeros nor poles. So this order doesn't change if we compute it from another coordinate chart through φ_β with $x_0 \in U_\beta$.

5.3.1. *Divisors.* Conclude: For a given ω, the formal sum $\sum_{x \in X} m_x x$ has meaning. Denote it (ω) or D_ω depending on the notational context. It is the *divisor* of ω. Similarly, for any meromorphic function and meromorphic tangent vector on X we may define its divisor (f) or D_f. Call any formal sum $D = \sum_{x \in X} m_x x$ a divisor, and m_x is its *support multiplicity* at x.

LEMMA 5.13. *On a connected Riemann surface X, let D be the divisor of a nonconstant meromorphic differential, function or tangent vector. Then, the points of nonzero support multiplicity for D have no accumulation point. So, if X is also compact, divisors of nonconstant meromorphic differentials, functions or tangent vectors have only a finite number of nonzero support multiplicities.*

PROOF. We do the case for differentials. The others are similar. Suppose $(\omega) = \sum_{x\in X} m_x x$ is the divisor of a differential and infinitely many of the m_x are nonzero. Then, this set of xs has an accumulation point, x_0. Let $(U_\alpha, \varphi_\alpha)$ be a coordinate chart containing x_0, so the statement is that on $\varphi_\alpha(U_\alpha)$ we have a meromorphic differential $f_\alpha(z_\alpha)\,dz_\alpha$ having an accumulation of zeros or poles at $\varphi_\alpha(x_0) = z'_\alpha$. As in Chap. 2 [9.8a], this implies f_α is identically zero (or ∞) and using connectedness, that the same holds for the differential, contrary to our assumption (for extra help, see the argument of Chap. 4 Lem. 2.1). □

If X is not compact, divisors as in Lem. 5.13 may have infinitely many nonzero support terms (as with a holomorphic nonpolynomial function in the complex plane $\mathbb{C}_z$). In fact, the next general result in the complex plane has a similar version for any noncompact Riemann surface attached to an algebraic function [**Ahl79**, p. 195].

PROPOSITION 5.14 (Weierstrass factorization). *Suppose $\{m_{x_i}\}_{i\in I}$ is any collection of nonzero integers attached to a sequence of distinct points $\{x_i \in \mathbb{C}_z\}_{i\in I}$ with no accumulation point in $\mathbb{C}_z$. Then, there is a holomorphic function $f(z)$ with $(f) = \sum_{i\in I} m_{x_i} x_i$. Also, $f(z)\,dz$ (resp. $f(z)\frac{\partial}{\partial z}$) is a holomorphic differential (resp. vector field) with exactly the same divisor.*

Still, our tool will be the investigation of differentials, functions, etc., that extend meromorphically to a natural compactification of X. So, we typically assume (unless otherwise said) that $m_x = 0$ except for finitely many $x \in X$. For such a divisor D, the sum $\sum_{x\in X} m_x$ is the *degree* $\deg(D)$ of D. A divisor D is *positive* (or $D \geq 0$) if all its support multiplicities are nonnegative. This definition gives a partial ordering on divisors: With $D = \sum_{x\in X} m_x x$ and $D' = \sum_{x\in X} m'_x x$, $D \geq D'$ if $m_x \geq m'_x$ for each $x \in X$. Equivalently, with the obvious subtraction of divisors, $D - D'$ is positive.

Multiplying two functions or a function and a differential gives an object with divisor having the sum of the constituent multiplicities: $(f\omega) = (f) + (\omega)$.

DEFINITION 5.15. Suppose X is a compact Riemann surface. We say two divisors D_1 and D_2 on X are linearly equivalent if $D_2 - D_1 = (f)$ for some meromorphic function $f : X \to \mathbb{P}^1_z$. This is an equivalence relation between divisors.

Our notation for the linear equivalence class of a divisor D on a compact Riemann surface will be $[D]$. On a compact Riemann surface, the divisor of a meromorphic function has degree 0 (Chap. 4 Lem. 2.1; see Ex. 5.17). Anticipating that, conclude there is a well-defined degree attached to a linear equivalence class of divisors. Finally, we have a crucial definition attached to a divisor for which the reader should practice the notation.

DEFINITION 5.16. For any divisor D on a Riemann surface, the linear system of D, $L(D)$, is the collection of meromorphic functions f for which $(f) + D \geq 0$.

5.3.2. *Relation between functions and differentials.* As in Ex. 5.12, any (nonconstant) meromorphic function on a Riemann surface X provides us a nontrivial meromorphic differential form. Further, assume ω_1, ω_2 are meromorphic differentials and ω_1 is not a constant multiple of ω_2. This produces a nonconstant meromorphic function $\psi : X \to \mathbb{P}^1_z$ by the formula

$$\psi \circ \varphi_\alpha^{-1}(z_\alpha) = \omega_{\alpha,1}/\omega_{\alpha,2}. \tag{5.9}$$

So, all nonconstant differentials are linearly equivalent, and (see Def. 5.15), on a compact Riemann surface, all have the same degree.

EXAMPLE 5.17. Consider the identity map $z : \mathbb{P}^1_z \to \mathbb{P}^1_z$ by $z \mapsto z$. Carefully consider what is $dz = \omega$ using Ex. 3.2.1. To clarify notation, denote φ_1 by φ_α and φ_2 by $\varphi_{\alpha'}$. Then, $\varphi_\alpha : \mathbb{C}_z \to \mathbb{C}_{z_\alpha}$ by $z \mapsto z$, and so $\omega_\alpha = \frac{dz_\alpha}{dz_\alpha} dz_\alpha = dz_\alpha$. Also, $\varphi_{\alpha'} : \mathbb{C}^*_z \cup \{\infty\} \to \mathbb{C}_{z_{\alpha'}}$ by $z \mapsto z^{-1}$. So,

$$\omega_{\alpha'} = \frac{dz_{\alpha'}^{-1}}{dz_{\alpha'}}\, dz_{\alpha'} = -z_{\alpha'}^{-2}\, dz_{\alpha'}. \tag{5.10}$$

The differential dz is meromorphic, not holomorphic, and it has degree -2. To see there are no nonconstant holomorphic differentials on $\mathbb{P}^1_z$, write such a differential as $g(z)\, dz$ with g a meromorphic function on $\mathbb{P}^1_z$. Liouville's Theorem says g has as many zeros as poles [**Ahl79**, p. 122]. So the degree of $g(z)\, dz$ also is -2, and $(g(z)\, dz)$ cannot be positive. A similar computation shows the vector space of holomorphic differentials on a complex torus has dimension 1 [9.8].

5.3.3. *Pulling back differentials.* Let $f : X_1 \to X_2$ be an analytic and surjective map between complex manifolds. Then, a meromorphic function $\psi : Y \to \mathbb{P}^1_z$ produces a meromorphic function $\psi \circ f \stackrel{\text{def}}{=} f^*(\psi) : X \to \mathbb{P}^1_z$ giving an embedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ (§4.1.2).

LEMMA 5.18. *We may extend f^* to embed meromorphic differentials $\mathcal{M}^1(Y)$ on Y into meromorphic differentials $\mathcal{M}^1(X)$. Further, this maps holomorphic differents $\Omega^1(Y)$ on Y into holomorphic differentials on X. Then φ^* has the following property. For $\omega \in \mathcal{M}^1(Y)$, suppose $\gamma \in \Pi_1(X, x_0)$ does not go through a pole of $\varphi^*(\omega)$. Then, $\int_\gamma \varphi^*(\omega) = \int_{\varphi_*(\gamma)} \omega$.*

PROOF. Use the notation of (4.1). To simplify we do this for the case of 1-dimensional complex manifolds, though the many variable case is just a slight addition to the notation. This is truely a local statement. Write ω as $h_{\alpha_2}(z_{\alpha_2})\, dz_{\alpha_2}$ on $\varphi_{\alpha_2}(f(U_{\alpha_1}) \cap U_{\alpha_2})$. Then, define $f^*(\omega)$ by

$$h_{\alpha_2}(\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1}(z_{\alpha_1}))\, d(\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1}(z_{\alpha_1})) \text{ on } U_{\alpha_1} \cap f^{-1}(U_{\alpha_2}).$$

The equality of the integrals is nothing more, after substituting for the coordinates of the path γ, than the change of variables formula Chap. 2 Lem. 2.3. □

5.4. Half-canonical differentials. Square-roots of differentials appear on a Riemann surface X when we seek a canonical choice of θ function attached to the surface. The case when X has genus 1 (Chap. 4 §6.5) will be our guide.

Riemann's θ functions often allow us to put coordinates (as in the initial discussion of §4) on such total familes. Whenever possible, we would like the construction of such coordinates to be canonical. Usually, however, constructing θ functions depends on choices. So, we are careful to note, for curves in families, how the construction varies with the points parametrizing the family members.

Riemann used θ functions to give coordinates for constructing objects, like differentials and functions on a Riemann surface. When the Riemann surface has genus 1 (or 0) there are natural choices for working with Riemann's coordinates. When, however, the genus exceeds 1, and the surface is not special, there are several $(2^{2g-1} - 2^{2g-2})$ potential choices of the *odd* θ function Riemann required to generalize Abel's Theorem. We will see that half-canonical differentials precisely differentiate between these choices.

5.4.1. *Cocycles.* For X an n-dimensional complex manifold, let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ be the coordinate chart, and $\{\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha,\beta\in I}$ the corresponding collection of transition functions (as in Def. 3.6). Each $\psi_{\beta,\alpha}$ then is a one-one analytic function on an open subset of $\mathbb{C}^n$ whose coordinates we label $z_{\alpha,1}, \dots, z_{\alpha,n}$. Denote the $n \times n$ *complex Jacobian matrix* for $\psi_{\beta,\alpha}$ by $J(\psi_{\beta,\alpha})$. Call the matrices $\{J(\psi_{\beta,\alpha})\}_{\alpha,\beta\in I}$ the (transformation) *cocycle* attached to meromorphic differentials. Similarly $\{J(\psi_{\beta,\alpha})^{-1}\}_{\alpha,\beta\in I}$ is the cocycle attached to meromorphic tangent vectors. Recall the notation for $n \times n$ matrices, $\mathbb{M}_n(R)$ with entries in an integral domain R and for the *invertible* matrices $\mathrm{GL}_n(R)$ with entries in R under multiplication. Cramer's rule says for each $A \in \mathbb{M}_n(R)$ there is an adjoint matrix A^* so that AA^* is the scalar matrix $\det(A)I_n$ given by the determinant of A. This shows the invertibility of $A \in \mathbb{M}_n(R)$ is equivalent to $\det(A)$ being a *unit* (in the multiplicatively invertible elements R^*) of R. Denote the $n \times n$ identity matrix (resp. zero matrix) in $\mathrm{GL}_n(R)$ by I_n (resp. $\mathbf{0}_n$).

Definition 5.19 (1-cycocle). Suppose $g_{\beta,\alpha} \in \mathrm{GL}_n(\mathcal{H}(U_\alpha \cap U_\beta))$, $\alpha, \beta \in I$. Assume also that $g_{\gamma,\beta} g_{\beta,\alpha} = g_{\gamma,\alpha}$ for all $\alpha, \beta, \gamma \in I$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (if this is nonempty). Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a multiplicative *1-cocycle with values in* $\mathcal{GL}_{n,X}$. Similarly, suppose $g_{\beta,\alpha} \in \mathbb{M}_n(\mathcal{H}(U_\alpha \cap U_\beta))$, $\alpha, \beta \in I$. Suppose $g_{\gamma,\beta} + g_{\beta,\alpha} = g_{\gamma,\alpha}$ for all $\alpha, \beta, \gamma \in I$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is an additive *1-cocycle with values in* $\mathcal{GL}_{n,X}$.

We also name (1-)cocycles for collections of subgroups in $\mathcal{GL}_{n,X}$ (resp. $\mathcal{M}_{n,X}$) for which it makes sense to multiply (resp. add) $g_{\gamma,\beta}$ and $g_{\beta,\alpha}$. So, for example, we may consider a multiplicative cocycle with values in $\{\pm I_n\}$ or an additive cocycle with values in $\mathbb{Z}I_n$. When there are 1-cocycles, there are also 0-chains and their associated 1-boundaries. We write the definition for GL_n, recognizing there are analogous versions for all other types of cocycles.

Definition 5.20 (1-boundary). With $u_\alpha \in \mathrm{GL}_n(\mathcal{H}(U_\alpha))$, $\alpha \in I$, suppose $g_{\beta,\alpha} = u_\beta(u_\alpha)^{-1}$ for all $\alpha, \beta, \gamma \in I$ in $U_\alpha \cap U_\beta$ (if nonempty). Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a 1-cocycle, called a *1-boundary* with values in $\mathcal{GL}_{n,X}$. Call the set $\{u_\alpha\}_{\alpha\in I}$ a *0-chain* with values in $\mathcal{GL}_{n,X}$.

5.4.2. *Half-canonical divisors.* Suppose ω is a meromorphic differential on a Riemann surface X, written locally as $f_\alpha(z_\alpha)dz_\alpha$ on simply connected domains U_α (Chap. 2 §8.3). Assume also the *square hypothesis*:

(5.11) The divisor of $f_\alpha(z_\alpha)$ has the form $2D_\alpha$ for U_α running over a subchart covering X.

Then, there is a branch $h_\alpha(z_\alpha)$ of square root (of $f_\alpha(z_\alpha)$) on U_α (Chap. 2 (8.1)). Of course, there are two of these; our notation means we have chosen one. Call the symbol $\tau_\alpha = h_\alpha(z_\alpha)\sqrt{dz_\alpha}$, a *half-canonical divisor* on U_α. The squares of these form a global differential on X. Denote the collection $\{h_\alpha(z_\alpha)\}_{\alpha\in I}$, by $\boldsymbol{h}$ and refer to it as a square-root of ω.

Lemma 5.21 (Half-canonical divisor). *The collection of divisors* $\{(h_\alpha(z_\alpha))\}_{\alpha\in I}$ *from a square root of* ω *give a well-defined divisor: a half-canonical divisor on* X.

PROOF. Let $D = (\omega)$ be the divisor of ω. Since, $h_\alpha^2 = f_{i,\alpha}$, the support multiplicities of D are all even integers. So, a square-root of ω defines $D_{1/2} = (\omega)/2$, a divisor uniquely given by the zeros and poles of the h_α s. □

Now consider how to decide, based on a square-root of ω, if there is an object $\omega_{1/2}$ with values at points on X whose divisor is $D_{1/2} = (\omega)/2$. Continue the transition function notation $\psi_{\beta,\alpha}$ from §5.4.1. This requires us to make sense, on $U_\alpha \cap U_\beta$, of equality between

$$\tau_\alpha(z_\alpha) = h_\alpha(z_\alpha)\sqrt{dz_\alpha} \text{ and } \tau_\beta(\psi_{\beta,\alpha}(z_\alpha)) = h_\beta(\psi_{\beta,\alpha}(z_\alpha))\sqrt{d\psi_{\beta,\alpha}(z_\alpha)}. \tag{5.12}$$

PROPOSITION 5.22. *Assume each component of $U_\alpha \cap U_\beta$, $(\alpha,\beta) \in I \times I$ is simply connected and for such, we have made a choice of $\sqrt{J(\psi_{\beta,\alpha})} = g_{\beta,\alpha}$ on $U_\alpha \cap U_\beta$. Then, independent of α with $x' \in U_\alpha$, setting the value of τ_α to $h_\alpha(\varphi_\alpha(x'))$ is well-defined if and only if $\{g_{\beta,\alpha}\}_{(\alpha,\beta)\in I\times I} = \boldsymbol{g}$ is a 1-cocycle. If there is a $\boldsymbol{g}$ that is a 1-cocycle, call the resulting half-canonical differential $\omega_{1/2,\boldsymbol{h},\boldsymbol{g}}$. Then, with $\boldsymbol{g}$ fixed, but $\boldsymbol{h}'$ varying over square-roots of ω, any pair of $\omega_{1/2,\boldsymbol{h}',\boldsymbol{g}}$ differ by a 1-boundary with values in $\{\pm 1\}$.*

PROOF. We need only add that the cocycle condition on $\boldsymbol{g}$ is necessary and sufficient for (5.12). For this check that if $x' \in U_\alpha \cap U_\beta \cap U_\gamma$, then all the values $h_\alpha(\varphi_\alpha(x'))$, $h_\beta(\varphi_\alpha(x'))$ and $h_\gamma(\varphi_\gamma(x'))$ at x' match up using $\boldsymbol{g}$. Comparing (5.12) for each of the pairs (α,β), (β,γ) and (α,γ) gives the cocycle condition. □

5.4.3. *Square-hypothesis for hyperelliptic curves.* Suppose the affine part of a hyperelliptic curve X, with compactification from Ex. 4.2.3, is $\{(z,w) \mid w^2 = h(z)\}$. We explicitly display differentials ω satisfying the square hypothesis of (5.11). For simplicity, assume h has odd degree and distinct zeros $z_1,\dots,z_{r-1}$ (with $z_r = \infty$). Denote the point on X over z_i by x_i, with x_∞ lying over $z = \infty$. As in [**Mum76**, p. 7], form the differentials

$$\omega_i = \frac{(z - z_i)^{\frac{1}{2}}}{(\prod_{j\neq i} z - z_j)^{\frac{1}{2}}}\, dz,\ i = 1,\dots,r-1.$$

Since $w = \sqrt{h(z)}$, the factor in front of the dz in ω_i is just $\frac{z-z_i}{w}$, a meromorphic function on X. The divisor of ω_i is therefore $2x_i - 2x_\infty = D_i$. For the check at a neighborhood of x_∞ over $z = \infty$, use $t = 1/\sqrt{z}$ as the uniformizing parameter on X. Consider the case $\deg(h) = 3$. Then, $(t^{-1} - z_i)(-2wt^3)\,dt$ has $t = 0$ as a pole of order 2. So, D_i is the same divisor as $(z - z_i)$.

Now consider the case $\deg(h) = r - 1$, $r \geq 6$ an even integer. Similarly, $(\omega_i) = 2x_i + 2(r/2-3)x_\infty$, as $\frac{z-z_i}{-2wt^3}\,dt$ has $t = 0$ as a zero of multiplicity $2(r/2-3)$.

6. Homotopy, monodromy and fundamental groups

Complex structure provides the notion of analytic continuation. We detect the effects of analytic continuation through monodromy action, a representation of some fundamental group. In practice this can be a permutation representation, a representation as automorphisms of a vector space or a representation into automorphisms of a more general group. The prototype use of monodromy is Riemann's Existence Theorem: We replace constructing a compact Riemann surface using charts with permutation representations of a fundamental group. For example, using classical generators (Chap. 4 Fig. 3) for the fundamental group of

$U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ gives an effective listing of Riemann surface covers (and their corresponding algebraic functions; Chap. 4 Cor. 2.8).

6.1. Homotopy of paths. Let $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, be two one-one simplicial paths in X with the same range, initial, and end points. The function $f(t) = \gamma_2^{-1} \circ \gamma_1$ is a simplicial path $f : [a_1, b_1] \to [a_2, b_2]$ for which $\frac{d}{dt}(f(t)) \ge 0$ (where the derivative is defined) and $\gamma_2(f(t)) = \gamma_1$. (Use the chain rule.) We give a more general statement.

DEFINITION 6.1 (Image equivalent paths). Let $\gamma : [a_1, b_1] \to X$ be a simplicial path in X, and let $f_1 : [a_2, b_2] \to [a_1, b_1]$ and $f_2 : [a_1, b_1] \to [a_2, b_2]$ be simplicial paths with $\frac{d}{dt}(f_i(t)) \ge 0$ where it is defined, $i = 1, 2$. Assume also $\gamma \circ f_1 \circ f_2(t) = \gamma(t)$ for $t \in [a_1, b_1]$. Call γ and $\gamma \circ f_1$ *image equivalent* paths. It is a simple exercise to show each path is image equivalent to a path $\gamma : [0, 1] \to X$.

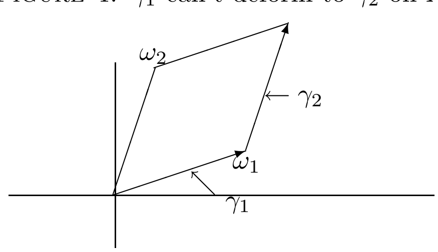
DEFINITION 6.2 (Homotopically equivalent paths). Consider a continuous map $F : [a, b] \times [0, 1] \to X$, and points $x_a, x_b \in X$, with the following properties: $F(t, s) = \gamma_s(t)$ is a path for each $s \in [0, 1]$ with initial point x_a and end point x_b. Call F a *homotopy* between γ_0 and γ_1 (or γ_0 and γ_1 are *homotopic*).

REMARK 6.3 (Warnings!). The end points of the paths γ_s remain fixed throughout a homotopy, or else all paths in a connected space would be homotopic.

Even if γ_0 and γ_1 are simplicial paths, we do not initially assume γ_s is also simplicial. Still, the argument of Chap. 2 Lem. 4.3 generalizes easily to any (union of) differentiable manifold(s) to say that any continuous path is homotopic to a simplicial path. Further, it is then image equivalent to a product of simplicial paths that are either constant or have nonzero derivative, and if it is a nonconstant path, you can toss out — up to equivalence — the constant paths. We use this statement freely [9.12]. It is common to think of both s and t as time parameters. It is compatible to consider the range of γ_0 as a physical object layed down parametrically. As a function of time, each point $\gamma_0(t)$ of the range of γ_0 moves to a different position $\gamma_s(t)$. So, F represents deforming an initial path, perhaps along which it is more efficient to accrue similar information from traversing γ_0.

In Fig. 4 the space X is the same as Fig. 3. Note: γ_1 and γ_2 are closed, beginning and ending at $0 \mod L \in \mathbb{C}/L$.

FIGURE 4. γ_1 can't deform to γ_2 on X



DEFINITION 6.4. Extend the definition of *homotopic paths*. We say two paths $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$ are *equivalent* (or *homotopic*) if γ_1 and γ_2 are image equivalent, respectively, to homotopic paths $\gamma_i^* : [a, b] \to X$, $i = 1, 2$, for some $a < b$. This is an equivalence relation.

6.2. Analytic continuation on a manifold. Suppose $f \in \mathcal{E}(D, z_0)$ is extensible in a domain D and $\gamma[a,b] \to D$ is a path. Chap. 2 Rem. 4.4 notes the production of a simplicial path γ^* in D for which the analytic continuations f_γ and f_{γ^*} are the same. Further, assume f is extensible as a holomorphic (rather than just meromorphic) function in D. Then, define F_γ for any antiderivative F of f (around z_0) as the analytic continuation F_{γ^*}. Chap. 2 Lem. 4.3 produces γ^* from γ by a succession of homotopies, between a piece of path on γ contained in a disk and a line segment joining two points on the boundary of the disk. Disks are a crucial case of the following definition. The simple lemma following it, hidden in the construction of γ^*, appears in most arguments about homotopy classes.

DEFINITION 6.5. Call a topological space X *contractible* (to $x_0 \in X$) if there is a continuous function $f : X \times [0,1] \to X$ satisfying $f(x, 0) = x$ and $f(x, 1) = x_0$ for each $x \in X$.

LEMMA 6.6. *A closed or open ball (or anything homeomorphic to such) in* $\mathbb{R}^n$ *is contractible. If* X *is contractible, then any two paths with the same endpoints are homotopic [9.12b].*

Analytic continuation of a meromorphic function (Chap. 2 Def. 4.1) extends to manifolds by imitating the other extensions to manifolds. Suppose X is a complex manifold with coordinate chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Consider any path $\gamma : [a,b] \to X$. Our notation follows the case for a dimension 1 complex manifold, though it extends easily to the general case.

By a disk (or ball) D on X we mean an open set in X which lies in one coordinate neighborhood U_α where $\varphi_\alpha(D)$ is a disk (or ball) in $\varphi_\alpha(U_\alpha) = V_\alpha$.

6.2.1. *Extensible functions on* X. Follow Chap. 2 §4.1 to extend analytic continuation of a function along a path to where the path is in a complex manifold.

DEFINITION 6.7 (Analytic continuation along a path). Suppose f is meromorphic in a neighborhood $U_{x_0} \subset X$ of $x_0 \in X$ and $\gamma : [a,b] \to X$ is a path based at x_0. Let $f^* : [a,b] \to \mathbb{P}^1_z$ be a continuous function with the following properties.

