

G-covers of \mathbb{P}^1 , Hurwitz spaces and modular towers

1. Classical Hurwitz space

u, r, g integers $r := 2g + 2n - 2$

$\mathcal{H}_{n,r}$: isomorphism classes of simple branched covers of \mathbb{P}^1 with degree r and n branch points

- Clebsch 1872
- Hurwitz 1891 (connectedness of $\mathcal{H}_{n,r}$)
- Severi 1921 $\mathcal{H}_{n,r} \rightarrow \mathcal{M}_g \Rightarrow$ connectedness of $\mathcal{M}_g / \mathbb{Q}$
- Fulton 1969 $\mathcal{H}_{n,r} / \mathbb{Z} \Rightarrow$ connectedness of $\mathcal{M}_g \otimes_{\mathbb{Z}} \mathbb{F}_p, p > g+1$
- Harris-Mumford 1982 modular compactification of $\mathcal{H}_{n,r}$

2. Hurwitz for G-covers (or variants)

- Fried 1977
- Fried-Velklein 1991 $\mathcal{H}_{r,s} / \mathbb{Q}$
- Newers 1998 $\mathcal{H}_{r,s} / \mathbb{Z}$ + modular compactification

Surveys:

- Velklein "Group as Galois groups"
Cambridge studies in adv. math. 53
Cambridge Univ. Press 1996
- Emsalem "Espaces de Hurwitz"
Séminaires et Conférences 5
SMF 2001
- Romagny-Newers "Hurwitz spaces"
Séminaires et Conférences 13
SMF 2006

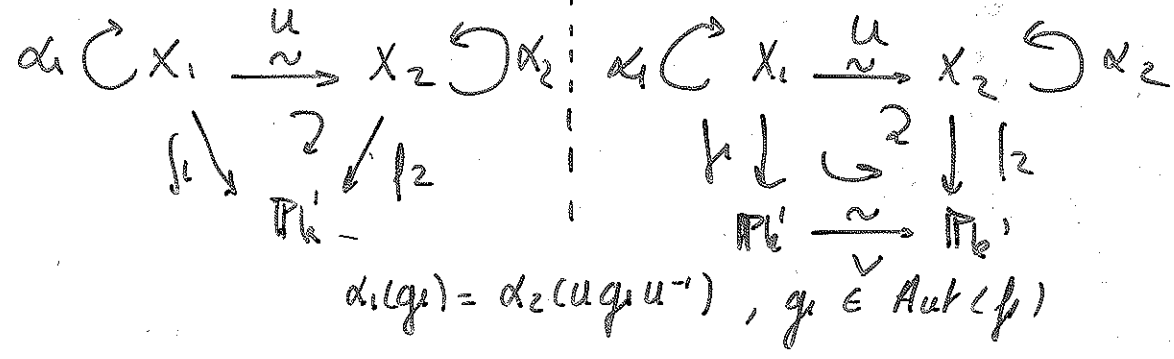
G-covers / G/PGL₂-covers

G-cover of P¹_k: pair (f, α) with

- f: X → P¹_k Galois cover i.e.
 - (i) f finite, flat morphism with X smooth, geo. irred., proj. curve / k
 - (ii) Aut(f) acts transitively on the fibers
- α: Aut(f) → G group isomorphism

G-covers

G/PGL₂-covers



Alternative description: $\underline{t} \in U_1(k)$

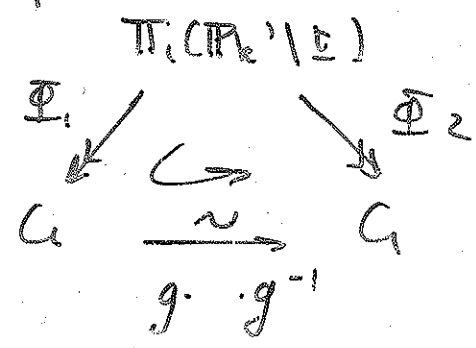
$M_{k, \underline{t}} / \overline{k(T)}$ max. alg. extension of $\overline{k(T)}$ (in $\overline{k(T)}$) unramified outside \underline{t}