(6.1a) $f^*(t) = f(\gamma(t))$ for t close to a (in $[a,b]$).

(6.1b) For each $t' \in [a,b]$, there is a neighborhood $U_{\gamma(t')}$ of $\gamma(t')$ and an analytic function $h_{t'} : U_{\gamma(t')} \to \mathbb{P}^1_z$ with $h_{t'}(\gamma(t)) = f^*(t)$ for t near t' (in $[a,b]$).

As before, $h_{t'}$ is the analytic continuation of f *to* t'. It is an analytic function in some neighborhood of $\gamma(t')$. Reference is usually to the *end* function $h_b = f_\gamma$, analytic in a neighborhood of $\gamma(b)$. This is the analytic continuation of f (along γ). As with analytic continuation along a path in $\mathbb{P}^1_z$, $f^*(t)$ determines all data for an analytic continuation. Also, it is unique: its difference from another function suiting (6.1) must be constant (restrict to coordinate neighborhoods of points of the path and apply Chap. 2 [9.8a]). Again, there is a related definition.

6.2.2. *Algebraic functions on* X. An analytic function $\hat{f} : X \to \mathbb{P}^1_z$ satisfying $\hat{f}(x) = f(x)$ for all $x \in U_{x_0}$ is an analytic continuation or *extension* of f to X.

DEFINITION 6.8. Denote by $\mathcal{E}(X, x_0)$ all functions meromorphic in a neighborhood of x_0 that analytically continue along every path in X based at x_0.

Further, suppose there is compact Riemann surface $\bar{X}$ with $X = \bar{X} \setminus \boldsymbol{x}$ where $\boldsymbol{x}$ is a finite set of points on $\bar{X}$. Chap. 4 shows, if such a $\bar{X}$ exists, it is unique up to analytic isomorphism. If $\boldsymbol{x}$ consists of r points, call such an X an *r-punctured Riemann surface*. Dropping reference to r, call it just a punctured Riemann surface. This tacitly assumes r is a finite number.

DEFINITION 6.9. Suppose X is a punctured Riemann surface. Then, $\mathcal{E}(X,x_0)^{\text{alg}}$ consists of the $f \in \mathcal{E}(X,x_0)$ for which both the following sets are finite.

(6.2a) All analytic continuations, $\mathcal{A}_f(X) = \{f_\gamma\}_{\gamma\in\Pi_1(X,x_0)}$ of f in X.
(6.2b) For $x' \in \boldsymbol{x}$, the limit endpoint values of f_γ along all $\gamma \in \Pi_1(X,x_0,x')$.

PROPOSITION 6.10. *Let D be a disk on X, and suppose $f : D \to \mathbb{P}^1_z$ is analytic. There is a partition $a = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = b$ of $[a,b]$, coordinate neighborhoods (U_i,φ_i), a disk D_i centered about $\gamma(t_i)$ in U_i and $f_i \in \mathcal{H}(D_i)$, $i = 1,\dots,n-1$, with these properties.*

(6.3a) $D_i \cap D_{i+1} \neq \emptyset$ *and* $f_i(z) = f_{i+1}(z)$ *for* $z \in D_i \cap D_{i+1}$.
(6.3b) $\gamma(t) \in D_i$ *for* $t \in [t_i,t_i^*]$, $\gamma(t) \in D_{i+1}$ *for* $t \in [t_i^*,t_{i+1}]$, $i = 0,\dots,n-1$.
(6.3c) $f_0(z) = f(z)$ *for* $z \in U_{z_0}$.

Further, let γ^* be the path along the consecutive line segments $\gamma(t_i)$ to $\gamma(t_i^*)$, then $\gamma(t_i^*)$ to $\gamma(t_{i+1})$, $i = 0,\dots,n-1$. Then, $f_{\gamma^*} = f_\gamma$.

PROOF. The proof reduces to that of Chap. 2 Lem. 4.3 by using the definition of function and coordinate charts on a complex manifold. □

PROPOSITION 6.11 (The general monodromy theorem). *Let $\gamma_1,\gamma_2 : [a,b] \to X$ be two paths with $\gamma_1(a) = \gamma_2(a) = x_0$ and $\gamma_1(b) = \gamma_2(b) = x_1$. Suppose γ_1 and γ_2 are homotopic on X. Let U_{x_0} be a neighborhood of x_0 and $f : U_{x_0} \to \mathbb{P}^1_z$ Then, $f_{\gamma_1} = f_{\gamma_2}$ (*[**Ahl79**, p. 295] *and* [**Con78**, p. 219]*).*

PROOF. Let $F : [a,b] \times [0,1] \to X$ be a homotopy between γ_1 and γ_2 fixing points $x_a = x_0, x_b = x_1 \in X$. A continuous function on a compact space is absolutely continuous. From absolute continuity of F there are partitions

$$a = s_0 < s_1 < \cdots < s_n = b \text{ of } [a,b] \text{ and } 0 = t_0 < t_1 < \cdots < t_m = 1 \text{ of } [0,1]$$

so that $F : [s_i,s_{i+1}] \times [t_j,t_{j+1}] \to X$ has range in a coordinate chart $U_{i,j}$ on X and $\varphi_{i,j} : U_{i,j} \to \mathbb{C}$ has range in a disk.

Suppose h is meromorphic in a neighborhood of $F(s_i,t_j)$ and extensible on the range of F on $[s_i,s_{i+1}] \times [t_j,t_{j+1}]$. Denote the product of the paths

$$s \mapsto F(s,t_j) = F_{ij,1},\ s \in [s_i,s_{i+1}] \text{ and } t \mapsto F(s_{i+1},t) = F_{i+1j,2},\ t \in [t_j,t_{j+1}]$$

by μ_{ij}^+. Similarly, let μ_{ij}^- be the product of paths $t \mapsto F(s_i,t) = F_{ij,2}$, $t \in [t_j,t_{j+1}]$ and $s \mapsto F(s,t_{j+1}) = F_{ij+1,1}$, $s \in [s_i,s_{i+1}]$. From Chap. 2 Lem. 4.6, $h_{\mu_{ij}^+} = h_{\mu_{ij}^-}$.

Write the path γ_1 as the product of the paths $F_{i0,1}$, $i = 0,\dots,m$. Similarly, γ_2 is the product of the paths $F_{in,1}$, $i = 0,\dots,m$. We give a sequence of paths (with the same endpoints) that starts with γ_1, and ends with γ_2. The terms of the sequence differ from path-to-path in the chain by a product of paths of form $(\mu_{ij}^+)^{-1}\mu_{ij}^-$ or of form $\gamma\gamma^{-1}$. This shows $f_{\gamma_1} = f_{\gamma_2}$. Simply replace $F_{i0,1}$ by

$$F_{i0,1}F_{i+10,2}F_{i+10,2}^{-1}(\mu_{i0}^+)^{-1}\mu_{i0}^-$$

for each $i = 1,\dots,m$. These substitutions lead from γ_1 to the path that is the product of $F_{i1,1}$, $i = 0,\dots,m$. Continue inductively to the path γ_2, which is the product of $F_{i1,n}$, $i = 0,\dots,m$. □

Chap. 2 §4.4 defines the product of two paths $\gamma_i : [a_i,b_i] \to X$, $i = 1,2$, for which the end point of γ_1 is the initial point of γ_2. Many treatments on fundamental groups (like [Ma; Chap. 2]) restrict the domain interval for a path to $[0,1]$. The treatment here aids computation of the *Artin braid group* (Chap. 4 [10.7], [**??**] and

Chap. 5). It has other virtues: If $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2, 3$, are three paths with $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$, $i = 1, 2$, then $\gamma_1(\gamma_2\gamma_3)$ and $(\gamma_1\gamma_2)\gamma_3$ are identical rather than just equivalent as in [**Ma67**, p. 59]. Thus, forming products is trivially *associative*.

6.3. Path equivalence classes form a group. We say $\gamma : [a, b] \to X$, a closed path with initial (and end) point $x_0 \in X$, is *based* at x_0. The set of paths based at x_0 is closed under taking products. Denote the (homotopy) equivalence class of γ by $[\gamma]$. Note: $[\gamma_1^*\gamma_2^*]$ is independent of the choice of $\gamma_i^* \in [\gamma_i]$, $i = 1, 2$. The function $\gamma : [a, b] \to X$ by $\gamma(t) = x_0$ is called a constant path; denote $[\gamma]$ by ϵ_{x_0}. The set of equivalence classes of paths in X based at x_0 is the *fundamental group* of X based at x_0.

THEOREM 6.12. *Equivalence classes of paths into X based at x_0 form a group, denoted $\pi_1(X, x_0)$, under the multiplication given by $[\gamma_1][\gamma_2] \stackrel{\text{def}}{=} [\gamma_1\gamma_2]$. The identity element is ϵ_{x_0}. The inverse of $[\gamma]$ is the class $[\gamma^{-1}]$ (*Chap. 2 *§4.4).*

PROOF. Consider $\gamma : [a, b] \to X$ and γ^{-1} as above. Let $s' = a+s(b-a)$ and consider the function $F : [a, 2b-a] \times [0, 1] \to X$ defined by

$$F(t, s) = \begin{cases} \gamma(t) & \text{for } t \in [a, s'] \\ \gamma(s') & \text{for } t \in [s', 2b-s'] \\ \gamma(2b-t) & \text{for } t \in [2b-s', 2b-a]. \end{cases} \tag{6.4}$$

So, F is a homotopy between $\gamma\gamma^{-1}$ and the constant path from $[a, 2b-a]$ into $\{x_0\}$.

From [9.12b], for $\gamma_0 : [a_0, b_0] \to \{x_0\}$, the paths $\gamma_0\gamma$ and $\gamma\gamma_0$ are equivalent to γ. Thus, $[\gamma][\gamma^{-1}] = \epsilon_{x_0}$, $[\gamma]\epsilon_{x_0} = [\gamma] = \epsilon_{x_0}[\gamma]$. This shows $\pi_1(X, x_0)$ is a group. □

The fundamental group *does* depend on the base point x_0, though its isomorphism class does not. Indeed, for $x_0, x_1 \in X$, let $\alpha : [a, b] \to X$ be a path with initial point x_0 and end point x_1. Define $\psi(x_0, x_1) : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by

$$\psi(x_0, x_1)([\gamma]) = [\alpha\gamma\alpha^{-1}] \text{ for each } [\gamma] \in \pi_1(X, x_1).$$

Check that $\psi(x_0, x_1)$ is a homomorphism of groups inverse to the homomorphism $\psi(x_1, x_0)$: $[\gamma] \in \pi_1(X, x_0) \mapsto [\alpha^{-1}\gamma\alpha] \in \pi_1(X, x_0)$. Note: The isomorphism $\pi(x_0, x_1)$ depends on the choice of α if $\pi_1(X, x_0)$ is not an abelian group.

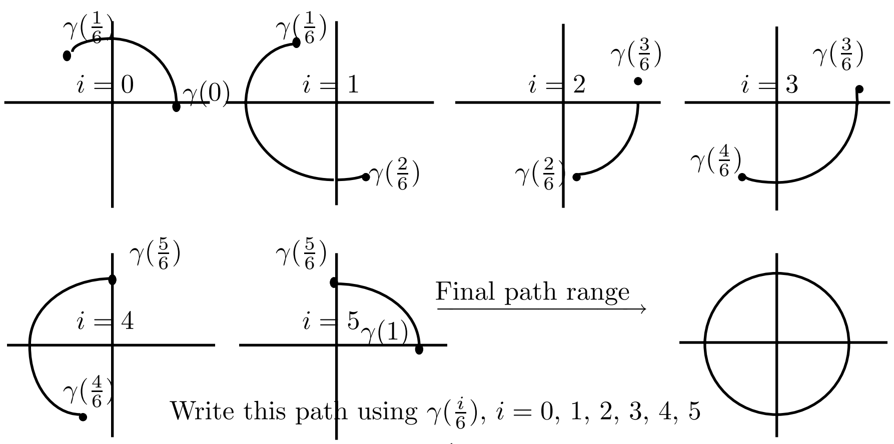
COROLLARY 6.13. *For $x_0, x_1 \in X$, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Still, we eventually come to fundamental groups of members of a *family* of topological spaces (Chap. 5), where all members have the same fundamental group. Our most profound (the *braid* and *Hurwitz monodromy*) groups appear to account for different identifications among these fundamental groups.

6.4. Fundamental group of a circle. For any differentiable manifold X, there is a natural map from the fundamental group $\pi_1(X, x_0)$ computed with piecewise differentiable paths to the fundamental group computed with continuous paths, $\pi_1(X, x_0)^{\text{cont}}$. This induces an isomorphism (though we don't exploit this seriously) from Rem. 6.3. This point shows in a comparison of the two fundamental groups when $X = S^1$, a *circle* which we take to be the unit circle in $\mathbb{C}_z$. We give two proofs that it is isomorphic to $\mathbb{Z}$. The first explicitly uses simplicial paths. The other uses the universal covering space (Lem. 8.4).

Consider the path $\gamma^*_{|_{[a,b]}} : [a, b] \to S^1$ by $t \mapsto \cos(2\pi t) + i\sin(2\pi t)$, $t \in [a, b]$. For $n \geq 0$ an integer, denote $\gamma^*_{|_{[0,n]}}$ by γ^*_n, and let S^1 be the image of γ^*_1. Denote the inverse of $\gamma^*_{|_{[0,1]}}$ by $(\gamma^*)^{-1}_{|_{[0,1]}}$. Since $(\gamma_1^*)^n = \gamma_n^*$ it is consistent to define γ^*_{-n} to be $(\gamma_1^{*-1})^n$. For $n = 0$ let γ_0^* be the constant path mapping to 1.

FIGURE 5. Homotopically speaking, a path going nowhere. Traversal for $t \in [\frac{i}{6}, \frac{i+1}{6}]$, $i = 0, 1, 2, 3, 4, 5$



THEOREM 6.14. *The group* $\pi_1(S^1, 1)$ *is infinite cyclic with generator* $[\gamma_1]$.

PROOF. From Rem. 6.3 any nonconstant path $\gamma : [a, b] \to S^1$ is equivalent (Def. 6.4) to a product of paths with nonzero derivative. Each such is then image equivalent to $(\gamma^*)^{\epsilon}_{|_{[r,s]}}$ for some $r < s$ and $\epsilon \in \{\pm 1\}$. So, we can write the path as $\prod_{i=1}^{\ell} (\gamma^*)^{\epsilon_i}_{|_{[r_i,s_i]}}$ with $s_i = r_{i+1}$. Suppose ϵ_i and ϵ_{i+1} have opposite sign. Further subdivide one of paths corresponding to i or to $i+1$ to assume $[r_i, s_i]$ and $[r_{i+1}, s_{i+1}]$ have the same length. From (6.4),

$$(\gamma^*)^{\epsilon_i}_{|_{[r_i,s_i]}} (\gamma^*)^{\epsilon_{i+1}}_{|_{[r_{i+1},s_{i+1}]}}$$

is equivalent to the constant path with image $(\gamma^*)^{\epsilon_i}(r_i)$ [9.12a]. Thus the whole path is equivalent to a path with a smaller ℓ. An induction on the integer $\sum_{i=1}^{\ell} |\epsilon_{i+1} - \epsilon_i|$ shows γ is equivalent to γ_n^* for some integer n.

The proof is complete if γ_n^* is inequivalent to γ_m^* for $m \neq n$. Decompose $\gamma : [a, b] \to S^1$ into its real and imaginary parts: $\gamma = \gamma_1 + i\,\gamma_2$ where $\gamma_i : [a, b] \to \mathbb{R}$, $i = 1, 2$. Define $\deg(\gamma)$ through the formula

$$\begin{aligned} 2\pi i \deg(\gamma) &= \int_a^b (\gamma_1(t), \gamma_2(t)) \cdot (\tfrac{d\gamma_1}{dt}(t), \tfrac{d\gamma_2}{dt}(t))\, dt \\ &\quad + i \int_a^b (-\gamma_2(t), \gamma_1(t)) \cdot (\tfrac{d\gamma_1}{dt}(t), \tfrac{d\gamma_2}{dt}(t))\, dt \end{aligned}$$

(as in Chap. 2 Lem. 2.3). By direct computation $\deg(\gamma_n^*) = n$.

If γ is homotopic to γ_n^*, then Chap. 2 Lem. 2.3 shows $\deg(\gamma) = n$. As $\deg(\gamma)$ depends only on $[\gamma]$ [9.12d], $[\gamma_n^*]$ is distinct from $[\gamma_m^*]$ for $n \neq m$. □

Chap. 4 computes fundamental groups of many spaces from Thm. 6.14.

Let $\gamma : [a, b] \to X_1$ be a (simplicial) path. Consider $f \circ \gamma : [a, b] \to X_2$, and for $x_1 \in X_1$, denote $f(x_1)$ by x_2. For $[\gamma] \in \pi_1(X_1, x_1)$, $[f \circ \gamma] \in \pi_1(X_2, f(x_1))$ is independent of the choice of γ representing $[\gamma]$. To a product of paths $\gamma_1\gamma_2$ in X_1, apply the formula $f \circ (\gamma_1\gamma_2) = (f \circ \gamma_1)(f \circ \gamma_2)$. This shows $[f \circ \gamma_1][f \circ \gamma_2] = [f \circ (\gamma_1\gamma_2)]$.

LEMMA 6.15. *Conclude: f induces a homomorphism of groups*

$$f_* : \pi_1(X_1, x_1) \to \pi_1(X_2, x_2).$$

If f is one-one and onto then f_* is an isomorphism of groups.

EXAMPLE 6.16. Let $X_1 = X_2 = S^1$ and consider $\cos(2\pi t) + i \sin(2\pi t) = z(t)$. For a fixed positive integer n define a function f by the formula $f(z(t)) = z(nt) = \cos(2\pi nt) + i \sin(2\pi nt)$. Thus $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$. Also, for γ_1^*, the generating path for $\pi_1(S^1, 1)$, $f \circ \gamma_1^*(t) = f(z(t))$. Therefore $f \circ \gamma_1^*$ is image equivalent to γ_n^*. Identify $\pi_1(S^1, 1)$ with $\mathbb{Z}$, the group of integers, by identifying the integer 1 with $[\gamma_1^*]$. Then, $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ sends the integer m to $f_*(m) = nm$. The image of f_* is the subgroup of $\pi_1(S^1, 1) = \mathbb{Z}$ that n generates.

6.5. Fundamental group of a product. Let (X, x_0) and (Y, y_0) be two differentiable manifolds with a base point. The projections onto each factor, $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$, induce homomorphisms

$$\mathrm{pr}_{X*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \text{ and } \mathrm{pr}_{Y*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0).$$

So, there is a homomorphism

$$(\mathrm{pr}_{X*}, \mathrm{pr}_{Y*}) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0). \tag{6.5}$$

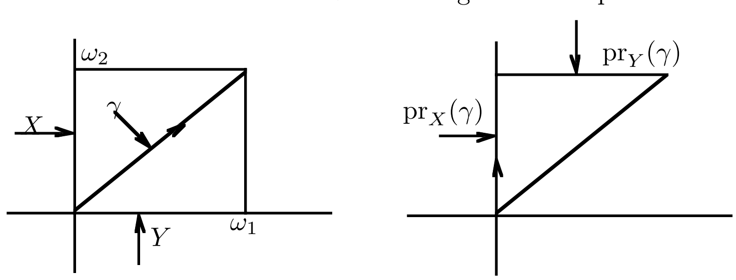
The right side is the product group with factors $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

THEOREM 6.17. *$\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ are isomorphic.*

PROOF. Let f^X (resp. f^Y) map $X \to X \times Y$ by $f^X(x) = (x, y_0)$ (resp. map $Y \to X \times Y$ by $f^Y(y) = (x_0, y)$). For $\gamma : [a, b] \to X \times Y$ consider the paths $(f^X \circ \mathrm{pr}_X \circ \gamma) = \psi^X : [a, b] \to X \times Y$ and $(f^Y \circ \mathrm{pr}_Y \circ \gamma) = \psi^Y : [a, b] \to X \times Y$.

We show the map taking $([\gamma_1], [\gamma_2]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ to $f_*^X([\gamma_1]) f_*^Y([\gamma_2])$ in $\pi_1(X \times Y, (x_0, y_0))$ is inverse to $(\mathrm{pr}_{X*}, \mathrm{pr}_{Y*})$. This only requires showing γ is equivalent to $\psi^X \psi^Y$. Fig. 6.5 illustrates this when $X = Y = S^1$ and $X \times Y$ is the complex torus of Fig. 3 with $\omega_1 = 1$ and $\omega_2 = i$ [9.5b].

FIGURE 6. The diagonal recomposes itself



Write $\gamma(t) = (\gamma^X(t), \gamma^Y(t))$ for $t \in [a, b]$ and assume $[a, b] = [0, 1]$. Then γ is image equivalent to the path $(\gamma^X(\frac{t}{2}), \gamma^Y(\frac{t}{2}))$ for $t \in [0, 2]$. Also, ψ^X is the path $t \mapsto (\gamma^X(t), y_0)$ for $t \in [0, 1]$ and $(x_0, \gamma^Y(t-1))$ for $t \in [1, 2]$. Here is a homotopy between these paths running over $s \in [0, 1]$:

$$\gamma_s(t) = \begin{cases} (\gamma^X(\frac{t}{2-s}), y_0) & \text{for } t \in [0, s] \\ \gamma^X(\frac{t}{2-s}), \gamma^Y(\frac{t-s}{2-s})) & \text{for } t \in [s, 2-s] \\ (x_0, \gamma^Y(\frac{t-s}{2-s})) & \text{for } t \in [2-s, 2]. \end{cases}$$

□

EXAMPLE 6.18 (Continuation of §3.2.2). Here $X^i = \mathbb{C}/L(\omega_1^i, \omega_2^i)$ is

$$\{t_1\omega_1^i + t_2\omega_2^i \mid 0 \le t_i < 1,\ i = 1, 2\}$$

where $\omega_1^i/\omega_2^i \in \mathbb{C} \setminus \mathbb{R}$, $i = 1, 2$. For the lattice $\{m_1\omega_1^i + m_2\omega_2^i \mid m_1, m_2 \in \mathbb{Z}\}$ use the letter L_i, $i = 1, 2$. For $z \in \mathbb{C}$, there is a unique $\omega \in L_i$ with $z - \omega \in X^i$. Then $z - \omega$ *represents* the coset $z \bmod L_i \stackrel{\text{def}}{=} \{z + u \mid u \in L_i\}$ (as in §7.1). Let $\pi^i : \mathbb{C} \to \mathbb{C}/L_i$ be the map that takes z to $z \bmod L_i$. Then π^i is an analytic map. It becomes a homomorphism of groups if we make X^i into a group using this addition formula:

$$z_1 \bmod L_i + z_2 \bmod L_i \stackrel{\text{def}}{=} z_1{+}z_2 \bmod L_i[9.9d].$$

Suppose $L_1 \subseteq L_2$. Then, for $z \in \mathbb{C}$, the set

$$(\pi^1)^{-1}(z \bmod L_1) = \{z + \omega \mid \omega \in L_1\}$$

is in $(\pi^2)^{-1}(z \bmod L_2)$. So, the map f taking $z \bmod L_1$ to $z \bmod L_2$ depends only on $z \bmod L_1$, not on z. Identify $\pi_1(X^i, 0)$ with L_i (as in [9.9g]). The induced map f_* is the inclusion L_1 into L_2. For each $x_2 \in X_2$ the cardinality of the set $f^{-1}(x_2)$ is the order of the quotient group L_2/L_1 [9.7d].

Note: These concepts work equally well for finite unions of manifolds.

7. Permutation representations and covers

Two types of group theory arise in analyzing algebraic functions from Riemann's viewpoint. One is the presentation of fundamental groups, as free groups on generators with relations. Elementary examples of that do appear in many topology books (here too, starting with Chap. 4 §1.1). The second type is less common: Analyzing homomorphisms of fundamental groups into other groups. Motivating problems and sufficient group theory show how *finite* and *profinite* group theory apply to the study of moduli of Riemann surfaces. The group theory starts with permutation representations and their associated group representations.