$\pi_1(\mathbb{P}^1_{\overline{k}}(\underline{t})) := \text{Gal}(M_{k, \underline{t}} / \overline{k(T)})$ geometric fund. group

$\pi_1(\mathbb{P}^1_k(\underline{t})) := \text{Gal}(M_{k, \underline{t}} / k(T))$ arithmetic fund. group

$$1 \rightarrow \pi_1(\mathbb{P}^1_{\overline{k}}(\underline{t})) \rightarrow \pi_1(\mathbb{P}^1_k(\underline{t})) \xrightarrow{\Delta} \Gamma_k \rightarrow 1$$

G-cover of P¹_k ↔ $\overline{\Phi}: \pi_1(\mathbb{P}^1_{\overline{k}}(\underline{t})) \rightarrow G$ cont. group epimorphism s.t. $\overline{\Phi}(\pi_1(\mathbb{P}^1_{\overline{k}}(\underline{t}))) = G$



Inertia canonical invariant

$\text{char}(k) = 0$, $(S_n)_{n \geq 1} \in \mathbb{K}$ $S_{nm} = S_m$ $n, m \geq 1$

$\underline{t} \in \text{Ur}(k)$, $t \in \underline{t}$

$P_t | t$ place of $\Pi_{k, \underline{t}}$ above t

$I(P_t | t)$ procyclic group of profinite order et

$u \in P_t$ uniformizing parameter

$\alpha_{P_t} : I(P_t | t) \hookrightarrow \mathbb{K}^\times$) cont. group monomorphism
 $\omega \longmapsto \frac{\omega(u)}{u} \text{ mod } P_t$

distinguished generator of $I(P_t | t)$: $\gamma_{P_t | t} = \alpha_{P_t}^{-1}(Set)$

$\implies C_t := \{ \gamma_{P_t | t} \}_{P_t | t}$
 $= \{ \sigma \gamma_{P_t | t} \sigma^{-1} \}_{\sigma \in \pi_1(\Pi_{\mathbb{K}} \setminus \underline{t})}$
 $= \underline{\text{inertia canonical class at } t}$

Lemma: $\forall \sigma \in \Gamma_{\mathbb{K}} \xrightarrow{s(\sigma)} \gamma_{P_t | t}$ is conjugate to $\gamma_{\sigma P_t | \sigma t}^{\chi(\sigma)}$ in $\pi_1(\Pi_{\mathbb{K}} \setminus \underline{t})$, where $\chi: \Gamma_{\mathbb{K}} \rightarrow \hat{\mathbb{Z}}^\times$ cyclotomic character of \mathbb{K} .

Thm (RET): $\exists P_{t_i} | t_i, i=1, \dots, r$ s.t.

$\rho : \pi_1(\Pi_{\mathbb{K}} \setminus \underline{t}) \xrightarrow{\sim} \langle \pi_1, \dots, \pi_r \mid \pi_1 \dots \pi_r = 1 \rangle^{\wedge}$) cont. group isomorphism
 $\gamma_{P_{t_i} | t_i} \longmapsto \pi_i$

Inertia canonical invariant:

(f, α) G -cover $\iff \Phi_f : \pi_1(\Pi_{\mathbb{K}} \setminus \underline{t}) \twoheadrightarrow G$ cont. group epi.

- $C_{t, f} = \Phi_f(C_t)$

= conjugacy class of $\Phi_f(\gamma_{P_t | t})$ in G

= inertia canonical class of f at t

- $\underline{C}_f = (C_{t, f})_{t \in \underline{t}}$

= inertia canonical invariant of f

Field of moduli, field of definition

(f.d) G -cover $\mathbb{P}^1 \xrightarrow{\Phi_f} \mathbb{P}^1$ $\rightarrow G$ cont. group epi.
 $\underline{t} \in \text{Ur}(k)$ k_0/k alg.