7.1. Permutation representations. Denote by $\{\boldsymbol{x}\} = \{x_1, \dots, x_n\}$ any set of n distinct elements. Let S_n be the collection of *permutations* of $\{\boldsymbol{x}\}$, and regard S_n as a group in the usual way. Multiplication of permutations corresponds to functional composition of maps on $\{\boldsymbol{x}\}$. Reminder: As the introduction states, we typically act with S_n on the *right* of elements from $\boldsymbol{x}$, though sometimes the presence of a second action forces us to act on the left.

7.1.1. *Permutation notation and actions.* Denote the identity element of S_n by 1. Here is an inefficient, though clear way to express the effect of $g \in S_n$:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ (1)g & (2)g & \cdots & (n)g \end{pmatrix}$$

where $k = (j)g$ is the integer subscript of the image of x_j under g.

EXAMPLE 7.1. Suppose $n = 16$, and the display of g is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 16 & 12 & 9 & 8 & 1 & 3 & 2 & 5 & 6 & 10 & 11 & 7 & 4 & 13 & 14 & 15 \end{pmatrix}.$$

The notation indicates g maps x_9 to x_6. *Disjoint cycle notation* for g represents it as a product of disjoint cycles of integers. It requires fewer symbols than the complete permutation notation. Also, it shortens computations in S_n by parsing the group action into memorable pieces. The disjoint cycle representation for g:

$$(1\,16\,15\,14\,13\,4\,8\,5)(2\,12\,7)(9\,6\,3).$$

The order of the disjoint cycles is unimportant; $(i)g$ goes to the right of i. That is, $(1)g = 16$ is right of 1, and the cycle closes at 5 because $(5)g$ is 1, back to the beginning. Exclude cycles of length 1 ($(10)g = 10$ gives a cycle (10)) for efficiency. An element of S_n is a *k-cycle*, $k > 1$ if it has one and only one cycle — of length k — of length bigger than 1.

For another unique, less orthodox way to write permutations see [9.17a].

Let G be any group. A degree n *permutation representation* of G is a homomorphism $T : G \to S_n$. Such a T is the same as giving an *action* of G on the set $S = \{x_1, \dots, x_n\}$.

With G a group and S a set, a *right action* is a function: $A = A_R : S \times G \to S$: $A(s, g) \mapsto (s)g$ with two action properties:

(7.1a) $(s)g_1g_2 = ((s)g_1)g_2$ for $s \in S$, $g_1, g_2 \in G$. Using A we would write this

$$A(A(s, g_1), g_2) = A(s, g_1g_2).$$

(7.1b) $(s)1_G = s$ for $s \in S$ (the identity in G leaves $s \in S$ fixed).

A *left action* is from a function $A_L : G \times S \to S$ with the action composite

$$A_L(g_1, A_L(g_2, s)) = A_L(g_1g_2, s).$$

An *orbit* of an action is the range of the set $s \times G$, under A, for some $s \in S$. The *kernel* of the action $\ker(A)$ consists of those $g \in G$ that act like the identity on S. The most important example is where G acts on the right cosets of a subgroup H of G. The set $Hg = \{hg\}_{h \in H}$ is a *right coset* of H in G. Two right cosets Hg and Hg' are either equal or have no elements in common. Assume there are exactly n distinct right cosets of H in G: $H, Hg_2, \dots, Hg_n$. Call n the *index* $(G : H)$ of H in G. Finding good representatives for cosets is an art (try [9.17c]).

The archetype of a right action: $A : (Hg', g) \mapsto Hg'g$, or $g \in G$ maps a right coset Hg' to $(Hg')g = Hg'g$. For any subgroup H there is both a set of right cosets of H and a set of left cosets of H. Only if H is normal in G are all right cosets also left cosets. The map $(g, g'H) \mapsto gg'H$ is a left action on left cosets. There are further actions of groups in [9.16]. We emphasize a right action because this is the natural action of fundamental groups acting on points as in Lem. 7.13.

Definition 7.2. Suppose G is a group with a normal subgroup H and another subgroup W. Assume $\langle H, W \rangle = G$ and $H \cap W = \{1\}$. We say G is the semi-direct product of H and W, written $H \times^s W$.

If $G = H \times^s W$, then elements of G act as automorphisms of H by conjugation. This is an action A: For $g \in G$, $A(g) : h \in H \mapsto g^{-1}hg \stackrel{\text{def}}{=} h^g$. This is a right action. The following lemma, in a left or right action form is in almost all graduate texts in algebra.

Lemma 7.3. *Each element of $H \times^s W$ has a unique expression as wh, $h \in H$, and $w \in W$. Suppose $A : W \to \mathrm{Aut}(H)$ is a homomorphism giving a right action of W on H. Then, there is a group G given as a semi-direct product of H and W. Multiplication in this group satisfies the formula $w_1h_1w_2h_2 = w_1w_2(h_1)A(w_2)h_2$.*

Remark 7.4 (Affine action). There is a memorable notation for multiplication by imitating matrix multiplication of lower triangular 2×2 matrices. Associate $w_i h_i$ with $\begin{pmatrix} w_i & 0 \\ h_i & 1 \end{pmatrix}$, $i = 1, 2$. Then, the multiplication in $H \times^s W$ imitates an expected matrix calculation:

$$\begin{pmatrix} w_1 & 0 \\ h_1 & 1 \end{pmatrix}\begin{pmatrix} w_2 & 0 \\ h_2 & 1 \end{pmatrix} = \begin{pmatrix} w_1 w_2 & 0 \\ (h_1)A(w_2)h_2 & 1 \end{pmatrix}.$$

Further, $H \times^s W$ acts as permutations of H. For its matrix form, replace $h' \in H$ by the *vector* $(h', 1)$: $(h', 1)\begin{pmatrix} w & 0 \\ h & 1 \end{pmatrix} = ((h')A(w)h, 1)$ or $h' \mapsto (h')A(w)h$. The left action version with upper triangular matrices has a little glitch in it, unless H is an abelian group. That, however, comes up often in important examples (see [9.6]).

7.1.2. *Transitive and intransitive representations.* We discuss concepts that use coset representations. Lem. 7.7 shows how to go from the definition of action to the language of homomorphisms. When using groups acting on manifolds we often translate from actions into representations.

Definition 7.5. The *right coset* representation $T_H : G \to S_n$, defined by the subgroup $H \le G$, comes from the formula

(7.2) for $g \in G$, $i \in \{1, 2, \dots, n\}$, $(i)T_H(g) = j$ with Hg_j the right coset equal to $Hg_i g$.

Denote the subgroup of elements $g \in G$ for which $T(g)$ fixes the integer j by $G(T, j) = G(j)$. For T a permutation representation, $\ker(T)$ is $\{g \in G \mid T(g) = 1_G\}$, the kernel of the action of G. Call T *faithful* if $\ker(T)$ consists only of 1_G. Also, T is transitive (G under T has one orbit) if for each $i \in \{1, 2, \dots, n\}$, there is $g_i \in G$ with $(1)T(g_i) = i$. Then, $G(1)g_i$ is the set of $g \in G$ taking 1 to i. By definition, $\ker(T)$ is $\bigcap_{i=1}^n G(i)$. Assume T is transitive and $(1)T(g_i) = i$, $i = 1, \dots, n$. Then, $g_i^{-1}G(1)g_i$, the *conjugate* of each element of $G(1)$ by g_i, equals $G(i)$. So, $G(1) \dots, G(n)$ is a complete list of conjugates of $G(1)$ in S_n.

Definition 7.6. Let T_i be a degree n permutation representation of G, $i = 1, 2$. Suppose there is $h \in S_n$ with $h^{-1}T_1(g)h = T_2(g)$ for each $g \in G$. Then T_1 is *permutation equivalent* to T_2: T_1 and T_2 are equivalent as permutation representations.

Lemma 7.7. *In notation above, G acts on (right) cosets of $H \le G$, permuting them, and $T_H : G \to S_n$ is a homomorphism. The kernel is those $g \in G$ that fix each coset. This is the same as the elements of $\cap_{g \in G} g^{-1}Hg$. Reordering cosets of H in G changes the representation T_H only up to permutation equivalence.*

Suppose A_S (resp. $A_{S'}$) is an action of G on S (resp. S') with S and S' disjoint sets. Then, there is an action of G on $S \times S'$, the *direct product* action: $A \times A' : (S \times S') \times G \to S \times S'$ by $g \in G : (s, s') \in S \times S' \mapsto ((s)g, (s')g)$. There is also an action of G on $S \cup S'$, the *direct sum* action: $A \oplus A' : (S \dot{\cup} S') \times G \to S \dot{\cup} S'$ by $g \in G : s \in S \dot{\cup} S' \mapsto (s)g$ given by A if $s \in S$, and by A' if $s \in S'$. For $T : G \to S_n$ an arbitrary permutation representation, partition $\{1, \dots, n\}$ into a disjoint union $X_1 \cup X_2 \cup \cdots X_t$ of the G orbits. Suppose $n_i = |X_i|$, $i = 1, \dots, n$.

Theorem 7.8. *Let $T_H : G \to S_n$ be the right coset representation associated to the subgroup H of G. Then T_H is a transitive representation with* $\ker(T_H)$ *equal to $\bigcap_{g \in G} g^{-1}Hg$. Conversely, if $T : G \to S_n$ is a transitive representation of G, then T is permutation equivalent to T_H with $H = G(1)$. Generally, in the notation* *above for T, $T = \oplus_{i=1}^t T_i : G \to \oplus_{i=1}^t S_{n_i}$ presents T as the direct sum of right coset representations corresponding to subgroups of G.*

PROOF. For each $i \in \{1, 2, \dots, n\}$, formula (7.2) shows $(1)T_H(\sigma_i) = i$. So T_H is transitive. The subgroup $\ker(T_H)$ consists of the $g' \in G$ such that $Hg_ig' = Hg_i$, $i = 1, \dots, n$: $g' \in g_i^{-1}Hg_i$, $i = 1, \dots, n$. Each element in G has the form hg_i for some $h \in H$ and $i \in \{1, 2, \dots, n\}$. So, $g' \in \ker(T_H)$ if and only if $g' \in \bigcap_{g \in G} g^{-1}Hg$.

Let $T : G \to S_n$ be an arbitrary transitive permutation representation. Choose $g_1, \dots, g_n$ so that $(1)T(g_i) = i$, $i = 1, \dots, n$. Thus, the cosets $G(1)g_1, \dots, G(1)g_n$ are distinct. Conclude that (7.2), with $G(1)$ replacing H, gives $T_{G(1)}$. As

$$\{g \in G \mid (i)T(g) = j\} = g_j^{-1}G(1)g_i,$$

$(i)T(g) = j$ exactly if $(i)T_{G(1)}(g) = j$. This means $T_{G(1)}$ and T are the same permutation representation. We made choices in selecting the g_j s. So, independent of choices, the representations are permutation equivalent.

Now suppose the representation is not transitive. Since the orbits are all distinct, there is a natural map from the representation to the direct sum representation on the collection of orbits. □

7.1.3. *Primitive representations and equivariant maps.* A subgroup $H \le G$ is *normal* if $g^{-1}Hg = H$ for each $g \in G$. Only then is the set of pairwise products $HgHg'$ of two cosets a single coset, equal to Hgg'. So, the cosets have a natural group multiplication. Denote this set by G/H: Each element $\bar{g} = g \mod H \in G/H$ denotes the coset Hg. For H any subgroup of G, the *normalizer* of H in G is $N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$. Similarly, define the *centralizer* of H in G:

$$\text{Cen}_G(H) = \{g \in G \mid g^{-1}hg = h \text{ for each } h \in H\} \text{ [9.15]}.$$

DEFINITION 7.9. Consider a transitive permutation representation $T : G \to S_n$ of G. Call T *primitive* if there are no groups properly between $G(1)$ and G. Let $G(1)$ be the subgroup of G that fixes 1. If T is transitive, then it is T is *doubly transitive* if for each $j \in \{2, \dots, n\}$ there is a $g \in G(1)$ with $(2)T(g) = j$: $G(1)$ is transitive on $\{2, \dots, n\}$.

When the notation shows G is in S_n, we drop the T notation for permutation representations. The *transitivity formula* for a chain of subgroups $K \le H \le G$ says that $(G : K) = (G : H)(H : K)$.

LEMMA 7.10. *Doubly transitive permutation representations are primitive.*

PROOF. Suppose $G \le S_n$ is doubly transitive. Let H be a subgroup of G properly containing $G(1)$. Choose $h \in H \setminus G(1)$. Then $(1)h = j \in \{2, \dots, n\}$. For any $j' \in \{2, \dots, n\}$, use double transitivity to produce g' with $(1)g' = 1$ and $(j)g' = j'$: $hg' \in H$ takes 1 to j'. So, the number of cosets of $G(1)$ in H is the same as the number of cosets of $G(1)$ in G. Apply the transitivity formula to the chain $G(1) < H \le G$ to conclude the index of H in G is 1 and T is primitive. □

Assume group G acts on two sets: It has an action A_S (resp. $A_{S'}$) on S (resp. S') with S and S' related by a function $f : S \to S'$. We say f commutes with (is *equivariant* for) these actions if $f((s, g)A_S) = (f(s), g)A_{S'}$ for $s \in S, g \in G$.

EXAMPLE 7.11 (Compatible permutation representations). For G a group and M a normal subgroup, let $u_M : G \to G/M$ be the natural homomorphism with kernel H. Suppose H_1 is a subgroup of G and H_2 is a subgroup of G/M for which

$f_M(H_1) \le H_2$. Then u_M induces a map $f_M : \{H_1 g \mid g \in G\} \to \{H_2 g \mid g \in G\}$. This map commutes with G acting on the cosets of H_1 and on the cosets of H_2.

7.1.4. *Representations from permutation representations.* [9.6] gives many examples of primitive groups that are not doubly transitive. For $g \in G$, some authors abuse notation to write $T(g) = (s_1) \cdots (s_t)$ where $s_1, \cdots, s_t$ are the integer lengths of the disjoint cycles of $T(g)$ (we usually omit cycles of length one) to indicate a cycle type (conjugacy class) in S_n. Denote the count of length one cycles in $T(g)$ by $\mathrm{t}(T(g))$, the *trace* of $T(g)$. For example, the permutation example of §7.1.1 has trace 2 and its cube has trace 5. We remind why $T(g)$ it is a trace.

Regard the formal symbols $\{x_1, \dots, x_n\}$ as basis vectors for a vector space V over a field F. Then each permutation $g \in S_n$ extends linearly to act on V. That is, applying $g \in G$ to $v = \sum_{i=1}^n a_i x_i \in V$ gives $\sum_{i=1}^n a_i x_{(i)g}$. Write the result of g on x_i to be $\sum_{j=1}^n a_{i,j} x_i$ with coefficients denoting what would appear in the ith position of a matrix M_g acting on the right of (row) vectors. When F has characteristic 0, the matrix M_g has trace $\sum_{i=1}^n a_{i,i}$, the count of the number of x_i s that g fixes. In each row and column the matrix M_g has exactly one non-zero entry and that is a 1. So, M_g is an element of the orthogonal group O_n: M_g times its transpose is the identity matrix. The determinant function is multiplicative on $n \times n$ matrices. Conclude that M_g has determinant $\mathrm{Det}(M_g) \stackrel{\text{def}}{=} \mathrm{Det}(g)$ equal to ± 1. When the field F has characteristic p, the count of the integers fixed by g is the trace mod p. We may revert, when acting with matrices to a traditional left-hand action.

The result is that a degree n permutation representation T of a group G produces a homomorphism $\rho_T : G \to \mathrm{GL}_n(F)$. If T is a faithful permutation representation, then ρ_T is a faithful group representation: Its kernel is trivial. Any homomorphism $\rho : G \to \mathrm{GL}_n(F)$ is called a *representation* of G over the field F. With $V = F^n$, we often write V_T to indicate we mean V with the action through T. Then, for any representation, extend this notation to use V_ρ. In fact, group theory doesn't restrict to just finite dimensional representations, though we will. Most situations regard permutation representations as the same if they are equivalent. If $M \in \mathrm{GL}_n(F)$, then the two permutation representations $g \mapsto \rho(g)$ and $g \mapsto M^{-1}\rho(g)M$ are *(representation) equivalent*. Though two permutation representations may be inequivalent, their corresponding representations might be equivalent (§8.6.2 and [9.20]).

The group representation attached to the sum of permutation representations is the action on the direct sum of the vector spaces. When F has characteristic 0, every permutation representation of degree exceeding 1 is the direct sum of the identity representation and another representation. These are the only summands if and only if the permutation representation is doubly transitive [9.19d]. Further, the group representation of the direct product of two permutation representations is their *tensor product*; the trace is the product of the constituent traces [9.19a]. The *group ring* of G over F has the notation $F[G]$. The *product* of $\sum_{g\in G} a_g g$ and $\sum_{g\in G} b_g g$ (with $a_g, b_g \in F$) is given by convolution: $\sum_{g\in G} c_g g$ with $c_g = \sum_{h\in G} a_h b_{h^{-1}g}$, $g \in G$. A representation ρ then produces a homomorphism of associative rings: $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \rho(g) \in \mathbb{M}_{\deg(\rho)}(F)$. Call an idempotent I in this ring G *invariant* if it commutes with multiplication by elements of G. That means the range of I is a G invariant space: I is a G invariant projection [9.19h].

7.2. Covering spaces. Let X and Y be differentiable (resp. analytic) manifolds. Assume $f : Y \to X$ is a differentiable (resp. analytic) map. We will often use that if f is one-one, and onto in a neighborhood of a point, then it has a differentiable (resp. analytic) *inverse* (Lem. 4.2). Suppose $\varphi : X \to X'$ is any map between spaces, and x_0 maps to x'_0 under φ. As in Lem. 6.15, this induces a homomorphism on fundamental groups $\varphi_* : \pi_1(X, x_0) \to \pi_1(X', x'_0)$ by mapping a closed path $\gamma : [a, b] \to X$ to $[\varphi \circ \gamma] \in \pi_1(X', x'_0)$. This makes sense because composing with φ preserves homotopy classes of paths into X. Though obvious, it doesn't trivialize computing the image of $\pi_1(X, x_0)$ under φ_*.

DEFINITION 7.12 (Covering space). The pair (Y, f) (or just Y if there is no confusion) is a *covering space* (or cover) of X if each point $x \in X$ has a connected neighborhood (Chap. 2 §2.2.2) U_x with this property: for each connected component V of $f^{-1}(U_x)$, restricting f to V is a one-one and onto map $V \to U_x$.

7.2.1. *Degree of a cover.* Assume X is connected, and $f : Y \to X$ is a cover. Then, the cardinality of the fibers $|f^{-1}(x)|$, $x \in X$, being locally constant, must actually be constant. This is the *degree* $\deg(f)$ of f. We say (Y, f) is *finite*, or that f is a finite cover if $\deg(f) < \infty$.

Two covers $f_i : Y_i \to X$, $i = 1, 2$ are *equivalent* (as covers of X) if there is a one-one and onto continuous map $\psi : Y_1 \to Y_2$ with $f_2 \circ \psi = f_1$ [9.21]. Note: For any covering space (Y, f) of X, U an open subset of X, and V a union of connected components of $f^{-1}(U)$, the restriction of f to V gives a cover $(V, f_{|_V})$ of U.

A framework for considering equivalence classes of finite covers of a manifold X is the goal remaining to this subsection. This immediately reduces to considering connected finite covers (Y, f); we assume Y is a connected space. The classification hinges on producing an equivalence class, $T(Y, f)$, of permutation representations (§7.1) from an equivalence classes of covers (Y, f). We do that now.

Note: Covers in this section are what topologists call covers. In *algebraic geometry* the word cover includes complex analytic maps of manifolds having some fibers that *ramify* (their cardinality is smaller than the degree). The phrase then includes, for example, any nonconstant analytic map $f : Y \to \mathbb{P}^1_z$, with Y a compact Riemann surface and $\deg(f) \geq 2$. As the fundamental group of $\mathbb{P}^1_z$ is trivial, such an f must ramify (Chap. 4 Thm. 1.8). By the end of Chap. 4, a cover will include any surjective analytic map between compact complex manifolds with finite (point sets in their) fibers. Reference back to this chapter will speak of the unramified covers corresponding to subgroups of fundamental groups as in Thm. 7.16.

7.2.2. *Covers and permutation representations.* Let $f : Y \to X$ be a cover with $\gamma : [a, b] \to X$ a path having initial point x_0 and end point x_1.

LEMMA 7.13 (Action of path lifting). *For $y' \in Y$ with $f(y') = x_0$, there is a unique path $\tilde{\gamma} : [a, b] \to Y$ with $f \circ \tilde{\gamma} = \gamma$: the lift of γ with initial point y'.*

So, γ produces a unique map $\gamma_* : f^{-1}(x_0) \to f^{-1}(x_1)$ depending only on the image of γ in $\pi_1(X, x_0, x_1)$. In particular, consider paths $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, with $\gamma_1(b_1) = \gamma_2(a_1)$ and $\gamma_1(a_1), \gamma_2(b_1), \gamma_2(b_2)$ respectively x_0, x_1, x_2. Then, there is a transitivity formula:

$$(\gamma_1 \cdot \gamma_2)_* = (\gamma_1)_* \circ (\gamma_2)_* : f^{-1}(x_0) \to f^{-1}(x_2). \tag{7.3}$$

PROOF. Each $\gamma(t)$ has a neighborhood U_t with f one-one on the connected components of $f^{-1}(U_t)$. The argument of Chap. 2 §3.3.2 works here as it did there, by assuming you have extended the path lifting $\tilde{\gamma}$ to an interval $[a, t']$ with $t' < b$.

Let $[r, s]$ be a closed nontrivial interval for which $t' \in [r, s]$ and there is neighborhood $U_{t'}$ of $\widetilde{(t')}$ containing $\gamma([r, s])$ with $U' \subset f^{-1}(U_{t'})$ a connected component on which f is one-one and $\gamma^*(t') \in U'$. For each $t \in [r, s]$ define $\tilde{\gamma}(t)$ to be the unique point of U' lying over $\gamma(t)$. Finish exactly as in Chap. 2 §3.3.2.

Now considering (7.3) Since the map γ_* is clearly continuous and varies continuously in a homotopy family, as a map on a finite set, it is a homotopy class invariant. So, γ_* depends only on the image of γ in $\pi_1(X, x_0, x_1)$. The path $\widetilde{\gamma_1 \cdot \gamma_2}$ starting at y' is the same as the path $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2$ where $\tilde{\gamma}_2$ is the unique path starting at the end point of $\tilde{\gamma}_1$. The formula (7.3) just says the endpoint of both of these paths are the same. □

Label the points of $f^{-1}(x_0)$ as $\boldsymbol{y} = \{y_1, \ldots, y_n\}$. Consider a path $\gamma : [a, b] \to X$ based at x_0. Then, the end point of the lift of γ with initial point y_j, $j = 1, \ldots, n$ associates to γ and $\boldsymbol{y}$ a unique labeling of $f^{-1}(\gamma(b))$. A closed path γ gives an element of S_n, $T_{\boldsymbol{y}}(\gamma)$, as follows:

(7.4) $(i)T_{\boldsymbol{y}}(\gamma) = j$ with y_j the end point of the lift of γ with initial point y_i.

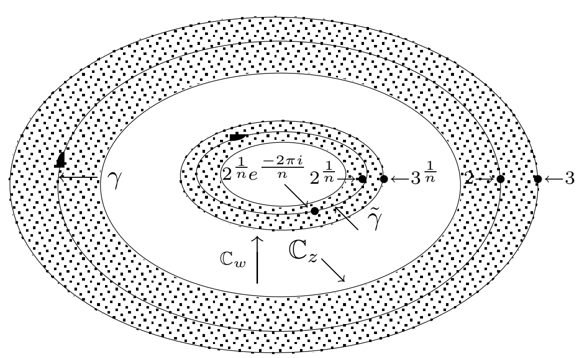
For $\gamma_1, \gamma_2 \in \Pi_1(X, x_0)$ (closed paths based at x_0) (7.3) gives

$$T_{\boldsymbol{y}}(\gamma_1\gamma_2) = T_{\boldsymbol{y}}(\gamma_1)T_{\boldsymbol{y}}(\gamma_2).$$

The right side consists of elements multiplied in S_n. So, $T_{\boldsymbol{y}}$ defines a permutation representation of $\pi_1(X, x_0)$ whose equivalence class we denote by $T(Y, f)$.

In Fig. 7, for example, $w \mapsto w^n = z$ gives the map $f : \mathbb{C}^*_w \to \mathbb{C}^*_z$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). A lift of γ (a clockwise circle, compatible with our choices in Chap. 4) is $\tilde{\gamma}$ going $\frac{1}{n}$ of the way around a clockwise circle. The associated permutation is an n-cycle of S_n representing that $\tilde{\gamma}$ goes from the lift $y' = 2^{1/n}$ of $\gamma(0) = 2$ to $y'' = 2^{\frac{1}{n}}e^{\frac{-2\pi i}{n}}$, the point on $\tilde{\gamma}$ lying $\frac{1}{n}$ of the way around from y'. §7.2.3 discusses a traditional picture representing the nth power map as if it were the projection on a real coordinate.

FIGURE 7. An n-cycle of path liftings



7.2.3. *Impossible pictures.* We discuss the problem of representing covers by pictures in $\mathbb{R}^3$. Consider the ramified cover $f : U_{w:0,\infty} \to U_{z:0,\infty}$ by $w \mapsto w^n$ in Fig. 7. Points of $U_{w:0,\infty}$ over $z \in U_{z:0,\infty}$ correspond on the graph of f to $\mathbb{C} \times \mathbb{C}$ points on the line with constant second coordinate z. You can't draw pictures in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4$. So first year complex variables texts try to represent $U_{w:0,\infty}$ and $U_{z:0,\infty}$ as subsets of $\mathbb{R}^3$.

Let (x_1, x_2, x_3) be coordinates for $\mathbb{R}^3$, and let $x_3 = 0$ represent $U_{z:0,\infty}$ sitting in $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Pictures try to represent an annulus around the origin in $U_{w:0,\infty}$ as a set M in $\mathbb{R}^3$ over an annulus D_0 in $U_{z:0,\infty}$. Then, points of M over $(x_1, x_2, 0) \in D_0$ are on the line in $\mathbb{R}^3$ whose points have first coordinates x_1 and x_2. That is, f appears as a coordinate projection. There is, however, no topological subspace M of $\mathbb{R}^3$ that can work! If there were, then a cylinder perpendicular to the plane $x_3 = 0$, with $(0,0,0)$ on its axis, would intersect M in a simple closed path winding n times around the cylinder. Represent such a path by $\gamma : [0,1] \to \mathbb{R}^3$ where $t \in [0,1]$ maps to

$$\gamma(t) = (\cos(2\pi nt), \sin(2\pi nt), x_3(2\pi nt)) \text{ and } x_3(2\pi n) = x_3(0).$$

Conclude: $w(t) = x_3(2\pi nt) - x_3(2\pi nt + 2\pi)$ is 0 for some value of t between 0 and $(n-1)/n$. So, the path isn't simple. The author has never seen such a picture attempt in the literature for any noncyclic cover, much less for more demanding nonsolvable groups. Still, we discuss this more in Chap. 4 §2.4 which also uses symbolic representations that assume we understand cyclic covers from their description in Chap. 2.

7.3. Pointed covers and a Galois correspondence. Let $f : Y \to X$ be a cover. Call the triple (Y, f, y') a *pointed cover* if $y' \in Y$. Then, we regard $f(x') = x_0$ as the base point for X, and (Y, f, y') is a pointed cover of (X, x_0).

DEFINITION 7.14. Suppose (Y, f_i, y_i'), $i = 1, 2$, are two pointed and connected covers of X. We say they are *compatibly pointed* (or compatible) if whenever we have covers $h : Z \to X$ and $h_j : Y_j \to Z$, with $h \circ h_j = f_j$, $j = 1, 2$, then $h_1(y_1) = h_2(y_2)$.

If it is clear a cover is pointed, we may refer just to the covering maps f_1 and f_2 to say these are compatible. Extension Lem. 8.1 shows the difference between a pointed cover on one hand, and a cover without a point on the other. Group theoretically this interprets as the difference between giving a subgroup of a group and giving a conjugacy class of subgroups.

7.3.1. *Fiber products of covers.* The basic theorems of Galois theory, including the construction of the Galois closure of a cover (§8.3), that translates geometrically using fiber products.

LEMMA 7.15. *Given connected covers $f_j : Y_j \to X$, $j = 1, 2$, of X, any connected component of $Y_1 \times_X Y_2$ is minimal among among connected covers (Y, f) of X factoring through each f_j. If the covers are compatibly pointed with $y_j' \in Y_j$, $j = 1, 2$, then a unique pointed component of $Y_1 \times_X Y_2$, $(Y, (f_1, f_2), (y_1', y_2'))$ is compatible with both (Y_i, f_i, y_i'), $i = 1, 2$.*

PROOF. Let Y be a connected component of $Y_1 \times_X Y_2$. Denote projection of Y on Y_j by pr_j. Consider any $(y_1, y_2) \in Y_1 \times_X Y_2$ lying over $x \in X$. Choose a neighborhood U_x of x for which there is a neighborhood $U_{y_j} \subset Y_j$ on which f_j maps one-one to U_x. Then, restricting (f_1, f_2) to $U_{y_1} \times_{U_x} U_{y_2}$ gives a one-one map that shows Y is a cover of X.

Now assume the covers are compatibly pointed. Let $x_0 \in X$ be $f_1(y_1') = f_2(y_2')$. Then, a unique component of $Y_1 \times_X Y_2$ contains (y_1', y_2'). □

Thm. 7.16 produces covers of any path-connected, locally path-connected space. For, however, our main applications where X is a (complex) manifold, it shows any cover of X is a (complex) manifold with a natural coordinate chart. It also says one cover of a space X dominates all others. This is the *universal* covering space $\tilde{X}$ corresponding to $H = \{1\} \leq \pi_1(X, x_0)$.

THEOREM 7.16 (Unramified Galois correspondence). *Let (Y, f, y') be a pointed cover of (X, x_0). This canonically corresponds to a subgroup $H_{Y,f,y'} \leq \pi_1(X, x_0)$ which we identify with $\pi_1(Y, y')$. The index $(\pi_1(X, x_0) : \pi(Y, y'))$ is $n = \deg(f)$. Any ordering $\boldsymbol{y} = \{y_1, \ldots, y_n\}$ on the fiber $f^{-1}(x_0)$ with $y_1 = y'$ corresponds to a transitive permutation representation $T_{Y,f,\boldsymbol{y}}$ in which the stabilizer of 1 is $H_{Y,f,y'}$. If $y'' \in f^{-1}(x_0)$, then $H_{Y,f,y'}$ and $H_{Y,f,y''}$ are conjugate subgroups of $\pi_1(X, x_0)$ and we identify y'' with a coset of H in $\pi_1(X, x_0)$.*

Conversely, each subgroup $H \leq \pi_1(X, X_0)$ of index n (possibly ∞) produces a canonical pointed (connected) degree n cover (Y_H, f_H, y'_H) of X. We regard y'_H as the H coset of the identity in $\pi_1(X, X_0)$. The fundamental group of Y_H maps one-one onto H under $(f_H)_*$.

Suppose H_1 and H_2 are two subgroups of $\pi_1(X, x_0)$. Then, the unique connected component of $Y_{H_1} \times_X Y_{H_2}$ containing (y'_{H_1}, y'_{H_2}) corresponds to the subgroup $H_1 \cap H_2$. The maximal pointed cover of X through which both f_1 and f_2 factor is $(Y_{\langle H_1,H_2\rangle}, f_{\langle H_1,H_2\rangle}, y'_{\langle H_1,H_2\rangle})$.

§7.3.2 consists of a proof of Thm. 7.16 and §8.1 has corollaries appropriate for covers that aren't pointed.

7.3.2. *Proof of Thm. 7.16.* Start with (Y, f, y'). Apply (7.4) to a closed path $\gamma : [a, b] \to X$ based at x_0. Use a specific ordering of $f^{-1}(x_0)$ with $y_1 = y'$. The lift of γ to a path with initial point y_1 is a closed path in Y based at y_1 if and only if $(1)T_{\boldsymbol{y}} = 1$. So we identify $\pi_1(Y, y_1)$ with $H(f, y_1)$, the subgroup of $\pi_1(X, x_0)$ stabilizing 1 under the map f_*.

Now consider how a subgroup H of $\pi_1(X, x_0)$ of index n canonically produces a degree n pointed cover of X. First: H produces an equivalence class of permutation representations of $\pi_1(X, x_0)$ of degree n (Thm. 7.8), with the coset of the identity corresponding to the integer 1 in the permutation representation.

Define Y_∞: As a set it is the collection of all equivalence classes of paths in X — not necessarily closed — with initial point x_0. For $\gamma \in Y_\infty$ let $f_\infty([\gamma])$ be the endpoint of γ. Define Y_H to be Y_∞ modulo the relation that equivalences

$$[\gamma_1] \text{ and } [\gamma_2] \text{ if } f_\infty([\gamma_1]) = f_\infty([\gamma_2]) \text{ and } [\gamma_1\gamma_2^{-1}] \in H.$$

Let $f_H : Y_H \to X$ be the map induced by f_∞ on the set Y_H. Now use that X is a connected manifold. For each $x \in X$ choose a path γ with initial point x_0 and endpoint x. A ball neighborhood U_x of x has this property: For $\gamma_1, \gamma_2 : [a', b'] \to U_x$, two paths with the same initial and endpoints, $\gamma_1\gamma_2^{-1}$ is equivalent to the constant path in U_x.

For each such pair (γ, U_x) consider the subset of Y_H represented by paths $\gamma\gamma_1$ with γ_1 a path in U_x with initial point x. Denote this subset by V_{γ,U_x}. We declare the topology on Y_H to have as a basis of open sets these V_{γ,U_x} s running over all pairs (x, U_x). For $y \in Y_H$ with $f_H(y) = x$, $f_H^{-1}(U_x)$ has n connected components, V_{γ_i,U_x}, $i = 1, \ldots, n$, where $[\gamma_1\gamma_i^{-1}]$ runs over distinct coset representatives of H in $\pi_1(X, x_0)$. With this topology (Y_H, f_H) satisfies Def. 7.12. It also has an atlas of open sets inherited from X. If we show Y_H is Hausdorff, then (Y_H, f_H) is a cover of X. As usual, since X is Hausdorff, we have only to find disjoint open sets around two points over the same point of X. We have done exactly that above.

To complete classifying pointed covers of X, we show the following. Given (Y, f, y') a connected cover and $H(f, y')$ the corresponding subgroup of $\pi_1(X, x_0)$, and $(Y_{H(f,y')}, f_{H(f,y')}, y'_{H(f,y')})$ the cover of X associated to $H(f, y')$, then

(7.5) (Y, f, y') is equivalent to $(Y_{H(f,y')}, f_{H(f,y')})$.

For $y \in Y$ let $\gamma^* : [a, b] \to Y$ be a path from y' to y, and let $\psi(y) = f_H(\gamma^*)$. Follow the defined maps to see $\psi : Y \to Y_{H(f,\boldsymbol{y})}$ is a one-one map giving (7.5).

Suppose (Y_H, f_H, y_H) is the canonical cover defined by $H \le \pi_1(X, x_0)$. Let (Y_H, f_H, y'') by the same cover, those with a different point, $y'' \in f_H^{-1}(x_0)$. Any $\gamma \in \pi_1(Y, y_H, y'')$ defines a coset $H[\gamma]$ of H in $\pi_1(X, x_0)$. Conversely, the elements of $\pi_1(X, x_0)$ that stabilize $H[\gamma]$ are exactly the elements of the conjugate subgroup $[\gamma^{-1}]H[\gamma]$. That shows that using different points attached to a fixed cover correspond to subgroups conjugate to H.

Now suppose H_1 and H_2 are two subgroups of $\pi_1(X, x_0)$. We must show properties attached to the equivalence of two categories: Pointed covers of (X, x_0) and subgroups of $\pi_1(X, x_0)$. The notion of fiber product is a categorical construction. So, the association between $H_1 \cap H_2$ and $(Y_{\langle H_1,H_2\rangle}, f_{\langle H_1,H_2\rangle}, y'_{\langle H_1,H_2\rangle})$ is that they are the fiber products of the two givens in their respective categories. Def. 1.3 notes the fiber product for subsets of a set is just their intersection. As the intersection of two subgroups is a subgroup, the fiber product from subgroups of a group is just their intersection. For saying fiber product is categorical, see [9.3a]. Similarly, the correspondence between $\langle H_1, H_2\rangle$ and $(Y_{\langle H_1,H_2\rangle}, f_{\langle H_1,H_2\rangle}, y'_{\langle H_1,H_2\rangle})$ is that these are the *pushouts* of the two givens in their respective categories [9.3c].

8. Group theory and covering spaces

We won't be able to make explicit computations with covers until Chap. 4. Still, the topics of this section come from practical experience with covers. Following a discussion of algebraic functions (§8.2) and a geometric approach to the Galois closure of a cover (§8.3), we consider the decomposing covers (§8.4) and the relation between covers and locally constant bundles (§8.5). A problem from this on computing components of covers shows the power of an elementary piece from finite group representations (§8.6)

8.1. Corollaries of Thm. 7.16. Suppose (Y_i, f_i, y'_i), $i = 1, 2$, are any two pointed covers of (X, x_0). By an isomorphism $g : (Y_1, f_1, y'_1) \to (Y_2, f_2, y'_2)$ between them, we mean an isomorphism between Y_1 and Y_2 with these properties:

(8.1a) $g(y'_1) = y'_2$ (g preserves basepoints); and

(8.1b) $f_2 \circ g = f_1$ (g commutes with projections).

The crucial point is that if two pointed covers are isomorphic, this isomorphism is unique. Suppose, however, we don't assume g preserves basepoints?

LEMMA 8.1 (Extension Lemma). *Consider a pair of covers (Y_i, f_i), $i = 1, 2$, without their basepoints, and any isomorphism g between them. Then, g maps the fiber $f_1^{-1}(x_0)$ one-one to $f_2^{-1}(x_0)$, and what g does to any one element of $f_1^{-1}(x_0)$ determines g. Further, isomorphisms between (Y_1, f_1) and (Y_2, f_2) correspond one-one with automorphisms $\mathrm{Aut}(Y_i, f_i)$ of (Y_i, f_i) (for either $i = 1$ or 2).*

Any automorphism of a cover (Y, f) of X lifts to an automorphism of the universal cover $(\tilde{X}, \tilde{f})$ of X. If X is a complex manifold, then $\mathrm{Aut}(Y, f)$ is a group of complex analytic isomorphisms.

PROOF. Assume g that maps $y'_1 \in f_1^{-1}(x_0)$ to $y'_2 \in f_1^{-1}(x_0)$. Then, g is an isomorphism between (Y_1, f_1, y'_1) and (Y_2, f_2, y'_2), and so it is unique. Let $A_{1,2}$ be the set of isomorphisms between (Y_1, f_1) and $Y_2, f_2)$. Then, we have an action of $\mathrm{Aut}(Y_1, f_1)$ (resp. $\mathrm{Aut}(Y_2, f_2)$) on the right (resp. left) of $A_{1,2}$:

$$\begin{aligned} A_1 &: A_{1,2} \times \mathrm{Aut}(Y_1, f_1) \to A_{1,2} \text{ by } \quad (g, \alpha) \mapsto g \circ \alpha; \text{and} \\ A_2 &: \mathrm{Aut}(Y_2, f_2) \times A_{1,2} \to A_{1,2} \text{ by } \quad (\beta, g) \mapsto \beta \circ g. \end{aligned}$$

For $g', g \in A_{1,2}$, $g^{-1}g' = \alpha$ is in $\mathrm{Aut}(Y_1, f_1)$. This shows $g \circ \alpha = g'$, and A_1 is transitive on $A_{1,2}$ (as in §7.1). Similarly, A_2 is transitive on $A_{1,2}$.

Now consider an automorphism α of (Y, f). Again, let (Y, f, y') with y' over x_0 be a corresponding pointed cover. Then, (Y, f, y') and $(Y, f, \alpha(y'))$ are pointed covers of (X, x_0). So, Thm. 7.16 shows they correspond to conjugate subgroups H and H_α: $H_\alpha = [\gamma^{-1}]H[\gamma]$ for some $[\gamma] \in \pi_1(X, x_0)$. A natural analytic isomorphism between (Y_H, f_H, y'_H) and $(Y_{H_\alpha}, f_{H_\alpha}, y'_{H_\alpha})$ comes by mapping the homotopy class of $[\gamma']$ defining a point of Y_{H_α} (in §7.3.2) to $[\gamma][\gamma']$. The new base point (the coset of $[\gamma]$) has stabilizer $[\gamma^{-1}]H[\gamma]$. This automorphism lifts to the universal covering space, because premultiplying by $[\gamma]$ also defines it there. □

DEFINITION 8.2. Let $T_{\boldsymbol{y}} : \pi_1(X, x_0) \to S_n$ be the representation of (7.4) associated to (Y, f). The image of $\pi_1(X, x_0)$ is called the *(geometric) monodromy group*, $G(Y, f)$, of the cover. It is isomorphic to $\pi_1(X, x_0)/\bigcap_{i=1}^n \pi_1(Y, y_i)$ (Thm. 7.8).

Covers (Y, f) of a manifold (X, x_0) have two extremes. For most, $\mathrm{Aut}(Y, f)$ consists only of the identity element: We say (Y, f) has no automorphisms. The other extreme is in this definition.

DEFINITION 8.3. If $\mathrm{Aut}(Y, f)$ is transitive on the the fiber $f^{-1}(x_0)$, we say (Y, f) is Galois.

The Galois situation is our main tool, though what constantly arises in practice is the situation with no automorphisms. §8.3 has the details for distinguishing these and all the cases in between. An example of the Galois situation is the universal cover of (X, x_0) where the automorphism group is isomorphic to the whole fundamental group of (X, x_0). The fiber $f^{-1}(x_0)$ in this case corresponds to the elements of $\pi_1(X, x_0)$, and by translation these give a permutation of the points. Automorphisms also give a permutation of $f^{-1}(x_0)$. Still, from Lem. 8.8, only when $\pi_1(X, x_0)$ is abelian can we expect to canonically identify these two groups of permutations. The next lemma revisits Chap. 2 Prop. 3.2. As previously, use the notation $\tilde{f} : \tilde{X} \to X$ for the universal cover of X with paths starting at x_0 representing its points.

LEMMA 8.4. *In the notation above, let* $[\gamma] \in \pi_1(X, x_0)$ *and let* $[\gamma']$ *represent a homotopy class of paths on* X *with* $\gamma' : [a, b] \to X$, $\gamma'(a) = x_0$ *and* $\gamma'(b) = x$. *Then, multiplication by* $[\gamma]^{-1}$ *on the left of* γ' *induces an automorphism of* $\tilde{X}$ *giving an action* $A_L : \pi_1(X, x_0) \times \tilde{X} \to \tilde{X}$. *Regard the fiber* $\tilde{f}^{-1}(x_0)$ *as elements of* $\pi_1(X, x_0)$. *Then, the usual right action of* $\pi_1(X, x_0)$ *gives the group structure identifying* $\pi_1(X, x_0)$ *with the monodromy group of* $\tilde{f}$.

The exponential map $\exp : \mathbb{R} \to S^1$ *by* $\theta \mapsto e^{2\pi i\theta}$ *presents* $\mathbb{R}$ *as the universal cover of* S^1 *with* $\mathbb{Z}$ *as its fundamental group. The path* γ_n^* *corresponds to* $n \in \mathbb{Z}$ *and the automorphisms of* $(\mathbb{R}, \exp)$ *identify with* $\mathbb{Z}$ *acting by translation. Similarly, the fundamental group of a complex torus* $\mathbb{C}^n/L$ *identifies with the lattice* L.

Proof. The universal covering space is unique up to homeomorphisms commuting with the map to X. One way to identify the fundamental group of a space X is to find any space $\tilde{X}$ with trivial fundamental group and a covering map $\tilde{f} : \tilde{X} \to X$. Given $x_0 \in X$, any other cover of X that has trivial fundamental group must be isomorphic to $(\tilde{X}, \tilde{f})$, and this isomorphism is unique up to composition on the left with an element of $(\tilde{X}, \tilde{f})$. Since $\mathbb{R}$ and $\mathbb{C}^n$ are contractible, they have trivial fundamental group (Lem. 6.6). The map $\theta \in \mathbb{R} \mapsto e^{2\pi i\theta}$ is a covering map with the elements of $\mathbb{R}$ over 1 given by the integers. The permutation of the fiber over 1 given by the path γ_n^* is translation by n. The argument is similar for a complex torus. □

The next corollary tells when a map between spaces extends to a map between covers of the spaces.

Corollary 8.5. *Suppose $\varphi : X \to X'$ is a differentiable map between complex manifolds mapping a point $x_0 \in X$ to $x_0' \in X'$. Let $\varphi_{H'} : Y'_{H'} \to X'$ be the cover defined by a subgroup $H' \le \pi_1(X', x_0')$. Then, there is a continuous (and so automatically differentiable) map $\psi : X \to Y'_{H'}$ with $\varphi_{H'} \circ \psi = \varphi$ if and only if the induced map $\varphi_* : \pi_1(X, x_0) \to \pi_1(X', x_0')$ has image in a conjugate of H'.*

Proof. Suppose the induced map φ_* has image in a conjugate $m^{-1}H'm$ of H'. Let γ^* be a representative path in X' for which $[\gamma^*] = m$. Then, let $\gamma : [a, b] \to X$ start at x_0 and end at x. Define $\psi_{m,H'} : X \to Y'_{H'}$ by $\psi(x)$ is the class $m \cdot [\varphi \circ \gamma] \in Y'_{H'}$: the product of m and the image under ψ of γ. To show the map doesn't depend on γ, we consider another closed path γ' from x_0 to x. We are done if the closed path $(\gamma^*)^{-1} \cdot \psi(\gamma \cdot (\gamma')^{-1}) \cdot \gamma^*$ in X' defines a closed path in $Y'_{H'}$. Since, however, $\gamma \cdot (\gamma')^{-1}$ is a closed path in X, its image under
phi_* is some $\rho \in m^{-1}H'm$ by hypothesis and the image of $(\gamma^*)^{-1} \cdot \psi(\gamma \cdot (\gamma')^{-1}) \cdot \gamma^*$ is therefore $m\rho m^{-1} \in H'$. From the definition of $Y'_{H'}$ this exactly says the image path is closed.

Conversely, suppose there is such a $\psi : X \to Y'_{H'}$. Then, closed paths in X have image under ψ in X' that lift to closed paths in $Y'_{H'}$. So, the image group $\psi_*(\pi_1(X, x_0)) = H^*$ is a subgroup of $\pi_1(X', x_0')$ whose corresponding cover Y'_{H^*} factors through $\psi_{H'} : Y'_{H^*} \to X'$. □

Suppose X is a connected complex manifold (like $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$). Define analytic continuation along a path from Def. 6.7. Consider the *extensible* functions $\mathcal{E}(X, x_0)$: complex analytic functions defined in a neighborhood of x_0 that have an analytic continuation along every path in X (as in Chap. 2 Def. 4.5). Let $\varphi : Y \to X$ be a cover with $y_0 \in Y$ lying over x_0. Let $\gamma : [a, b] \to X$ be a path starting at x_0 with $\gamma^\dagger : [a, b] \to Y$ its unique path lift starting at y_0 (Lem. 7.13).