Field of definition (f.o.d.): k is a f.o.d. for f
 \iff (1) $\exists f|_{k_0}: X_{k_0} \rightarrow \mathbb{P}^1_{k_0}$ G -cover of $\mathbb{P}^1_{k_0}$ s.t. $f|_{k_0 \times_k k} \simeq_{G, \underline{t}}$

\iff (2) $1 \rightarrow \Pi_1(\mathbb{P}^1 | \underline{t}) \xrightarrow{\Phi_f} \Pi_1(\mathbb{P}^1 | \underline{t}) \rightarrow \Gamma_k \rightarrow 1$
 $\Phi_f \downarrow \swarrow \exists \Phi_{f,k} \searrow$
 G

Field of moduli (f.o.m.): k_0 is a f.o.m. for f

\iff (1) $k_0 = \bar{k}^{\Gamma_{f,k}}$

with $\Gamma_{f,k} := \{ \sigma \in \Gamma_k \mid \sigma|_f \simeq_{G, \underline{t}} f \}$ $\langle d, j, i \rangle \Gamma_k$

\iff (2) $\exists h_{f,k_0}: \Gamma_{k_0} \rightarrow G$ s.t.

$$\Phi_f(\sigma(\gamma)) = h_{f,k_0}(\sigma) \Phi_f(\gamma) h_{f,k_0}(\sigma)^{-1}, \quad \sigma \in \Gamma_{k_0}, \gamma \in \Pi_1(\mathbb{P}^1 | \underline{t})$$

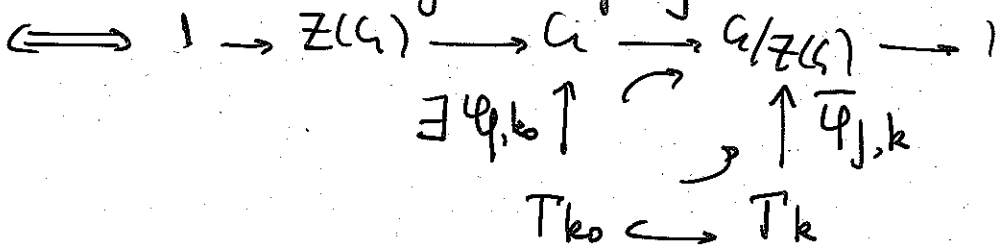
Cohomological obstruction:

- $\Phi_{f,k}: \Gamma_k \rightarrow G/Z(G)$
 $\sigma \mapsto h_{f,k}(\sigma) \text{ mod } Z(G)$) group morphism

- $\omega_{f,k}: \Gamma_k \times \Gamma_k \rightarrow Z(G)$
 $(\sigma, \tau) \mapsto h_{f,k}(\sigma\tau)^{-1} h_{f,k}(\sigma) h_{f,k}(\tau)$) 2-cochain

- k_0/k field ext. $\text{Res}_k^{k_0}([\omega_{f,k}]) = 0$ in $H^2(k_0, Z(G))$

$\iff k_0$ is a f.o.d. for f



③ $[\omega_{f,k}] \in H^2(k, Z(G))$ coh. obstruction for f to be defined / k

The functors $H(\underline{C}), H^{rd}(\underline{C})$

$r \geq 3$ integer
 G finite group
 $\underline{C} = (C_1, \dots, C_r)$ r -tuple of non trivial conj. classes in G
 $\Delta_{\underline{C}} := \{ \sigma \in T_{\mathbb{Q}}^r \mid \underline{C}^{\chi(\sigma)} = \underline{C} \text{ up to re-ordering} \} \subset \text{c.f.i. } T_{\mathbb{Q}}^r$

$\mathbb{Q}_{\underline{C}} := \overline{\mathbb{Q}}^{\Delta_{\underline{C}}}$

$\mathcal{Y}_{\underline{C}} :=$ category of field extensions $k/\mathbb{Q}_{\underline{C}}$

Rem: $(f: X \rightarrow \mathbb{P}^1_k, \alpha)$ G -cover of \mathbb{P}^1_k with group G
 and inertia canonical invariant $\underline{C} \implies k \supset \mathbb{Q}_{\underline{C}}$