Proposition 8.6. *There is an isomorphism (of rings) between $\mathcal{E}(Y, y_0)$ and $\mathcal{E}(X, x_0)$. In particular, for $(\tilde{X}, \tilde{x}_0)$ the universal cover of (X, x_0), holomorphic functions on $\tilde{X}$ form a ring isomorphic to $\mathcal{E}(X, x_0)$. If φ is a finite cover of punctured Riemann surfaces, this induces an analytic isomorphism between $\mathcal{E}(Y, y_0)^{\mathrm{alg}}$ and $\mathcal{E}(X, x_0)^{\mathrm{alg}}$. These results hold with extensible meromorphic replacing extensible holomorphic functions.*

Proof. Since φ is a cover, there is a disk neighborhood U_{x_0} of x_0 and a component U_{y_0} of $\varphi^{-1}(U_{x_0})$ with $y_0 \in U_{y_0}$ on which φ maps one-one. So, restriction of a function $f \in \mathcal{E}(X, x_0)$ to U_{x_0} transports by φ^{-1} to a function $f \in U_{y_0}$. There is no harm in using the same notation to extend f along $\gamma^\dagger : [a,b] \to Y$ starting at y_0. Let γ be $\varphi \circ \gamma^\dagger$, and let $f^* : [a,b] \to \mathbb{P}^1_z$ be the continuous function defining the analytic continuation along γ. Define the analytic continuation of f along $\gamma^\dagger$ to be the same, f^*. This shows f is extensible in Y. Clearly, if f is algebraic (on X) it will also be algebraic on Y. □

8.2. The problem of identifying algebraic functions explicitly. Suppose $\tilde\varphi : \tilde X \to X$ is the universal covering space of a complex manifold X and $\tilde x$ lies over $x_0 \in X$. Then, similar to formation of complex torii and other quotient manifolds, it is natural to regard points of X as the orbits of the action of $\pi_1(X, x_0)$ on $\tilde X$. Riemann's approach was to identify the universal covering space of a Riemann surface as a simply connnected domain on the Riemann sphere. Consider the case of Prop. 8.6 when $Y = \tilde X$ and $X = U_{\boldsymbol z}$, with $|\boldsymbol z| \ge 3$. Riemann's Uniformization Theorem says $\tilde X$ is analytically isomorphic to a disk Δ in such a way that the map extends continuously to the boundaries (Chap. 4 Def. 6.8 for an elementary proof, or [**Spr57**, Thm. 9.6] for the more general case). So, $\mathcal{E}(U_{\boldsymbol z}, z_0)$ is ring isomorphic to convergent functions in a disk. We find it convenient to replace a disk by the analytically isomorphic upper half plane $\mathbb{H}$. This is the same exact space independent of $(z_0, \boldsymbol z)$. What changes, however, with $\boldsymbol z$ is the identification of algebraic functions $F_{\boldsymbol z}$. Suppose $\varphi_{\boldsymbol z} : \mathbb{H} \to U_{\boldsymbol z}$ is this uniformization.

Elements of $\mathrm{PGL}_2(\mathbb{R})$ with positive determinant (Chap. 2 [9.14d]; this identifies with $\mathrm{PSL}_2(\mathbb{R})$) represent the action of complex analytic isomorphisms of $\mathbb{H}$. As $\boldsymbol z$ varies, a different subgroup $\Gamma_{\boldsymbol z}$ (though abstractly isomorphic as a group) of $\mathrm{PSL}_2(\mathbb{R})$ defines $U_{\boldsymbol z}$ as a quotient of $\mathbb{H}$.

Prop. 8.6 identifies extensible (meromorphic) algebraic functions on $U_{\boldsymbol z}$ with certain meromorphic functions $\mathcal{F}_{\boldsymbol z}$ on $\mathbb{H}$. Though, which ones? Given g^* meromorphic on $\mathbb{H}$, composing it with an analytic isomorphism of $\mathbb{H}$ produces a new meromorphic function on $\mathbb{H}$. We call the compositions of g^* with elements of $\Gamma_{\boldsymbol z}$ *transforms* by $\Gamma_{\boldsymbol z}$.

PROPOSITION 8.7. *Suppose f, meromorphic on $\mathbb{H}$, has only finitely many transforms under the action of $\Gamma_{\boldsymbol z}$ and a unique limit value as it approaches any point in $\mathbb{R} \cup \{\infty\}$. Then, f defines an algebraic element of $\mathcal{E}(U_{\boldsymbol z}, z_0)$ and conversely.*

OUTLINE. Let $\tilde x \in \mathbb{H}$ lie over $z_0 \in U_{\boldsymbol z}$. From Prop. 8.6, any meromorphic extensible function g on $U_{\boldsymbol z}$ identifies with a meromorphic function g^* on $\mathbb{H}$. Further, the analytic continuation of g around $[\gamma] \in \pi_1(U_{\boldsymbol z}, z_0)$ produces g^*_γ, the result of composing g^* with the analytic isomorphism of $\mathbb{H}$ associated to γ. If g is algebraic, then it has only finitely many analytic continuations, so the different transforms g^*_γ, running over $\gamma \in \pi_1(U_{\boldsymbol z}, z_0)$ are finite in number. Conversely, if the number of transforms of a meromorphic function g^* on $\mathbb{H}$ are finite in number, then the identification of g^* with $g \in \mathcal{E}(U_{\boldsymbol z}, z_0)$ gives a function with only finitely many analytic continuations. □

8.3. Galois theory and covering spaces. Use notation from Lem. 8.1: (Y, f) is a cover of X.

8.3.1. *Identifying automorphisms of a cover.* Having $\mathrm{Aut}(Y, f)$ act on a fiber $\{y_1, \dots, y_n\} = f^{-1}(x_0)$ induces a homomorphism $\Lambda_{\boldsymbol y} : \mathrm{Aut}(Y, f) \to S_n$.

It is a mistake to confuse the Galois (geometric monodromy) group of a cover with its automorphism group, even if the cover is Galois. The next lemma efficiently differentiates $\mathrm{Aut}(Y,f)$ from $G(Y,f)$. It shows that having chosen a *right* action for $G(Y,f)$ forces using a left action of $\mathrm{Aut}(Y,f)$ on the set $\{1,\dots,n\}$.

LEMMA 8.8. *Let (Y,f) be a connected cover of X. The homomorphism $\Lambda_{\boldsymbol{y}}$ injects $\mathrm{Aut}(Y,f)$ onto the centralizer $\mathrm{Cen}_{S_n}(G(Y,f))$ of $G(Y,f)$ in S_n. This is isomorphic to $N_{\pi_1(X,x_0)}(\pi_1(Y,y_1))/\pi_1(Y,y_1)$ (§7.1) and $|\mathrm{Aut}(Y,f)|\le n$ with equality if and only if $\pi_1(Y,y_1)$ is normal in $\pi_1(X,x_0)$.*

PROOF. For $y\in Y$ let $\gamma^*:[a,b]\to Y$ be a path with initial point y_i and endpoint y. Consider $\psi\in\mathrm{Aut}(Y,f)$. Then $\psi\circ\gamma^*:[a,b]\to Y$ is the (unique) lift of $f\circ\gamma^*$ with initial point $\psi(y_i)$. So, if $i=1$ and $\psi(y_1)=y_1$, then $\psi\circ\gamma^*=\gamma^*$. Thus $\psi(y)=y$ for each $y\in Y$, and $\Lambda_{\boldsymbol{y}}$ is injective. This alone shows $|\mathrm{Aut}(Y,f)|\le n$.

In the above, assume $\gamma=f\circ\gamma^*$ is a closed path. If the endpoint of γ^* is y_j, then the endpoint of $\psi\circ\gamma^*$ is $\psi(y_j)$. Thus

$$(i)\Lambda_{\boldsymbol{y}}(\psi)^{-1}\circ T_{\boldsymbol{y}}(\gamma)\circ\Lambda_{\boldsymbol{y}}(\psi)=(i)T_{\boldsymbol{y}}(\gamma).$$

Equivalently, $\Lambda_{\boldsymbol{y}}(\psi)\in\mathrm{Cen}_{S_n}(G(Y,f))$. Conversely, for $\alpha\in\mathrm{Cen}_{S_n}(G(Y,f))$ define α to be a permutation of the points $\{y_1,\dots,y_n\}$ from its action on $\{1,\dots,n\}$. Still, use an action on the left: If $(i)\alpha=j$, write $\alpha(y_i)=y_j$). Our goal is to create an automorphism—also called α—on Y that extends this action on the fiber over x_0.

Take $i=1$ and γ^* as in the first paragraph above. Define ψ_{α,γ^*}:

(8.2) $\psi_{\alpha,\gamma^*}(y)$ is the endpoint of the lift of $f\circ\gamma^*$ with initial point $\alpha(y_1)$.

If we show $\psi_{\alpha,\gamma^*}(y)$ is independent of γ^* having endpoint y, then ψ_{α,γ^*} defines an element $\psi_\alpha\in\mathrm{Aut}(Y,f)$. For this purpose let γ^1 (resp., γ^2) be a path in Y with initial (resp., end) point y and end (resp., initial) point y_1. If $\psi_{\alpha,\gamma^*}(y)\ne\psi_{\alpha,\gamma^1}(y)$, then $\psi_{\alpha,\gamma^*\gamma^2}(y_1)\ne\psi_{\alpha,\gamma^1\gamma^2}(y_1)$. Therefore, $\psi_{\alpha,\gamma^*}(y)$ is independent of γ^* if and only if $\psi_{\alpha,\gamma^*}(y_1)$ is independent of γ^* for $\gamma^*\in\pi_1(Y,y_1)$. That is, we must show $\alpha(y_1)$ is the endpoint of the lift of $f\circ\gamma$ with initial point $\alpha(y_1)$ for each $\gamma\in\pi_1(Y,y_1)$.

With $\alpha(y_1)=y_j$, this is equivalent to $((1)\alpha)T(Y,f)(f\circ\gamma)=j$. (The right action of α on 1 is intentional—α did come from S_n.) For γ a closed path on Y with initial point y_1, $(1)T(Y,f)(f\circ\gamma)=1$ is automatic. Apply α to the right side of this and use that α commutes with $T(Y,f)(f\circ\gamma)$ to conclude from [9.15b]. Recall: $G(1)$ is the subgroup of $G(Y,f)$ leaving 1 fixed. Thm. 7.16 identifies $N_{\pi_1(X,x_0)}(\pi_1(Y,y_1))/\pi_1(Y,y_1)$ with $N_{G(Y,f)}(G(1))/G(1)$. □

8.3.2. *Fiber products and Galois closure.* We say a connected cover (Y,f) of X is a *Galois cover* (or is Galois) if $|\mathrm{Aut}(Y,f)|$ equals $n=\deg(f)$. By Lem. 8.8 this holds if and only if $\pi_1(Y,y_1)$ is a normal subgroup of $\pi_1(X,x_0)$). Each cover (Y,f) produces a Galois cover $(\hat{Y},\hat{f})$ of X called the Galois closure of (Y,f). If $H\le\pi_1(X,x_0)$ corresponds to Y, then $\cap g^{-1}Hg$ corresponds to $(\hat{Y},\hat{f})$. We use fiber products to give an alternate construction of it (Def. 1.3). It correctly displays the automorphism group action. We again warn: Don't confuse it with the geometric monodromy group, though they are isomorphic for a Galois cover.

Denote the fiber product of $Y\to X$ taken $n=\deg(f)$ times by

$$Y_X^n\stackrel{\text{def}}{=}Y\times_X\times\cdots\times_X Y.$$

Points of Y_X^n are n-tuples $(y_1',\dots,y_n')\in Y^n$ for which $f(y_i)=f(y_j)$ for all i and j. The *fat diagonal*, $\Delta_{Y,f,n}$, is the subset of n-tuples of Y_X^n with at least two equal coordinate entries. Remove it to form $Y_X^n\setminus\Delta_{Y,f,n}=U_{Y,f,n}$. We use a copy of S_n acting on the *left* of $\{1,\dots,n\}$ to give an action of automorphisms on this set:

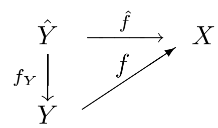
(8.3) for $\sigma \in S_n$ and $\boldsymbol{y}' = (y'_1, \ldots, y'_n) \in U_{Y,f,n}$, α_σ maps $\boldsymbol{y}'$ to

$$(y'_{\sigma(1)}, \ldots, y'_{\sigma(n)}) = \alpha_\sigma(\boldsymbol{y}').$$

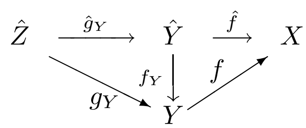
Restrict the natural map of Y^n_X to $X$to $U_{Y,f,n}$ to present $U_{Y,f,n}$ as a degree $n!$ cover of X with automorphism group containing S_n. The action of S_n is transitive on points mapping to x_0. Yet, $U_{Y,f,n}$ may not be connected. (We don't consider it a Galois cover of X.) Decompose $U_{Y,f,n}$ into connected components $\hat{Y}_1, \ldots, \hat{Y}_t$. Let $\hat{f}_i$ be the restriction to $\hat{Y}_i$ of the projection map $U_{Y,f,n} \to X$, $i = 1, \ldots, t$. A computation shows $\deg(\hat{f}_i) = |G(Y, f)|$ [9.22].

THEOREM 8.9. *The covers $(\hat{Y}_i, \hat{f}_i)$ are equivalent as covers of X, $i = 1, \ldots, t$. Characterize members $(\hat{Y}, \hat{f})$ of this equivalence class from these properties.*

(8.4a) *$(\hat{Y}, \hat{f})$ is a Galois cover of X, with its group a transitive subgroup of S_n.*
(8.4b) *There is a commutative diagram of covers of X:*



(8.4c) *For any Galois cover $\hat{g} : \hat{Z} \to X$ factoring through Y by $g_Y : \hat{Z} \to Y$, there is commutative diagram of covers of X:*

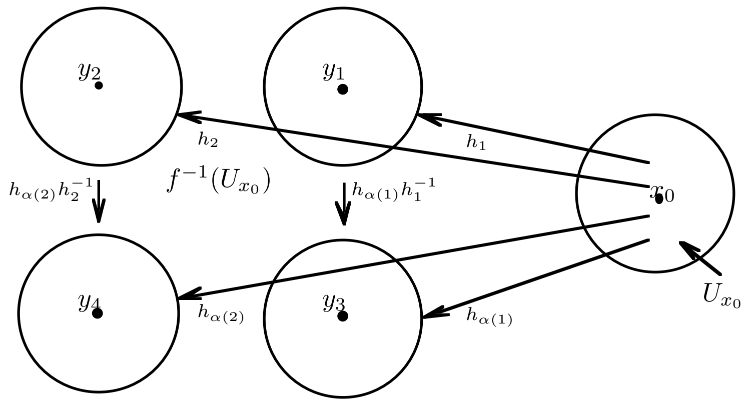


PROOF. Choose $y_1 \in Y$ lying over $x_0 \in X$. Thm. 7.3.2 identifies the subgroup of $\pi_1(X, x_0)$ corresponding to (Y, f) with $\pi_1(Y, y_1)$. It also identifies its conjugates (in $\pi_1(X, x_0)$) $\pi_1(Y, y_i)$ with y_i running over $f^{-1}(x_0)$. A Galois cover corresponds to a normal subgroup of $\pi_1(X, x_0)$. So, the *smallest* Galois cover mapping through (Y, f) corresponds to the *largest* normal subgroup, $H = \bigcap_{i=1}^n \pi_1(Y, y_i)$, of $\pi_1(X, x_0)$ contained in $\pi_1(Y, y_1)$. So, there is a cover with property (8.4c).

Let $\mathrm{pr}_1 : Y^n_X \to Y$ be projection onto the first factor, and let $f_{Y,i}$ be the restriction of pr_1 to $\hat{Y}_i$. Then, with $(\hat{Y}, \hat{f})$ (resp., f_Y) replaced by $(\hat{Y}_i, \hat{f}_i)$ (resp., $f_{Y,i}$) properties (8.4a) and (8.4b) hold, $i = 1, \ldots, t$. This shows the map $h : \hat{Y}_i \to Y$ has degree 1: $(\hat{Y}_i, \hat{f}_i)$ and $(\hat{Y}, \hat{f})$ are equivalent covers of X. The proof is complete. □

Fig. 8 shows four discs on a degree 4 cover of $U_{\boldsymbol{z}}$ lying over a disk U_{z_0} around the base point. Assume the cover has monodromy group S_4. (Like that from a general degree 4 polynomial $f \in \mathbb{C}[w]$.) We visibly can see the action of any element $\alpha \in S_4$ on the four points of $f^{-1}(z_0)$ extend to the four disjoint disks over U_{z_0}. Yet, there is no continuous extension of any nonidentity α to $f^{-1}(U_{z_0})$. Lem. 8.8 says such extending αs must centralize the monodromy. We stipulated, however, this is S_4, a group with trivial center.

FIGURE 8. $\alpha = (1\,2)(3\,4) \in S_4$ tries, but fails, to be an automorphism of Y: The four discs on the left constitute $f^{-1}(U_{x_0})$



8.3.3. *Galois closure orbits.* Chap. 2 [9.5] has Galois exercises based on using fields. We now explain how these have analogs where we replace field extensions of a given field by covers of a given space. One tricky point: Composite of two fields makes sense only if there is given a priori a field L containing them both. As with the comments from §4.2.3 on local holomorphic functions, the next lemma shows fiber product of covers is dual to tensor product of fields. This analogy will come through even more when we deal with the field of meromorphic functions on a cover in Chap. 4 Prop. 2.10.

LEMMA 8.10. *Let K_i, $i = 1, 2$, be two finite extensions of a field K (having 0 characteristic). The ring $K_1 \otimes_K K_2$ is the direct sum of field extension of K. These summands are, up to isomorphism of extensions of K, in one-one correspondence with all compositions of K_1 and K_2.*

PROOF. Since the characteristic is 0 (only need separable extensions), the primitive element theorem says $K_2 = K(\alpha)$ for some $\alpha \in K_2$. Up to isomorphism of extensions, K_2/K is $K[x]/(f_2(x))$ with f_2 the irreducible polynomial for α over K. Factor f_2 as $\prod_{i=1}^{u} g_i(x)$ over K_1, with the g_i s monic and distinct. (Again use characteristic 0, or just that irreducible polynomials have no repeated roots.) Now apply Lem. 4.8 to write $K_1 \otimes K_2 = K_1[x]/(f_2(x))$ as $\oplus_{i=1}^{u} K_1[x]/(g_i(x))$. Since each of the g_i s is irreducible over K_1, each of the summands is a field. So each summand is a field generated by extensions of K isomorphic to K_1 and K_2.

Conversely, suppose L is a field containing K_1 and generated by K_1 and $K' = K(\alpha')/K$ with α' the image of α in an isomorphism of K_2/K with it. Then, L is isomorphic to one of the summands of $K_1 \otimes K_2$. This concludes the proof. $\square$

Suppose L_i/K (resp. $f_i : Y_i \to X$) is a field extension (resp. connected cover) of finite degree n_i, with G_i its Galois closure group and $\hat{L}_i/K$ (resp. $\hat{f}_i : \hat{Y}_i \to X$ its Galois closure field (resp. cover), $i = 1, 2$. As in Chap. 2 [9.6a], consider the fiber product H_f of G_1 and G_2 over the Galois group of the well-defined field extension $\hat{L}_1 \cap \hat{L}_2$. Then, $G(\hat{L}_1 \cdot \hat{L}_2/K)$ is H_f. The restriction of elements of H_f to $\hat{L}_i$ produces a permutation representation T_i, $i = 1, 2$. Now consider the direct product representation T_f of H_f induced from T_1 and T_2 (§7.1.2). The next lemma, in this analogy, shows different composites of field extensions correspond to the different components of the fiber product of the covers over X. The proof shows also that inequivalent composite extensions $L_1 \cdot L_2$ correspond one-one to orbits of T_f (compare with Chap. 2 [9.6c]).

LEMMA 8.11. *Let $g : Y \to X$ be the maximal cover through which $\hat{f}_i$, $i = 1, 2$, both factor. Then, g is a Galois cover. If M is its group, this induces homomorphisms $f_{i*} : G_i \to M$. Denote the fiber product of these group homomorphisms by H_c. Then, any connected component $\hat{Y}_{1,2}$ of $\hat{Y}_1 \times_X \hat{Y}_2$ (as a cover of X) is the minimal Galois cover of X factoring through $\hat{f}_i$, $i = 1, 2$. The group of this cover is H_c, a subgroup of $S_{n_1} \times S_{n_2}$ (acting on pairs (i, j), $1 \le i \le n_1, 1 \le j \le n_2$). Orbits of T_c correspond one-one to the components of $Y_1 \times_X Y_2$.*

PROOF. Let $\mathcal{C}_{\text{Gal}}$ be the category of Galois covers of X up to isomorphism commuting with the map to X. Similarly, let $\mathcal{C}_{\text{Nor}}$ be the category of normal subgroups of $\pi_1(X, x_0)$. The first part of the lemma is an equivalencing of fiber products in each of these categories (as at the end of the proof of Thm. 7.16). The fiber product for two normal subgroups of $\pi_1(X, x_0)$ is their intersection, which identifies the quotient as H_c in this case. Since the fiber product $\hat{Y}_1 \times_X \hat{Y}_2$ may not be connected, and therefore not Galois, this cannot be the fiber product in the category of Galois covers of X. A connected component, however, of it defines an equivalence class of connected and Galois covers. It is this that is the fiber product in the category $\mathcal{C}_{\text{Gal}}$.

Now consider the statement on orbits of T_c. Since H_c factors through G_i, with its representation T_i, $i = 1, 2$, it makes sense to form the direct (tensor) product T_c of T_1 and T_2. Direct summands in the category of permutation representations correspond to components of covers in the category of covers of X. Since permutation representations correspond to equivalence classes of covers, to show the statement on orbits we have only to show that the direct product permutation representation T_c corresponds to the fiber product $Y_1 \times_X Y_2$. This is the equivalence of direct product in their respective categories. □

8.4. Imprimitive covers and wreath products. Suppose $f : Y \to X$ is a (connected) cover, and f factors through another cover $f_1 : Y_1 \to X$. That gives a series of covers $Y \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$. We say $f_1 \circ f_2$ is a decomposition of f if $\deg(f_i) > 1$, $i = 1, 2$. If there is no such decomposition of f, we say it is *indecomposable* or *primitive*. Equivalence two decompositions if the their corresponding covers $f_1 : Y_1 \to X$ are equivalent to give equivalence classes of decompositions. As $G(Y, f) \le S_n$, denote the subgroup stabilizing 1 by $G(Y, f)(1)$.

LEMMA 8.12. *The monodromy group $G(Y, f)$ is a primitive subgroup of S_n if and only if f is primitive (Def. 7.9). Equivalence classes of decompositions of f correspond one-one with subgroups properly between $G(Y, f)$ and $G(Y, f)(1)$.*

PROOF. Choose a basepoint $y_1 \in Y$ to apply Thm. 7.16. Groups between $G(Y, f)$ and $H_1 = \{g \in G(Y, f) \mid (1)g = 1\}$ correspond one-one to decompositions of f. In particular, f is primitive if and only if there no decomposition of f. □

Suppose G and H are groups, with $G_1 \le G$ and $H_1 \le H$. Let $T_{G_1} : G \to S_n$ and $T_{H_1} : H \to S_m$ be corresponding coset representations. Use T_{G_1} to have G act on H^n, the product of n copies of H:

(8.5) $g \in G$ acts by $(h_1, \dots, h_n) \mapsto (h_{(1)T_{G_1}(g)}, \dots, h_{(n)T_{G_1}(g)})$.

This gives a natural permutation representation $T_{H\wr G}: H\wr G \overset{\text{def}}{=} H^n \times^s G \to S_{nm}$ acting on a set $L = \{1_1, \dots, 1_m, 2_1, \dots, 2_m, \dots, n_1, \dots, n_m\}$ by this formula:

$$(i_j)T_{H\wr G}(h_1, \dots, h_n, g) = (i)T_G(g)_{(j)T_H(h_i)}.$$

Call $T_{H\wr G}$ the *wreath product* representation of T_G and T_H. Then, $H \wr G$ is the wreath product of G and H, though this assumes we know the corresponding permutations representations. Now consider how the wreath occurs in covering theory.

DEFINITION 8.13. Suppose $\psi: \hat{G} \to G$ is a cover of groups. Let T_{G_1} (resp. $T_{\hat{G}_1}$) be a faithful permutation representation of G (resp. $\hat{G}$). Call $T_{\hat{G}_1}$ an *extension* of T_{G_1} if ψ maps some conjugate of $\hat{G}_1$ maps surjectively to G_1: $T_{\hat{G}_1}$ extends T_{G_1}.

LEMMA 8.14. *Suppose $f: Y \to X$ is a (connected) cover, and f factors as a series of covers $Y \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$. Let G_{f_i} be the group of the Galois closure of f_i, with T_{f_i} the corresponding permutation representations, $i = 1, 2$. Use similar notation for f. Then, T_f extends T_{f_1}, G_f is a transitive subgroup of $G_{f_2} \wr G_{f_1}$ and $G_{f_1}(1)$ maps surjectively to the group G_{f_2}. Further, $G_f = G_{f_2} \wr G_{f_1}$ if and only if the kernel of $G_f \to G_{f_1}$ is isomorphic to $G_{f_2}^{deg(f_1)}$.*

PROOF. Choose a base point in $y_0 \in Y$ and therefore image base points in Y_1 and X. Apply Thm. 7.16 to identify G_f (resp. G_{f_2}, G_{f_1}) with permutation representations of $\pi_1(X, f(y_0))$ (resp. $\pi_1(Y_1, f_2(y_0)), \pi_1(X, f(y_0))$) given by the cosets of $\pi_1(Y, y_0)$ (resp. $\pi_1(Y, y_0), \pi_1(Y, y_0)$). So, the permutation representation of G_f (resp. G_{f_1}) comes from the image $G_f(1)$ (resp. $G_{f_1}(1)$ of $\pi_1(Y, y_0)$ in G_f (resp. G_{f_1}). As $G_f(1)$ and $G_{f_1}(1)$ are images of the same group, this shows T_f extends T_{f_1}. All coset permutation representations are transitive. That shows G_f is transitive.

With $x_0 = f(y_0)$, let $W = y_1, \dots, y_{\deg(f_1)}$ be the points of Y_1 lying over x_0. Similarly, let $W_i = \{y_{i,j_i}\}_{j_i=1,\dots,\deg(f_2)}$ be the points of Y lying over y_i. Intersecting the conjugates of $G_f(1)$ gives the kernel of $G_f \to G_{f_1}$. So, K acts as permutations on each W_i, $i = 1, \dots, \deg(f_1)$. Restricting the action of $G_f(1)$ to W_1 gives the group G_{f_2} in the representation T_{f_2}. Similarly, using the natural identification of all the sets W_i, the kernel of $G_f \to G_{f_1}$ is isomorphic to a subgroup K of $G_{f_2}^{\deg(f_1)}$. This identifies G_f with a subgroup of the wreath product. Since the order of G_f is $|G_{f_1}||K|$, the index of G_f in $G_{f_2} \wr G_{f_1}$ equals $(G_{f_2}^{\deg(f_1)} : K)$. This gives the last statement line of the lemma. □

8.5. Representations and groupoids. Rather than define *groupoid* generally, we present a classical case for later use. The idea is that of Deligne and Grothendieck. Deligne has a notion of (fundamental group) *realizations*. We think of these as ways a space declares its presence through types of analytic continuation. This helps us to explain the *profinite* fundamental group of a complex manifold (Chap. 4 §7.2). Mastering the *Hurwitz monodromy group* in Chap. 5 simplifies if we understand how a fundamental group depends on a base point. That leads to generalizing what will serve as a base point. Tangential base points (Chap. 2 §8.4) are an example. We get much mileage from a particularly significant parameter space, the classical j-line (Chap. 4 §6.8). This follows [**De89**, §10] which used the related λ-line.

8.5.1. *A law of composition.* Suppose $\mathcal{C}_X$ is the category of unramified covers of an complex manifold X. For $\varphi: Y \to X$ an unramified cover and $\psi: X' \to X$

any map of complex manifolds, there is a natural contravariant map $\psi^* : \mathcal{C}_X \to \mathcal{C}_{X'}$ through fiber products: $\psi^*(\varphi) =: X' \times_X Y \to X'$.

LEMMA 8.15. *The map ψ^* preserves fiber products. For $\varphi_1, \varphi_2 \in \mathcal{E}_X$:*

$$\psi^*(\varphi_1 \times_X \varphi_2) = \psi^*(\varphi_1) \times_{X'} \psi^*(\varphi_2).$$

PROOF. Seeing this set theoretically makes it clear the cover structures are compatible. First: Identify $(Y_1 \times_X Y_2) \times_X X'$ with $(Y_1 \times_X X') \times_{X'} (Y_2 \times_X X')$ by mapping (y_1, y_2, x') all lying over a given $x \in X$ to $((y_1, x'), (y_2, x'))$. Then, both maps send this element to x'. □

Let $\hat{\varphi} : \hat{Y} \to X$ be the Galois closure of this cover. Suppose this has group G. Then G acts faithfully and transitively on the fibers of $\hat{\varphi}$. On $\hat{Y} \times \hat{Y} \to X \times X$ let G act diagonally: $(\hat{y}_1, \hat{y}_2)g \stackrel{\text{def}}{=} ((\hat{y}_1)g, (\hat{y}_2)g)$.

Denote $\hat{Y} \times \hat{Y}/G$, the orbits of the action of G, by $\mathcal{G}$. Let $\mathcal{G}_{i,j}$ be the pullback of $\mathcal{G}$ to $X \times X \times X$ induced from the projection of $X \times X \times X$ on its (i, j) factors. For example, $\mathcal{G}_{1,2}$ consists of triples $(\hat{y}_1, \hat{y}_2, x_3)$ with $\hat{y}_i \in \hat{Y}$, $i = 1, 2$, and $x_3 \in X$.

This gives a *composition law* $\mathcal{G}_{1,2} \times \mathcal{G}_{2,3} \to \mathcal{G}_{1,3}$ respecting fibers over $X \times X \times X$. Here is what that means. For $(x_1, x_2, x_3) \in X \times X \times X$, let $(\hat{y}_1, \hat{y}_2, \hat{x}_3)$ (resp. $(x_1, \hat{y}_2', \hat{y}_3')$) represent a point of $\mathcal{G}_{x_1,x_2}$ the fiber of $\mathcal{G}_{1,2}$ (resp. $\mathcal{G}_{2,3}$) over (x_1, x_2) (resp. (x_2, x_3)). The composition law $\mathcal{G}_{x_1,x_2} \times \mathcal{G}_{x_2,x_3} \to \mathcal{G}_{x_1,x_3}$ uses the following formula. There is a unique $g \in G$ taking $\hat{y}_2'$ to $\hat{y}_2$. Define the product of $(\hat{y}_1, \hat{y}_2, x_3)$ and $(x_1, \hat{y}_2', \hat{y}_3')$ to be $(\hat{y}_1, x_2, (\hat{y}_3')g)$.

We say $\mathcal{G} = \hat{Y} \times \hat{Y}/G \to X \times X$ is a *groupoid*. Most significant is that it induces a groupoid in $\mathcal{F}_{X'}$ by pullback, for each $\psi : X' \to X$.

8.5.2. *Fundamental groupoid.* There is a *fundamental groupoid* that dominates all (classical) groupoids over X. We define this directly, as it will appear in Chap. 5.

Consider this data: $x_1, x_2 \in X$, and D_i a simply connected (path-connected) neighborhood of x_i on X, $i = 1, 2$. Suppose $x_i' \in D_i$, $i = 1, 2$. To read the next lemma correctly, emphasize the word *canonical*.

LEMMA 8.16. *There is a canonical isomorphism (dependent on (D_1, D_2)):*

$$\psi_{D_1,D_2} : \pi_1(X, x_1, x_2) \to \pi_1(X, x_1', x_2').$$

PROOF. For γ_i any path from x_i to x_i' in D_i, $i = 1, 2$, map $\gamma \in \pi_1(X, x_1, x_2)$ to $[\gamma_1^{-1} \cdot \gamma \cdot \gamma_2] = [\gamma_1^{-1}][\gamma][\gamma_2] \in \pi_1(X, x_1', x_2')$. Under the hypotheses, $[\gamma_i]$ depends only on x_i, x_i', D_i and not the particular choice of path. That shows the lemma. We will, however, confront repeatedly the dependence of ψ_{D_1,D_2} on (D_1, D_2). □

DEFINITION 8.17. The fundamental groupoid $\mathcal{P}_X$ of X consists of the disjoint union $\dot{\cup}_{x_1,x_2 \in X} \pi_1(X, x_1, x_2)$. The composition law for $\pi_1(X, x_1, x_2) \times \pi_1(X, x_2, x_3)$ is the usual path multipication: $[\gamma_{1,2}] \in \pi_1(X, x_1, x_2)$ times $[\gamma_{2,3}] \in \pi_1(X, x_2, x_3)$ is $[\gamma_{1,2}][\gamma_{2,3}] \in \pi_1(X, x_1, x_3)$.

Restriction of $\mathcal{P}_X$ to the diagonal of $X \times X$ is the *local system* of fundamental groups $\dot{\cup}_{x_1 \in X} \pi_1(X, x_1)$. For $x \in X$, restrict $\mathcal{P}_X$ to $X \times \{x\} \subset X \times X$ to get the universal cover of (X, x). Now we trace through an action of a groupoid on various locally constant sets.

8.5.3. *Action of a groupoid.* We recognized already that the category $\mathcal{C}_X$ consists of *locally constant finite sets* on X. That means, given $f : Y \to X$ an unramified cover, the topology on Y comes from an open cover $\mathcal{U}$ of X so that $f_U : Y_U \to U$ makes of Y_U a finite collection of disjoint copies of U. Generalizing the notion of covers allows defining related locally constant structures. We concentrate here on $\mathbb{V}_X$, the category of *locally constant —or flat —vector bundles* on X. Suppose V is a vector space over $\mathbb{C}$ (say, $\mathbb{C}^n$). Then, there is a natural fiber preserving addition and scalar multiplication with the expected properties on $V \times U$. An object $\mathcal{V} \in \mathbb{V}_X$ consists of an analytic map $L : \mathcal{V} \to X$ of manifolds with an open cover $\mathcal{U}$ having the following properties.

(8.6a) For $U \in \mathcal{U}$, there is an analytic isomorphism $\psi_U : \mathcal{V}_{U_i} \to V \times U_{\gamma(t_i)}$ so that $L_U : \mathcal{V}_U \to U$ and $\mathrm{pr}_U \circ \psi_U : \mathcal{V}_U \to U$ are the same.

(8.6b) Local constancy: For $U, U' \in \mathcal{U}$, with $U \cap U'$, an element of $\mathrm{GL}_n(\mathbb{C})$ gives $\psi_U^{-1} \circ \psi_{U'}$ restricted to $V \times (U \cap U')$ along each fiber.

(8.6c) A fiber preserving complex analytic addition and multiplication by $\mathbb{C}$ on $\mathcal{V}$ restricts over each $U \in \mathcal{U}$ to that structure on $V \times U$.

Note the right action in (8.6b). We say $\mathcal{V}$ is a *rank* n (locally constant, or *flat*) bundle. Two flat bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ are *bundle isomorphic* if there is a compatible open cover $\mathcal{U}$ for both and a fiber preserving analytic isomorphism $\psi : \mathcal{V}_1 \to \mathcal{V}_2$. Suppose ψ intertwines (8.6) for $\mathcal{V}_2$ relative to $\mathcal{U}$ to that for $\mathcal{V}_1$ so that for each $U \in \mathcal{U}$, an element $g_U \in \mathrm{GL}_n(\mathbb{C})$ gives $\psi_{1,U}^{-1} \circ \psi \circ \psi_{2,U}$. Then, ψ is a *flat isomorphism*. Warning: Some bundle isomorphisms have no corresponding flat isomorphism.

Example 8.18 (Flat bundle from a cover). Let $f : Y \to X$ be a degree n cover (element of $\mathcal{C}_X$). For each $x \in X$, denote the space spanned over $\mathbb{C}$ by the points of $f^{-1}(x)$ by V_x. We explain why $\mathcal{V}_f \stackrel{\text{def}}{=} \dot\cup_{x \in X} V_x$ is a locally constant vector bundle on X by taking L_f to be the natural projection. Suppose $U \le X$ is open, $x' \in U$ and f_U identifies Y_U with $\dot\cup_{y' \in f^{-1}(x')} U_{y'}$ where $U_y \le Y$ maps one-one onto U. This means we have n sections to the map f_U. We also call these $y'_1, \dots, y'_n$. So, for each $x \in U$, $\{y'_i(x)\}_{i=1}^n$ is a basis for V_x. Then, we have a natural analytic manifold topology on $\mathcal{V}_f$ by identifying $\mathcal{V}_{f,U}$ with $\mathbb{C}^n \times U$ by mapping the standard basis of $\mathbb{C}^n$ to $y'_1(x), \dots, y'_n(x)$ running over $x \in U$.

Suppose $\mathcal{P}$ is a groupoid on X and $\mathcal{V} \in \mathbb{V}$. Regard $\mathcal{P}$ as a locally constant bundle of sets over $X \times X$. Consider the fiber products $\mathrm{pr}_i^*(\mathcal{V}) \stackrel{\text{def}}{=} \mathcal{V} \times_X (X \times X)$, using $\mathrm{pr}_i : X \times X \to X$, projection on the ith factor, $i = 1, 2$. We say $\mathcal{P}$ acts on $\mathcal{V}$ if there is a fiber preserving analytic map

$$A_X : \mathrm{pr}_1^*(\mathcal{V}) \times_{X \times X} \mathcal{P} \to \mathrm{pr}_2^*(\mathcal{V}). \tag{8.7}$$

Regard each term $\mathcal{P}$, $\mathrm{pr}_1^*(\mathcal{V})$ and $\mathrm{pr}_2^*(\mathcal{V})$ as a locally constant bundle over $X \times X$.

Denote the vector space $\mathbb{C}^n$ (with its canonical basis understood) by V, so that there is an action of $\mathrm{GL}_n(\mathbb{C})$ on the right of V. (To adjust to a left action on $\mathrm{GL}_n(\mathbb{C})$, see Ex. [9.16f].) For $x_0 \in X$, and n a positive integer, consider pairs $(\mathcal{V}, m_{x_0})$ with $\mathcal{V}$ a flat bundle of rank n, and m_{x_0} a fixed vector space isomorphism of $\mathcal{V}_{x_0}$ with V, by $\mathbb{V}_{x_0,n}$. Compose m_{x_0} with any element of $\mathrm{GL}_n(\mathbb{C})$ gives a natural action of $\mathrm{GL}_n(\mathbb{C})$ on the pairs $(\mathcal{V}, m_{x_0})$.

Proposition 8.19. *The fundamental groupoid $\mathcal{P}_X$ acts on every $\mathcal{V} \in \mathbb{V}_X$. Each $(\mathcal{V}, m_{x_0}) \in \mathbb{V}_{x_0,n}$ produces $\alpha_{\mathcal{V},m_{x_0}} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_n(\mathbb{C}))$ and the map*

$(\mathcal{V}, m_{x_0}) \mapsto \alpha_{\mathcal{V},m_{x_0}}$ is one-one and onto. Flat rank n bundles up to flat isomorphism correspond to elements of $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_n(\mathbb{C}))/G$.

PROOF. Consider $(x, x') \in X \times X$ and $[\gamma] \in \pi_1(X, x, x')$. We give an action: $A_X((v, x, x'), [\gamma]) = (v', x, x')$ with $v \in \mathcal{V}_x$ and $v' \in \mathcal{V}_{x'}$. The construction, for a given path γ, is exactly as in the proof of Lem. 7.13. We set appropriate notation.

If $\gamma : [a, b] \to X$, then there is a partition $t_0 = a < t_1 < \cdots < t_n = b$ and contractible open subsets $U_{\gamma(t_i)}$, $i = 0, \dots, n$, with $U_{\gamma(t_i)} \cap U_{\gamma(t_{i+1})}$ contractible, $i = 0, \dots, n-1$, so the following holds.

(8.8a) $\psi_{U_i} : \mathcal{V}_{U_i} \to V \times U_{\gamma(t_i)}$ is one of the maps given by (8.6).

(8.8b) $\gamma_{[t_{i-1},t_{i+1}]} \le U_{\gamma(t_i)}$, $i = 0, \dots, n$, with the provisos $t_{-1} = a$ and $t_{n+1} = b$.

Since the path γ has the information about the endpoints in it, we may simplify notation by rewriting our expression for A_X as $A_X(v, [\gamma]) = v'$ with v (resp.v') in the beginning (resp. end) point of γ. Inductively define $A_X(v, \gamma_{[t_0,t_{k+1}]}) = v_{k+1}$:

$$A_X(A_X(v, [\gamma_{[t_0,t_k]}]), [\gamma_{[t_k,t_{k+1}]}]) = A_X(v_k, [\gamma_{[t_k,t_{k+1}]}]) = (v_k)(\psi_{U_k})^{-1} \circ \psi_{U_{k+1}}.$$

That defines the action for a particular path. We need to know the result doesn't depend on the partition, nor on the homotopy class of γ. Starting from the definition of the action on γ with a partition, apply the General Monodromy Theorem 6.11 proof. (Our contractibility assumptions on the U_i s allow us to use this proof.) Line-for-line this shows A_X depends only on the homotopy class $[\gamma]$ and not on γ.

Define $\alpha_{\mathcal{V}}$ as $\prod_{k=0}^{n-1} (\psi_{U_k})^{-1} \circ \psi_{U_{k+1}}$. We use that the constituent elements are in $\mathrm{GL}_n(\mathbb{C})$ (locally constant as a function of $x \in X$), and that the result is independent of the homotopy class of the path to see it is a homomorphism. Now consider when two flat bundles are flat isomorphic.

Notice that the collection of isomorphisms $\psi_U : \mathcal{V}_U \to V \times U$ gives a cocycle condition: For U, U', U'' intersecting nontrivially,

$$(\psi_U^{-1} \circ \psi_{U'}) \circ (\psi_{U'}^{-1} \circ \psi_{U''}) = \psi_U^{-1} \circ \psi_{U''}.$$

Apply Lem. 2.2 to see that $\mathcal{V}$ identifies with the disjoint union of $\cup_{U \in \mathcal{U}} V \times U$ modulo the equivalence of points on $V \times U$ with $V \times U'$ on the overlap of $U \cap U'$ by $\psi_U^{-1} \circ \psi_{U'}$. Using this, a flat isomorphism between $\mathcal{V}_1$ and $\mathcal{V}_2$ interprets as the existence of $g_U \in \mathrm{GL}_n(\mathbb{C})$ for which

$$g_U^{-1} \circ \psi_{1,U}^{-1} \circ \psi_{1,U'} \circ g_{U'} = \psi_{2,U}^{-1} \circ \psi_{2,U'}.$$

In running around any path given by a sequence of U_i s, the conclusion is that $\alpha_{\mathcal{V}_1}$ differs from $\alpha_{\mathcal{V}_2}$ on this path by conjugation by g_{U_0}. That effect is determined by its effect on m_{x_0}. This concludes the proof of the theorem. □

8.6. Complete reducibility and covers with equivalent flat bundles. Flat bundles appear in a few well-known papers long ago. [**Gun67**, p. 97], from which the author first heard of these subjects many years ago, cites [**We38**] and [**At57**]. Riemann knew of the distinction between holomorphic vector bundles and flat bundles through his investigation general ordinary differential equations versus differential equations with ordinary singular points. This topic appears in Chap. 4. An advanced reader will note we have yet to define general holomorphic bundles.

8.6.1. *Decomposing the representations of a cover.* A cover $f : Y \to X$ has a flat bundle on X associated with it (Ex. 8.18). Let $\rho_X \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_n(\mathbb{C}))$ be the associated homomorphism. We explore the natural map $\mathcal{E}_X \to \mathcal{V}_X$, especially noting it is not injective. [**Sch70**] and [**Fri73**] are sources for practical problems in which this becomes significant. In particular, Chap. 4 [10.11] uses Riemann's Existence Theorem on the groups of [9.20] to produce primitive, inequivalent covers whose fibers products are reducible. This is a chance to introduce the significant topic of *complete reducibility* for fundamental groups representations.

DEFINITION 8.20. Let G be a group and F a field. Suppose $\rho : G \to GL_n(F)$ is a representation of G. Then, ρ has an *invariant subspace* $V \le F^n$ if $\rho(g)$ maps V into V for each $g \in G$. A representation is *irreducible* if it has no invariant subspace. Two invariant subspaces V and W (for φ) are *complements* if V and W span F^n, and $V \cap W = \{0\}$. Call ρ completely reducible if every ρ invariant subspace V has a complement.

Recall: With R a ring, $r \in R$ is an *idempotent* if $r^2 = r$. Idempotents in $\mathbb{M}_n(F)$ are the matrices of projection onto subspaces of F^n.

LEMMA 8.21. *Suppose V is a ρ invariant subspace. If F has characteristic 0, then V has a complement.*

PROOF. Let $P : F^n \to V$ be any projection onto V: Choose a basis $v_1, \dots, v_k$ of V, extend to a basis $v_1, \dots, v_n$ of V, and define P by $\sum_{i=1}^n a_i v_i \mapsto \sum_{i=1}^k a_i v_i$. Then, $P^2 = P$ and P is an *idempotent*. So, too is $I_n - P$=P', and it defines a complementary space by projection. If P commutes with the action of G, then $I_n - P$ would also be a G invariant subspace. To get this, *average over G*: Replace P with $P_G = \frac{1}{|G|}\sum_{g\in G} \rho(g)^{-1} P \rho(g)$. Since each term $\rho(g)^{-1} P \rho(g)$ acts like the identity on V, for $v \in V$, $(v)P_G = \frac{1}{|G|}\sum_{g\in G} (v)\rho(g)^{-1} P \rho(g) = v$. □

[9.19] applies the complete reducibility of finite group representations when F has zero characteristic. Complete reducibility does in general if either G is infinite or F has positive characteristic [9.17]. If a representation ρ is completely reducible, then we may write F^n as $\oplus_{i=1}^k V_i$, a direct sum of invariant and irreducible subspaces for the action of G. Another notation for this is $\rho = (\rho_1, \dots, \rho_k)$ with ρ_i restriction of ρ to the space V_i: ρ is the direct sum of the actions of the ρ_i, $i = 1, \dots, k$.

The notation $\mathbf{1}_G$ is for the one-dimensional representation of G where the action of G leaves each vector fixed. Given any representation ρ there is natural *conjugate representation* $\bar\rho$: $g \mapsto \bar\rho(g)$ by applying $\bar{\ }$ to each entry of $\rho(g)$.

8.6.2. *Components of fiber products.* Suppose $f_i : Y_i \to X$ is a connected cover of degree n_i, with $\rho_{f_i} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_{n_i}(\mathbb{C}))$ the corresponding element from Prop. 8.19, $i = 1, 2$. Then, ρ_{f_1} and ρ_{f_2} induce the tensor product representation $\rho_{f_1} \otimes \rho_{f_2} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_{n_1 n_2}(\mathbb{C}))$. Let G_i be the group of a Galois closure $\hat{Y}_i \to X$ of f_i, $i = 1, 2$. Lem. 8.11 shows each of these representations factors through a faithful representation of $G = G_1 \times_H G_2$ for some group H that is a quotient of both G_1 and G_2. Here G is the group of the minimal Galois cover of X factoring through f_1 and f_2. Use the notation $\rho_{f_1} \otimes \rho_{f_2}$ for this representation, too. Since G is a finite group, each representation is completely reducible. Any representation of G_i induces a representation of G through the canonical projection of G onto G_i. Write $\rho_{f_i} = \oplus_{j=1}^{k_i} V_{i,j}$, $i = 1, 2$, indicating the irreducible representations of G coming from those of G_i, $i = 1, 2$.

PROPOSITION 8.22. *The number of connected components of the fiber product $Y_1 \times_X Y_2$ is the same as the number of times the identity appears in $\rho_{f_1} \otimes \rho_{f_2}$. In turn, this is the same as the number of distinct pairs (j, j') where $V_{1,j}$ is equivalent to the conjugate of $V_{2,j'}$.*

If $G = G_1 = G_2$, and $\rho_1 = \rho_2$, $Y_1 \times_X Y_2$ has at least two connected components. In this case it has precisely two if and only if the permutation representation associated with f_1 (or with f_2) is doubly transitive.

PROOF. Apply Lem. 8.11 to conclude there are as many connected components in $Y_1 \times_X Y_2$ as the number of orbits in the direct product applied to G of the permutation representations attached to f_1 and f_2. This counts the appearances of the identity in the corresponding representation which in turn counts the number of appearances of the identity in $\rho_{f_1} \otimes \rho_{f_2}$. Use the representation theory reminders in [9.19b] to see this also counts the number of pairs (j, j') listed in the statement of the proposition. This completes the first part of the proof.

Suppose $\rho_T = \oplus_{j=1}^k V_{T,j}$ is the decomposition of $\rho_1 \otimes \rho_2$ given in the statement into irreducible representations (over $\mathbb{C}$). A permutation representation is the same as its conjugate. So, for each $V_{T,j}$, its conjugate also appears in the summands of ρ_T. If $\rho_1 = \rho_2$ and $G = G_1 = G_2$, besides the identity in both ρ_1 and ρ_2, there must exist at least one other pair indexed by (j, j') of conjugate representations. From [9.19d], $k = 2$ if and only if the permutation representation is doubly transitive. If, however, $k \geq 3$, there will be at least three pairs (j, j') indicating corresponding pairs of conjugate represenations. This concludes the proof. □

9. Exercises

We apply group theory exercises here to geometric applications in Chap. 4. [**FH91**] contains a hurried encyclopedic account of classical representations. Yet, it doesn't cover our later needs. [**Ben91**] (very concise) and older relaxed texts like [**Ha63**] work for Riemann surface applications requiring deeper group theory. We have exercises that prepare some characteristic p representations. These appear in *Modular Towers* (Chap. 5). Representation theory changes as much as Riemann surface theory. As [**Lam98**, p. 369] notes, it is about 100 years old. Even such topics as *higher characters* from its beginnings — unlike linear characters these do determine the group — have still an uncertain place in the theory.

9.1. Constructing manifolds. Call a topological space a *pre-manifold* if it has coordinate charts, but is not necessarily Hausdorff. We characterize Hausdorff.

(9.1a) Show the space of Ex. 2.4 is not Hausdorff.

(9.1b) Prove Lemma 2.5 using the argument before it.

(9.1c) Let $\{(X_{\alpha_i}, \varphi_{\alpha_i})\}_{\alpha_i \in I_i}$ (resp., $\{(Z_\alpha, \varphi_\alpha)\}_{\alpha \in I}$) be topological data for X_i (resp., Z), $i = 1, 2$. Let $f_i : X_i \to Z$, $i = 1, 2$ be continuous. Show

$$\{(X_{\alpha_i} \times X_{\alpha_j}) \cap (X_1 \times_Z X_2), (\varphi_{\alpha_i}, \varphi_{\alpha_j})\}_{(\alpha_i, \alpha_j) \in I_1 \times I_2}$$

gives topologizing data on $X_1 \times_Z X_2$ with continuous projections $\mathrm{pr}_i : W \stackrel{\mathrm{def}}{=} X_1 \times_Z X_2 \to X_i$, $i = 1, 2$. Further, W is Hausdorff if X_1, X_2 and Z are. Use this to prove Lemma 4.3.

(9.1d) Let $f : X \to Y$ be continuous, with X and Y pre-manifolds. Let $\gamma : [0, 1] \to Y$ be a path. If a continuous $\gamma_1 : [0, 1) \to X$ lies over $\gamma_{[0,1)}$ ($f \circ \gamma_1(t) = \gamma(t)$ for $t \in [0, 1)$). Show: For all pairs (γ, γ_1), there is at most one extension of γ_1 to a path $\gamma_1^* : [0,1] \to Y$ if and only if the diagonal in $X \times_Y X$ is closed. Call an f satisfying this *separated*.

(9.1e) With f in d) separated, consider extending γ_1 to $\gamma_1^* : [0,1] \to Y$. Show: Such γ_1^* exists (for each γ_1) if and only if f is a proper map (§2.2).

Consider some manifolds (differentiable) from vector calculus.

(9.2a) If X_i is n_i-dimensional, $i = 1, 2$, show $X_1 \times X_2$ is $n_1{+}n_2$-dimensional.

(9.2b) The n-sphere is $S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$. Here is some data for defining a manifold structure on S^n:

$$U^+ = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$$

and $R_{\boldsymbol{x}}$ is any rotation of the sphere that that takes $\boldsymbol{x}$ to $(0, \dots, 0, 1)$. Let $U_{\boldsymbol{x}}$ be the image of U^+ under $R_{\boldsymbol{x}}^{-1}$, and define $\varphi_{\boldsymbol{x}}$ to be $\text{pr} \circ R_{\boldsymbol{x}}$ where $\text{pr}(\boldsymbol{x}) = (x_1, \dots, x_n)$. Show the $(U_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}})$'s are a differentiable atlas on S^n.

(9.2c) Consider $f \in \mathbb{R}[x_1, \dots, x_n]$ and the set $X_f = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\}$. Let $X_f^0 = \{\boldsymbol{x} \in X_f \mid \nabla(f)(\boldsymbol{x}) \neq 0\}$ (Lemma 3.2). State a differentiable version of the *implicit function theorem* [**Rud76**, p. 224] from Chap. 2 §6.2.

(9.2d) Assume $n = 3$ in c) and two open sets U_1 and U_2 with these properties: $\frac{\partial f}{\partial x_1}$ is nonzero in U_1 and $\frac{\partial f}{\partial x_3}$ is nonzero in U_2. Apply c) to conclude there is a differentiable transition function $\varphi_2 \circ \varphi_1^{-1}$ for the pair (U_1, U_2).

(9.2e) If X_f^0 is nonempty, show it is a differentiable $n-1$ dimensional manifold.

(9.2f) State a complex analog of c) for $f \in \mathbb{C}[z_1, \dots, z_n]$ using complex partials. How does this show the complex version of X_f^0 is an $n-1$ dimensianal analytic manifold?

(9.2g) Apply the fundamental theorem of algebra [**Ahl79**, p. 122] to show the manifold in f) cannot be compact.

Fiber products and pushouts are categorical constructions. Chap. 4 [10.9] continues this exploration.

(9.3a) The fiber product of two maps $f_i : Y_i \to X$, $i = 1, 2$, satisfies the following universal property: If $f : Y \to X$ factors through each of the f_i s, then f factors through (f_1, f_2). Further, (f_1, f_2) is universal for this property.

(9.3b) The pushout for $f_i : Y_i \to X$, $i = 1, 2$, satisfies a reverse diagram to the fiber product. It is the maximal object through which both f_i, $i = 1, 2$, factor. For subsets of a set, the pushout would be the union. Show the pushout of pointed covers is exactly as given in Thm. 7.16.

(9.3c) For subgroups of a group, the union is not a group. Show the subgroup generated by the two groups is the pushout.

9.2. Complex structure and torii. Going from $\mathbb{R}$ to $\mathbb{C}$ is partly a linear algebra constraint. Use the identifications $\{L_n\}_{n=1}^{\infty}$ of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ in §3.1.2. Consider replacing $\{L_n\}_{n=1}^{\infty}$ by the sequence $\{L'_n\}_{n=1}^{\infty}$ of linear (invertible) maps (from $\mathbb{R}^{2n} \to \mathbb{C}^n$). Denote $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$ by J_n.

(9.4a) Show with L'_n in place of L_n, though the functions labeled analytic in any neighborhood of an analytic manifold X will change, the set of n-dimensional analytic manifolds remains the same.

(9.4b) Show, for analytic manifolds X and Y (possibly of different dimensions), the set of analytic maps X to Y using $\{L_n\}_{n=1}^{\infty}$ map naturally to the corresponding set using $\{L'_n\}_{n=1}^{\infty}$.

(9.4c) Show $\{L'_n\}_{n=1}^{\infty}$ gives the same analytic functions on each analytic manifold as $\{L_n\}_{n=1}^{\infty}$ if and only if $L'_n = B_n \circ L_n$ with $B_n \in \mathrm{GL}_n(\mathbb{C})$ for all n. Further, this is equivalent to $L'_n \circ J_n = i \cdot L'_n$ for all n. Hint: Check on $\mathbb{C}$ linear combinations of $z_1, \dots, z_n$ in $\mathbb{C}^n$ using L'_n. Also: Invertible $\mathbb{R}$ linear maps $\mathbb{C}^n \to \mathbb{C}^n$ are in $\mathrm{GL}_n(\mathbb{C})$ if and only if they commute with i.

(9.4d) Consider the case $L = L_n : \mathbb{R}^{2n} \to \mathbb{C}^n$ by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 - iy_1, \dots, x_n - iy_n) = (\bar{z}_1, \dots, \bar{z}_n)$$

for examples where using $\{L'_n\}_{n=1}^{\infty}$ changes a given analytic structure. Hint: See Chap. 4 §7.7.1.

Consider the topology of the torus of Fig. 3.

(9.5a) Show the complex torus $\mathbb{C}/L(\omega_1, \omega_2)$ of § 3.2.2 is compact.

(9.5b) Suppose $R > 3r$ with $r, R \in \mathbb{R}$. The *torus*, $T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}} =$, with radii (r, R) centered at $\boldsymbol{x}_0 = (0,0,0) \in \mathbb{R}^3$ and perpendicular to $\boldsymbol{v} = (0,0,1)$ has this underlying set of points:

$$\{\boldsymbol{x}_0 + R(\cos(\theta), \sin(\theta), 0) + r(\cos(\theta)\cos(\beta), \sin(\theta)\cos(\beta), \sin(\beta))\}_{\theta,\beta \in [0,2\pi]}.$$

Show $T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}}$ is differentiably isomorphic to $\mathbb{C}/L(\omega_1, \omega_2)$.

(9.5c) Consider the two torii in Fig. 2: Assume one is $T = T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}}$, the other $T' = T_{r,R;\boldsymbol{x}'_0,\boldsymbol{v}'}$ for vectors $\boldsymbol{x}'_0, \boldsymbol{v}' \in \mathbb{R}^3$ and $T \cap T' = \emptyset$. Call T and T' *unknotted* if for any $C > 0$ there is a continuous function

$$F : [0,1] \times \mathbb{R}^3 \setminus T \to \mathbb{R}^3 \setminus T$$

with $F(0, y) = y$ for $y \in \mathbb{R}^3 \setminus T$ and $|F(1, y)| > C$ for $y \in T'$. Otherwise they are knotted. Show there are two knotted torii in $\mathbb{R}^3$.

(9.5d) Regard $\mathbb{R}^3$ as in in $\mathbb{R}^4$: It is the set of $\boldsymbol{x} \in \mathbb{R}^4$ with $x_4 = 0$. Extend the definitions above to show any pair of torii in $\mathbb{R}^3$ is unknotted in $\mathbb{R}^4$.

We start discussing the nature of the lattice attached to a complex torus.

(9.6a) Let $\mathbb{C}/L(\omega_1, \omega_2) = X$ be a complex torus with lattice $L(\omega_1, \omega_2) = L$ as in Ex. 6.18. For $z_1, z_2 \in \mathbb{C}$ define $m(z_1 \bmod L, z_2 \bmod L)$ to be $z_1 + z_2 \bmod L$. Define the inverse of $z \bmod L$ to be $-z \bmod L$. Show X is a differentiable group with multiplication m.

(9.6b) For $t \in \mathbb{R}$, let $z(t) = \cos(2\pi t) + \sqrt{-1}\sin(2\pi t)$. Use $f : X \to S^1 \times S^1$ by $t_1\omega_1 + t_2\omega_2 \mapsto (z_1(t), z_2(t))$ to conclude that $\pi_1(X, 0 \bmod L)$ identifies with L as a group isomorphic to $\mathbb{Z}^2$, pairs of integers.

(9.6c) Suppose $x_1, x_2 \in S^1$ generate an infinite group $\langle x_1, x_2 \rangle$. Consider the collection $T_N = \{x_1^j x_2^{j'}\}_{-N \le j,j' \le N}$ for large N to conclude 1 is a limit point for $\langle x_1, x_2 \rangle$. Conclude: $w_1, w_2 \in \mathbb{C}$, $\mathbb{C}/L(w_1, w_2)$ satisfies the conditions of Lem. 2.3 only if w_1, w_2 lie on different lines through the origin.

Consider comparing two lattices of complex torii. With $L_i = L(\omega_{1,i}, \omega_{2,i})$, $i = 1, 2$, continue Ex. 6.18. Assume $\lambda_i = \frac{\omega_{1,i}}{\omega_{2,i}} \in \mathbb{C} \setminus \mathbb{R}$, $i = 1, 2$.

(9.7a) Assume $\lambda_2 = \frac{a\lambda_1 + b}{c\lambda_1 + d}$ for some $a, b, c, d \in \mathbb{Z}$ for $ad - bc = 1$. Show $\mathbb{C}/L_1 = X_1$ and $\mathbb{C}/L_2 = X_2$ are analytically isomorphic. Hint: Map $t_1\omega_{1,1} + t_2\omega_{2,1}$ to $t_1(a\omega_{1,1} + b\omega_{2,1})\alpha + t_2(c\omega_{1,1} + d\omega_{2,1})\alpha$ with $\alpha \in \mathbb{C}$ satisfying

$$(a\omega_{1,1} + b\omega_{2,1})\alpha = \omega_{1,2} \text{ and } (c\omega_{1,1} + d\omega_{2,1})\alpha = \omega_{2,2}.$$

(9.7b) Why assume $ad - bc = 1$ in a)? Why must we have a, b, c, d in $\mathbb{Z}$, rather than just $a, b, c, d \in \mathbb{R}$?

(9.7c) Suppose $L_1 \subset L_2$. Consider $f : X_1 \to X_2$ given in Ex. 6.18. Show there exist $\omega_1, \omega_2 \in L_2$ and $n_1, n_2 \in \mathbb{Z}$ with these properties: $L(\omega_1, \omega_2) = L_1$; and the complex numbers

$$z(k_1, k_2) = (\tfrac{k_1}{n_1})\omega_1 + (\tfrac{k_2}{n_2})\omega_2,\ 0 \le k_i \le n_i,\ i = 1, 2,$$

give the $n_1 n_2$ distinct elements z mod L_1 mapping to 0 mod L_2. Hint: Apply the Elementary Divisor Theorem Chap. 2 [9.15] to get a basis $\{\boldsymbol{u}_i\}_{i=1}^2$ of L_2 and integers $n_1, \dots, n_2$ so that $\{n_i \boldsymbol{u}_i\}_{i=1}^2$ generates L_1.

(9.7d) Conclude for $x \in X_1$ that $x + z(k_1, k_2)$ mod L_1 are the distinct elements of X_1 mapping $f(x)$ under f.

Now we describe holomorphic differentials on a complex torus.

(9.8a) Let L be a lattice in $\mathbb{C}_z$. Define ω_α on one of the local coordinate charts $\varphi_\alpha(U_\alpha) \subset \mathbb{C}_z$ for $\mathbb{C}/L$ to be the differential dz (As in Ex. 6.18). Show this defines a global differential form ω_L on $\mathbb{C}/L$, and the divisor of this form is 0. Hint: Use that the transition functions, on connected subsets of $\varphi_\alpha(U_\alpha \cap U_\beta)$ have the form $z \mapsto z + \beta$.

(9.8b) Accept without proof that any meromorphic function has divisor of degree 0. Conclude: Holomorphic differentials on $\mathbb{C}/L$ have degree 0 divisor; so they are constant multiples of ω_L.

(9.8c) A g dimensional complex torus has the form $A = \mathbb{C}^g/L$ where L is a $\mathbb{Z}$ module having dimension $2g$ and such that $\mathbb{R}L = \mathbb{C}^g$ (a *lattice*). Imitate b) to show holomorphic differentials on A form a dimension g vector space.

[9.8c] considers complex torii. Since $\mathbb{C}^g$ is contractible, $\pi_1(A, \mathbf{0})$ identifies with L. We now see all differentiable groups have an abelian fundamental group.

(9.9a) Suppose that $\gamma_{0,i}$ and $\gamma_{1,i}$ are homotopic paths in a space X, $i = 1, 2$, and that the end point of $\gamma_{0,1}$ is equal to the initial point of $\gamma_{0,2}$. Show $\gamma_{0,1}\gamma_{0,2}$ is homotopic to $\gamma_{1,1}\gamma_{1,2}$.

(9.9b) Show the associative rule for multiplying paths.

(9.9c) Let ψ_1 and ψ_2 be two isomorphisms between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ as in Corollary 1.19. Show $\psi_2^{-1} \circ \psi_1$ is an *inner automorphism* of $\pi_1(X, x_0)$. That is, it is given by conjugation by an element of $\pi_1(X, x_0)$.

(9.9d) A group G is *differentiable* G if it is a differentiable manifold, and its multiplication and inverse are both differentiable maps. Similarly, there is the notion of *analytic group*. Show a complex torus $\mathbb{C}^g/L$ (L a lattice) is an analytic group.

(9.9e) Suppose M is a subvariety of $\mathrm{GL}_n(\mathbb{C})$ (defined by a finite number of equations in the n^2 coordinates of the entries), closed under multiplication and inverse. Show M is an analytic group.

(9.9f) For G a differentiable group consider $f_1 : G \to (G, 1)$ (resp. $f_2 : G \to (1, G)$) by $g \mapsto (g, 1)$ (resp., $g \mapsto (1, g)$). Show for $[\gamma_1], [\gamma_2] \in \pi_1(G, 1)$:

$$m_*((f_1)_*[\gamma_1])(f_2)_*[\gamma_2]) = [\gamma_1][\gamma_2].$$

(9.9g) Continuing b), show $\pi_1(G, 1)$ is an abelian group. Conclude: A differentiable manifold X with a nonabelian fundamental group (as often in Chap. 4) has no differentiable group structure.

9.3. $\mathbb{P}^n$ compactification. Use the notation of §4.3.

(9.10a) Consider $h \in \mathbb{C}(w)$, $h = h_1/h_2$, with $(h_1, h_2) = 1$. Let $m = h_2(w)z - h_1(z)$ as in Ex. 4.7. Show the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification of $\{(z, w) \mid m(z, w) = 0,\ z \notin \boldsymbol{z}\}$ is a manifold.

(9.10b) Consider $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$. Induct on n to show $\mathbb{P}^n = U_0 \dot\cup \mathbb{P}^{n+1}$. Inductively define a topology: neighborhoods of $\boldsymbol{x} \in \mathbb{P}^{n-1}$ are the image in $\mathbb{P}^n$ of neighborhoods of $(0, v_1, \dots, v_n) \in \mathbb{C}^{n+1}$.

(9.10c) Prove directly in $\mathbb{P}^n$: Any infinite sequence has a limit point. Hint: Any infinite sequence has an infinite subsequence in U_i for some i.

Fiber products help construct new manifolds from old. Consider some aspects of this. Use the notation of §4.2.3.

(9.11a) Generalize the $\mathbb{P}^2$ compactification of $h(w) - g(z)$ from Ex. 4.3.3.

(9.11b) Conclude the proof of Prop. 4.9 by noting $\mathcal{L}^h_{z'}[(z - z')^{1/e_1}, (z - z')^{1/e_2}]$ is a proper subring of $\mathcal{P}^h_{z',[e_1,e_2]}$, though its quotient field equals $\mathcal{P}_{z',[e_1,e_2]}$.

(9.11c) Finish the hyperelliptic case according to Ex. 4.3.3: $\mathbb{P}^1 \times \mathbb{P}^1$-compactification gives a manifold while no $\mathbb{P}^2$-compactification ever does.

(9.11d) Apply b) to $f : X \to \mathbb{P}^1_z$ of degree at least 3. Then, $V = X \times_{\mathbb{P}^1_z} X$ contains the diagonal Δ and it consists of the union of this and another compact set V'. Show V' has a manifold structure from its embedding in $X \times X$ if and only if there is only one ramified point over each branch point of f and that ramification order is 2. That is, f is a *simple-branched* cover.

(9.11e) Show global meromorphic functions on $\mathbb{P}^n$ are ratios of (same degree) homogeneous polynomials in the coordinates of $\mathbb{P}^n$. Show there is no analytic map $\psi : \mathbb{P}^2 \to \mathbb{P}^1$. Hint: A ratio of same degree polynomials has a singularity at common zeros.

(9.11f) Assume $\bar{X} \subset \mathrm{pr}^2_{z,w,u}$ is a compact manifold, and $(z_0, w_0, u_0) \in \bar{X}$ is the intersection of L_1 and L_2 in Prop. 4.13. Show there is no other value $z'_0 \neq z_0$ so $L_1 - z'_0 L_2$ is tangent to $\bar{X}$. Hint: Otherwise, $u' = (L_1 - z'_0 L_2)/z$ and $w' = (L_1 - z'_1 L_2)/z$ give local coordinates for $\bar{X}$ in a neighborhood of $(0, 0) \in\in \mathbb{C}_{u'} \times \mathbb{C}_{w'}$ though both functions ramify at $(0, 0)$.

9.4. Paths and vector fields. Let X be a manifold.

(9.12a) Show each (simplicial) path $\gamma : [a, b] \to X$ is image equivalent to $\gamma_1 : [0, 1] \to X$. Show each nonconstant path is image equivalent to a path constant on no interval.

(9.12b) Assume X is contractible (Def. 5.8). Suppose $\gamma : [a, b] \to X$ is a path with initial point x_0 and endpoint x_1. Form the function $G : [a, b] \times [0, 1] \to X$ by $G(t, s) = f(\gamma(t), s)$. Use this to show *all* paths in X with initial point x_0 and endpoint x_1 are homotopic.

(9.12c) Let $\gamma : [a, b] \to \mathbb{R}^n$ be a simplicial path. Let $\boldsymbol{f} = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be defined and continuous on the image of $[a, b]$. Consider

$$\sum_{i=1}^{n} \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt \stackrel{\text{def}}{=} \int_\gamma \boldsymbol{f} \cdot d\boldsymbol{x},$$

the *line integral of* $\boldsymbol{f}$ *along* γ.

(9.12d) If γ_1 and γ are image equivalent paths in $\mathbb{R}^n$, show line integrals along them are equal (use change of variables formula from Chap. 2 Lem. 2.3).

(9.12e) Let $F : [a, b] \times [0, 1] \to \mathbb{R}^n$ be a homotopy between paths γ_0 and γ_1 (write $F(t, s) = \gamma_s(t)$) in $\mathbb{R}^n$. Assume $\boldsymbol{f}$ is continuous on the image of F. Show the line integral of $\boldsymbol{f}$ along γ_s is a continous function of s.

For a differentiable path $\gamma : [0, 1] \to U$ with U open in $\mathbb{R}^n$, there may not exist a vector field T_U having γ as an integral curve, though *locally* this is so.

(9.13a) If T_U exists explain why $\gamma(t_1) = \gamma(t_2)$ implies $\frac{d\gamma}{dt}(t_1) = \frac{d\gamma}{dt}(t_2)$.

(9.13b) Let V be a neighborhood of the line segment $t \to (t, 0, \ldots, 0) \in \mathbb{R}^n_{\boldsymbol{t}}$, $t \in [0, 1]$. Assume there is a one-one differentiable $\Gamma : V \to U$ with $\Gamma(t, 0, \ldots, 0) = \gamma(t)$. Show $\frac{\partial\gamma}{\partial t_1}(\boldsymbol{t})$ (applying $\frac{\partial}{\partial t_1}$ to all coordinates of Γ) produces a vector field on $\Gamma(V)$ with γ an integral curve.

(9.13c) Assume $\frac{d\gamma}{dt}$ is never 0. Consider $H_t = \{\boldsymbol{w} \in \mathbb{R}^n \mid \boldsymbol{w} \cdot \frac{d\gamma}{dt} = 0\}$. Find differentiable one-one $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(\boldsymbol{t}) = \gamma(t_1) + \boldsymbol{w}(t_1, t_2, \ldots, t_n)$ with $\boldsymbol{w}(t_1, t_2, \ldots, t_n) \in H_{t_1}$ linear in $(t_2, \ldots, t_n)$ (t_1 fixed). Hint: Apply the chain rule.

(9.13d) How does F give Γ in b)?

Returning to (5.3) we relate

$$(f_{\alpha,1}(\boldsymbol{y}_\alpha), \ldots, f_{\alpha,n})(\boldsymbol{y}_\alpha) \text{ to } (f_{\beta,i}, \ldots, f_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$$

.

(9.14a) Apply both sides of (5.3) to the coordinate function $y_{\beta,j}$ to get

$$f_{\beta,j}(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)) = \sum_{i=1}^{n} f_{\alpha,i} \frac{\partial \psi_{\beta,\alpha,j}}{\partial y_{\alpha,i}}(\boldsymbol{y}_\alpha)$$

where $\psi_{\beta,\alpha,j}$ is the jth coordinate of $\psi_{\beta,\alpha}$. That is, the f_β s are the result of applying the Jacobian matrix of $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$ to the f_α s.

(9.14b) Consider the case $\psi = \psi_{(x,y),(r,\theta)} : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ by $(r, \theta) \mapsto (x, y)$. Express $\frac{\partial}{\partial x}$ as $f_r \frac{\partial}{\partial r} + f_\theta \frac{\partial}{\partial \theta}$ by applying both to $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Do the same for $\frac{\partial}{\partial y}$, expressing it as $f'_r \frac{\partial}{\partial r} + f'_\theta \frac{\partial}{\partial \theta}$. Applying $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ to $f(x, y)$ and evaluating at $(r\cos(\theta), r\sin(\theta))$ is the same as applying $J(\psi_{(x,y),(r,\theta)})(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ to $f(r\cos(\theta), r\sin(\theta))$.

(9.14c) Generalize b) to say (as in (5.4))

$$J(\psi_{\boldsymbol{y}_\beta,\boldsymbol{y}_\alpha})^{-1}\Big(\frac{\partial}{\partial y_{\alpha,1}}, \ldots, \frac{\partial}{\partial y_{\alpha,n}}\Big) = \Big(\frac{\partial}{\partial y_{\beta,1}}, \ldots, \frac{\partial}{\partial y_{\beta,n}}\Big).$$

9.5. Permutation group properties. Suppose $G \le S_n$ is transitive. Def. 7.9 defines primitive subgroup of S_n.

(9.15a) For $g \in N_G(G(1))$, multiplication of g on the *left* of the distinct right cosets $G(1)\sigma_1, \ldots, G(1)\sigma_n$ of $G(1)$ permutes these cosets. Conclude: This induces a homomorphism $\psi : N_G(G(1))/G(1) \to \mathrm{Cen}_{S_n}(G)$.

(9.15b) Show ψ is an isomorphism because both groups have order equal

$$|\{i \in \{1, 2, \ldots, n\} \mid \sigma(i) = i \text{ for each } \sigma \in G(1)\}|.$$

(9.15c) Show $N_G(G(1))/G(1)$ (or $\mathrm{Cen}_{S_n}(G)$) is trivial if G is primitive and $G(1)$ is nontrivial.

(9.15d) Show a nontrivial normal subgroup of a primitive group is transitive.

(9.15e) Show a primitive subgroup of S_n containing a 2-cycle is S_n. Conclude any transitive group generated by 2-cycles is S_n. Hint: Consider the normal subgroup generated by the conjugates of the 2-cycle.

Let G be a centerless group, $\mathrm{Aut}(G)$ its automorphisms and $T : G \to S_n$ faithful transitive permutation representation.

(9.16a) Explain this from [**Isa94**, p. 43]: In general neither $(gH)A(g') \stackrel{\mathrm{def}}{=} gHg'$ nor $(gH)A(g') \stackrel{\mathrm{def}}{=} (gg')H$ define an action on left cosets of H in G.

(9.16b) Let S be the collection of conjugates of a subgroup H of the group G, with the action by conjugation by elements of G: $S = g^{-1}Hg_{g\in G}$ and the right action of $g' \in G \mapsto (g')^{-1}g^{-1}Hgg'$. What is the coset representation associated with this transitive action, and when is it faithful?

(9.16c) Show (conjugation by) G is normal in $\mathrm{Aut}(G)$. The *outer automorphism* group $\mathrm{Out}(G)$ of G is the quotient $\mathrm{Aut}(G)/G$. Show the natural map $\psi_T : N_{S_n}(G) \to \mathrm{Out}(G)$ has kernel $\mathrm{Cen}_{S_n}(G)$ (§7.1.3; compare with [9.15c]).

(9.16d) Denote the image of ψ_T in $\mathrm{Out}(G)$ by $\mathrm{Out}_T(G)$. Show $\mathrm{Out}_T(G) = \mathrm{Out}(G)$ if and only if $G(T,1)$ (§7.1.2) has exactly n images under $\mathrm{Aut}(G)$. Hint: Associate to $\alpha \in \mathrm{Aut}(G)$ an element of S_n defined up to $\mathrm{Cen}_{S_n}(G)$ if it maps among the conjugates of $G(T,1)$. Show [9.20b] gives examples where T is doubly transitive and $\mathrm{Out}(G) \neq \mathrm{Out}_T(G)$.

(9.16e) Case: $G = A_n$ (resp. $G = S_n$), $n \geq 4$, in its standard representation T. Show $\mathrm{Out}(S_n) = \{1\}$ (resp. $\mathrm{Out}_T(A_n) = \mathrm{Out}(A_n) = \mathbb{Z}/2$) if and only if S_n (resp. A_n) has exactly n transitive subgroups of index n under $\mathrm{Aut}(G)$. Hint: Intransitive subgroups have small orders. (See [9.17b].)

(9.16f) Set notation in the proof of Prop. 8.19 to change to a left action of GL_n.

We will need the following facts later.

(9.17a) For each i, $2 \leq i \leq n$, consider $L_i = \{1_n, (1\,i), (2\,i), ..., (i-1\,i)\} \subset S_n$ (1_n indicates the identity). Show each $x \in S_n$ has a unique product representation as $x = x_1x_2\dots x_n$ with $x_i \in L_i$. (This gives a technique to generate random elements of S_n with uniform distribution.) Hint: For $g \in S_n$ if $(n)g = i$, let $h = g(i\,n)$ and induct on n.

(9.17b) [**Isa94**, p. 79-80] bases $\mathrm{Out}(S_n) = \{1\}$, if $n \neq 6$, on two observations:
- If $\alpha \in \mathrm{Aut}(S_n)$ permutes transpositions, then conjugating by some $g \in S_n$ gives α. Hint: Elements of $(L_i)\alpha$ in a) then have a unique integer of common support.
- If $n \neq 6$, among elements of order 2, the conjugacy class of transpositions has a unique cardinality.

(9.17c) Let $T_H : G \to S_n$ be a permutation representation. Show all cosets of H have the form Hg^i, $i = 0, \dots, n-1$, if and only if g is an n-cycle in T_H.

(9.17d) Suppose F has characteristic p which also divides the order of finite group G. Show a faithful permutation representation of G cannot be completely reducible. Hint: Reduce to $G = \langle g \rangle$ with g having order p.

(9.17e) Suppose G is a free group on $r \geq 2$ generators. Find representations $\varphi : G \to \mathrm{GL}_r(\mathbb{C})$ that are not completely reducible. Hint: Map G into an upper-triangular, not diagonal, matrix group.

9.6. Affine groups as permutation representations. Let $H \leq GL_k(F)$ with $F = \mathbb{F}_q$. Regard $G = \{\left(\begin{smallmatrix} A & v \\ 0 & 1 \end{smallmatrix}\right) \mid A \in H,\ v \in V = \mathbb{F}_q^k\}$ as a group $V \times^s H$ as in Rem. 7.4. Note: If a nonabelian group replaced $\mathbb{F}_q^k$, then $A(v') + v$ should more naturally be written $v + A(v')$.

(9.18a) Suppose $\{0\} < V_1 < V$ is an H invariant space. Then, $V_1 \times^s H$ is a subgroup of G properly containing H. Show conversely, a group properly between H and G has the form $V_1 \times^s H$ with H invariant V_1.

(9.18b) Embed V in G by $v \mapsto \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$. Have G act on V by $\begin{pmatrix} A' & v' \\ 0 & 1 \end{pmatrix}$ maps $v \mapsto A(v) + v' = v^*$: equivalent to $\begin{pmatrix} A' & v' \\ 0 & 1 \end{pmatrix}$ multiplies $\binom{v}{1}$ to $\binom{v^*}{1}$. Show this gives a faithful transitive permutation representation of G.

(9.18c) From a) the representation of b) is primitive if and only if H acts irreducibly. Suppose $H = \langle A \rangle$ has a single matrix generator, which we use to makes V into an $F[z]$ module by having $f(z) \in F[z]$ map $v \in V$ to $f(A)(v)$. The *elementary divisor theorem* (Chap. 2 §9.15) says $V \equiv \oplus_{i=1}^t F[z]/(f_i)$ (as an $F[z]$ module). Example: If $v = (a, b) \in F^2$, and $A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, $f(z) = z^2 + z + 1$, then $f(z)(v) = \begin{pmatrix} 6 & 12 \\ 4 & 10 \end{pmatrix}(v) = (6a + 12b, 4a + 10b)$. We can uniquely choose the f_is monic so that $f_1|f_2|\cdots|f_t$. Show G is primitive if and only if $t = 1$ and f_1 is an irreducible polynomial.

(9.18d) The multiplicative group $\mathbb{F}_{p^n}^*$ is cyclic. Let α be a generator, and A the matrix of α acting on $\mathbb{F}_p^n$ by regarding it as $\mathbb{F}_{p^n}$. In equation form: $v \in \mathbb{F}_{p^n} \mapsto \alpha v \in \mathbb{F}_{p^n}$. Show $V \times^s \langle A \rangle$ is doubly transitive on F.

(9.18e) From Def. 7.9, b) is doubly transitive if and only if H is transitive on $V \setminus \{0\}$. When $H = \langle A \rangle$, show G is doubly transitive if and only if, for some isomorphism of $\mathbb{F}_{p^n}$ and $(\mathbb{F}_p)^n$, A acts like multiplication by $\alpha \in \mathbb{F}_{p^n}^*$.

9.7. Group representations. In this exercise consider representations over any field containing $\mathbb{Q}$.

(9.19a) Show that the direct product of two permutation representations as a group representation is the tensor product of the two group representations. Therefore the trace is the product of the traces.

(9.19b) Finish showing the number of orbits is the same as the number of appearances of the identity.

(9.19c) Let $T_i : G \to S_{n_i}$, $i = 1, 2$, be permutation representations for which $\mathrm{t}(T_1(g)) = \mathrm{t}(T_2(g))$ for each $g \in G$ (as in §7.1). Show $n_1 = n_2$ and $T_1(g)$ and $T_2(g)$ are conjugate in S_{n_1} for each $g \in G$. Hint: Induct on the length of the highest disjoint cycles and compare $\mathrm{t}(T_1(g))$ and $\mathrm{t}(T_1(g^r))$ for some prime r dividing the order of g.

(9.19d) Show $\frac{1}{|G|}\sum_{g\in G} \mathrm{t}(T(g))$ counts the orbits of a permutation representation T. Hint: Put the additive operator t on the outside of the sum by regarding $T(g)$ as a permutation matrix. Each orbit I gives a 1-dimensional invariant subspace spanned by $\sum_{i\in I} x_i$ (as in §7.1.4).

(9.19e) Show the collection of $L_{\mathrm{C}} = \sum_{u\in C} u$ with C a conjugacy class of G, span the G invariant idempotents of $\mathbb{C}[G]$. For ρ any representation, $\frac{1}{|G|}\sum_{g\in G} \mathrm{t}(\rho(g))$ counts appearances of $\mathbf{1}_G$ in ρ. Hint: $\frac{1}{|G|}\sum_{g\in G} \rho(g)$ is an idempotent, and its trace equals the dimension of its range.

(9.19f) *Orthogonality Relations*: Let ρ_V and ρ_W be representations of G on respective spaces V and W. Show $\mathrm{t}(\rho_{V^*\otimes W}(g)) = \mathrm{t}(\rho_V(g))\mathrm{t}(\rho_W(g))$ gives

$$\sum_{g\in G} \mathrm{t}(\bar{\rho}_V(g))\mathrm{t}(\rho_W(g)) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V,W).$$

Further, this dimension gives the appearances of $\mathbf{1}_G$ in $\mathrm{Hom}_{\mathbb{C}}(V,W)$. Hint: For ρ' irreducible, $\mathbf{1}_G$ appears exactly once in $\mathrm{Hom}_{\mathbb{C}}(V_{\rho'},V_{\rho'})$.

(9.19g) Show V_T is $\mathbf{1}\oplus V'$ with V' irreducible if and only if T is doubly transitive. Hint: Apply d) to count appearances of $\mathbf{1}_G$ in $V_T\otimes V_T$; use that $\mathbf{1}_G$ appears in $\rho_1\otimes\rho_2$ with ρ_1,ρ_2 irreducible only if $\rho_2=\bar{\rho}_1$.

(9.19h) Suppose the representation $T: G\to S_n$ is doubly transitive. Show G does not contain a subgroup H of degree $m<n$ intransitive in T. Hint: Count appearances of $\mathbf{1}_G$ in $V_T\otimes V_{T_H}$ using d).

Denote the finite field of $q=p^r$ for p a prime by $\mathbb{F}_q$. Let $G=\mathrm{GL}_k(F)$ be the $k\times k$ invertible matrices with coefficients in the field $F=\mathbb{F}_q$. Write $\mathbb{P}^{k-1}(F)$ for lines through the origin in $\mathbb{F}_q^k$: $\{\alpha\boldsymbol{v}\mid \alpha\in F\}$ for some $\boldsymbol{v}\in\mathbb{F}_q^k\setminus\{\mathbf{0}\}$. Then, G has a permutation action $T_{k,F}$ on $\mathbb{P}^{k-1}(F)$ induced from its action on $\mathbb{F}_q^k$. Let $\psi:\mathbb{F}_q^k\to\mathbb{F}_q$ be a nonconstant linear map (*linear functional*). Denote linear functionals up to multiplication by elements of $\mathbb{F}_q\setminus\{0\}$ by $\hat{\mathbb{P}}^{k-1}(F)$, with a permutation action $\hat{T}_{k,F}$: For $\psi\in\mathbb{P}^{k-1}(F)$ and $A\in\mathrm{GL}_k(F)$, $\psi^A(\boldsymbol{v})\overset{\text{def}}{=}\psi((\boldsymbol{v})A^{-1})$ for $\boldsymbol{v}\in\mathbb{F}_q^k$.

(9.20a) Show $T_{k,F}$ is doubly transitive of degree $n(q)=\frac{(q^k-1)}{(q-1)}$ for $k>1$.

(9.20b) Show $\hat{T}_{k,F}$ also has degree $n(q)$ and is doubly transitive, though $T_{k,F}$ and $\hat{T}_{k,F}$ are not permutation equivalent. Hint: Show the stabilizer in G of a hyperplane in $\mathbb{P}^{k-1}(F)$ fixes no point.

(9.20c) Show $\mathrm{t}(\hat{T}_{k,F}(g))=\mathrm{t}(T_{k,F}(g))$, so $\hat{T}_{k,F}$ and $T_{k,F}$ are equivalent as representations [9.19c]. Hint: $\hat{T}_{k,F}(g)$ is induced from the transpose of g, and a matrix and its transpose are conjugate.

(9.20d) As in [9.6d], identify $\mathbb{F}_q^k$ with $\mathbb{F}_{q^k}$ as vector spaces over $\mathbb{F}_q$ to find $\alpha\in\mathbb{F}_{q^k}$ producing $A\in\mathrm{GL}_k(F)$ with $T_{k,F}(A)$ and $\hat{T}_{k,F}(A)$ both $n(q)$-cycles.

(9.20e) Assume: T_1,T_2 are inequivalent degree n doubly transitive representations of a group G; they are equivalent as group representations; and $T_1(g)=T_2(g)=(1\,2\,\dots\,n)$ for some $g\in G$. Let D be the orbit of 1 under $G(T_1,1)$ in the representation T_2. Use double transitivity to show D is a *difference set*: $\{d_i-d_j\mid d_i\neq d_j\in D\}$ contains each nonzero integer mod n with the same multiplicity t [**Fri73**]. Further, $t\cdot(n-1)=|D|\cdot(|D|-1)$. Example: For $k=3$, $q=2$, $n=7$ in b), $D=\{1,2,4\}$ and $t=1$.

9.8. Easy Galois covers.

(9.21a) Suppose X and Y_i are differentiable manifolds, and that $f_i: Y_i\to X$ are covering maps, $i=1,\,2$. Assume $\psi: Y_1\to Y_2$ is any *continuous* map with $f_2\circ\psi=f_1$. Show ψ is a map of differentiable manifolds. Also: ψ is analytic if X is a complex manifold.

(9.21b) Let $f: Y\to X$ be a finite cover of degree n. Use that X is connected to show $|f^{-1}(x)|$ is n for each $x\in X$.

(9.21c) Consider $X_1=\{x+\sqrt{-1}\,y\in S^1\mid y>0\}$ and $X_2=S^1$. Show, for $n>0$, the map of Ex. 6.16 restricted to X_1 is not a covering map.

(9.21d) Follow the notation of Ex. 6.18 and of [9.7]. Let L and L_i, with $L_i \subseteq L$, $i = 1, 2$, be lattices. Show that if $f_i : X_i = \mathbb{C}/L_i \to \mathbb{C}/L$ by $z \bmod L_i \mapsto z \bmod L$, then the covers (X_i, f_i) are equivalent if and only if $L_1 = L_2$.

(9.21e) Let $X_i = X$, $i = 1, \dots, n$, and let Y be the disjoint union of the X_i's. What is the automorphism group of the cover $Y \to X$ obtained by mapping each point of Y to its corresponding point in X?

(9.21f) Let $f : Y \to X$ be a cover and consider a subgroup G of $\mathrm{Aut}(Y, f)$ of order equal to $\deg(f)$. Assume that, for some point $x_0 \in X$, G acts transitively on the set $f^{-1}(x_0)$. Show f restricted to any connected component of Y gives a Galois cover of X.

(9.21g) Let $X = Y = \mathbb{C} \setminus \{0\}$. Show $f : Y \to X$ by $z \mapsto z^n$ is a Galois cover. Hint: Consider $\psi_k : z \mapsto e^{2\pi\sqrt{-1}\,k}z$, $0 \le k \le n-1$.

(9.21h) Let X_i, $i = 1, 2$, be as in [9.7c] with $L_1 \subset L_2$. Show $f : X_1 \to X_2$ in Ex. 6.18 is a Galois cover. Hint: Consider $\psi_{k_1,k_2} : z \bmod L_1 \mapsto z + z(k_1, k_2) \bmod L_1$.

(9.21i) Consider a) with $f \in \mathbb{C}[y]$ and $f(y) = y^n + c_{n-2}y^{n-2} + \cdots + c_1 y$. Assume the greatest common divisor of the set $\{n \text{ and } i \text{ with } c_i \ne 0\}$ is 1. Show $\mathrm{Aut}(Y, \mathrm{pr})$ is trivial. Hint: Apply Liouville's Theorem [**Ahl79**, p. 122] to see elements of $\mathrm{Aut}(Y, f)$ have the form $y \mapsto ay + b$ for some $a, b \in \mathbb{C}$.

9.9. Imprimitive and Frattini covers. This discussion on imprimitivity continues in Chap. 4 [10.12]

(9.22a) Let $\pi_1(X, x_0)$ be the fundamental group of a connected differentiable manifold X. Let $H\sigma_1, \dots, H\sigma_n$ be the distinct cosets of a subgroup $H \le \pi_1(X, x_0)$ of index n corresponding to the cover (Y, f) (with fiber $\{y_1, \dots, y_n\}$ over x_0). Consider the points of $U_{Y,f,n}$ (§8.3.2) over x_0 that connect by a path to $(y_1, \dots, y_n)$. Show these correspond to distinct n-tuples of cosets: $\{(H\sigma_1\sigma, H\sigma_2\sigma, \dots, H\sigma_n\sigma) \mid \sigma \in \pi_1(X, x_0)\}$. Why is this the same as $|G|$? Conclude $\deg(\hat{f}_i) = |G(Y, f)|$ (as prior to Thm. 8.9).

(9.22b) Show components of $Y \times_X Y$ of degree 1 over Y correspond to elements of $\mathrm{Aut}(Y, f)$ (Lem. 8.8). If $f : Y \to X$ has automorphisms, and f is not a cyclic Galois cover of prime degree, show $G(Y, f)$ is imprimitive. How does [9.21i] give explicit imprimitive covers with no automorphisms?

(9.22c) Show (Y, f) decomposes if and only if $Y \times_X Y \to X$ properly factors through a fiber product of form $Y' \times_X Y'$. If so, show $Y' \times_X Y' \setminus \Delta$ is a nontrivial component of $Y \times_X Y$.

Let $K \subset \hat{L} \subset \hat{M}$ be a chain of fields with $\hat{M}/K$ (resp. $\hat{L}/K$) Galois with group G^* (resp. G). This is a *Frattini chain* if the only subfield $K \le T \le \hat{M}$ with $T \cap \hat{L} = K$, is $T = K$. Denote restriction of elements of G^* to $\hat{L}$ by $\mathrm{rest} : G^* \to G$.

(9.23a) Suppose $T = \hat{M}^H$ is the fixed field of a subgroup H of G^*. Show $T \cap \hat{L} = K$ is equivalent to $\mathrm{rest} : H \twoheadrightarrow G$. Hint: Use that $T \cap \hat{L} = K$ allows extending any automophism of $\hat{L}$ to $T \cdot \hat{L}$ to be the identity on T.

(9.23b) Show a) is equivalent to this group statement: If $H \le G^*$ and $\mathrm{rest}(H) = G$, then $H = G^*$ (the map $\mathrm{rest} : G^* \to G$ is a *Frattini cover*). Hint: $\mathrm{rest}(H) = G$ is equivalent to $\hat{M}^H \cap \hat{L} = K$.

(9.23c) Suppose $\hat{X} \to \hat{Y} \to Z$ is a sequence of covers with $\psi_X : \hat{X} \to Z$ Galois with group G^* and $\psi_Y : \hat{Y} \to Z$ Galois with group G. Let $\psi : G^* \to G$ be the natural map and assume ψ is a Frattini cover. Show the equivalence with this. For any sequence $\hat{X} \to W \to Z$ of covers with $W \neq Z$, there is a proper cover of Z that $W \to Z$ and $\hat{Y} \to Z$ factor through.

9.10. Laplacian. The *Laplace operator* $\nabla^2 = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y}$ on $\mathbb{R}^2_{x,y}$ acts on $C^\infty(\mathbb{R}^2)$. It generalize to a Riemann surface X (see Chap. 4 §10.9 for $\wedge$ product). Locally in $z = x + iy$, write a differential 1-form (not necessarily holomorphic) on an open set $U \subset \mathbb{C}$ as $\omega = p(x, y)\, dx + q(x, y)\, dy$. Consider $*\omega = -qdx + pdy$. Write $w = u + iv$ for the real and imaginary components of the variable for $\mathbb{C}_w$.

(9.24a) With $z = f(w)$, suppose $f : V \subset \mathbb{C}_w \to U \subset \mathbb{C}_z$ is analytic, one-one and onto from V to U. Write $w = u(x, y) + iv(x, y)$ as the local inverse of f. Express ω as $\Omega_(u, v) = p(x(u, v), y(u, v))\, dx(u, v) + q(x(u, v)\, y(u, v))\, dy(u, v)$. Show $*\Omega_(u, v) = -Q(u, v)\, du + P(u, v)\, dv$ equals $*\omega$ expressed in u and v. Hint: Apply the Cauchy-Riemann equations: $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$ and $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$.

(9.24b) Conclude from a): On any Riemann surface X, $*$ defines a linear map on differentiable 1-forms.

(9.24c) Show these further properties of $*$: Its square is multiplication by -1, $\omega_\wedge * \omega_= (p^2 + q^2) dx \wedge dy$, and $*\omega = i\omega$ if ω is holomorphic. Conclude: ω is holomorphic if and only if $d\omega = 0$ and $*\omega = i\omega$.

(9.24d) Consider $*d = -\frac{\partial}{\partial y} dx + \frac{\partial}{\partial x} dy$ acting on differentiable functions. So, for f differentiable on X, $*df = *d(f)$ is well-defined, and it extends to 1-forms: $p(x, y)\, dx + q(x, y)\, dy \mapsto *d(p(x, y)) \wedge dx + *d(q(x, y)) \wedge dy$. Show $*d(\omega)$ is $-d * \omega$. Define $\nabla^2(f)$ by $d * df = \nabla^2(f) dx \wedge dy$. Argue why this defines a (complex) Laplacian ∇^2_X on a 1-dimensional complex manifold.

(9.24e) Suppose differentiable f on X has a corresponding $\lambda \in \mathbb{C}$ with $\nabla^2_X(f) = \lambda f\, dx \wedge dy$ (everywhere locally). Call λ an *eigenvalue* of ∇^2_X. If $f_i : Y_i \to X$ are inequivalent covers of X, with equivalent locally flat bundles (Defn8.6), over X, $i = 1, 2$, show their Laplacians have the same eigenvalues.