Not: For all $k/\mathbb{Q}_{\underline{C}}$

$H(\underline{C})(k) = G$ -
 $H^{rd}(\underline{C})(k) = G/PGL_2$ -
) isomorphism classes of G -covers
 of \mathbb{P}^1_k with group G and i.c.i. \underline{C}

$H^{rd}(\underline{C})(k) = H(\underline{C})(k) / PGL_2(k)$

$\implies \Pi: H(\underline{C}) \rightarrow H^{rd}(\underline{C})$ morphism of categories
 fibered in groupoids / $\mathcal{Y}_{\underline{C}}$

$H(\underline{C}) \rightarrow \mathcal{Y}_{\underline{C}}$ Hurwitz moduli functor
 $H^{rd}(\underline{C}) \rightarrow \mathcal{Y}_{\underline{C}}$ Hurwitz reduced moduli functor

Coarse and fine moduli spaces

$\mathcal{Y} = \mathcal{Y}_{\subseteq}$, Sch

$F: \mathcal{C} \rightarrow \mathcal{Y}$ category fibered in groupoids / \mathcal{Y}

Fine moduli space for F : pair (X, Θ) with

- $X \in \mathcal{Y}$
- $\mathcal{C} \xrightarrow{\Theta} X$ isomorphism of categories fibered in groupoids / \mathcal{Y}

Coarse moduli space (+) for F : pair (X, Θ) with

- $X \in \mathcal{Y}$
- $\mathcal{C} \xrightarrow{\Theta} X$ morphism of categories fibered in groupoids

verifying the three following conditions

(i) For any pair (Y, \mathcal{T}) with $Y \in \mathcal{Y}$ and $\mathcal{C} \xrightarrow{\mathcal{T}} Y$ morphism of categories fibered in groupoids

$$\exists! X \xrightarrow{\varphi} Y \in \text{Hom}_{\mathcal{Y}}(X, Y) \text{ s.t. } \begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{T}} & Y \\ \Theta \downarrow & \swarrow \varphi_0 & \\ \mathcal{C} & \xrightarrow{\Theta} & X \end{array}$$

(ii) $k = \bar{k}$ field, $k \in \mathcal{Y}$ $\mathcal{C}(k) \xrightarrow{\sim} X(k)$ bijection

(iii) (+) k field, $k \in \mathcal{Y}$ $\forall c \in \mathcal{C}(\bar{k}), \sigma \in \Gamma_k$
 $\sigma \cdot \Theta(\bar{k})(c) = \Theta(\bar{k})(\sigma c)$

$$\begin{aligned} &\implies \Theta(\bar{k})^{-1}(X(k)) \\ &\xleftrightarrow{(4.1)} \{c \in \mathcal{C}(\bar{k}) \text{ with f.o.m. } k\} \end{aligned}$$

Hurwitz spaces

$$\psi: H(\Sigma)(\mathbb{C}) \longrightarrow \mathcal{U}_r(\mathbb{C}) \quad \text{ramification divisor map}$$

$$[(p, \alpha)] \longmapsto \underline{E}(p)$$

$$\text{R.E.T.} \implies \psi^{-1}(E) \xrightarrow{(u,1)} \bar{n}_i(\Sigma)$$

$$n_i(\Sigma) := \left\{ g = (g_1, \dots, g_r) \in G^r \mid \begin{array}{l} G = \langle g_1, \dots, g_r \rangle \\ g_1 \dots g_r = 1 \\ g_i \in C_{\sigma(i)}, i=1, \dots, r, \sigma \in \mathcal{S}_r \end{array} \right\}$$

$$\bar{n}_i(\Sigma) := n_i(\Sigma) / \text{Inn}(G) \quad \underline{\text{Nielsen classes}}$$

Thm (Fried-Veeklein 91): The analytic space $H(\Sigma)(\mathbb{C})$ can be endowed in a unique way with an algebraic structure of affine variety $\mathcal{H}(\Sigma) \text{ def } / \mathcal{O}_{\Sigma}$ and such that the analytic cover ψ is induced by a finite étale cover $\psi: \mathcal{H}(\Sigma) \longrightarrow \mathcal{U}_r \text{ def } / \mathcal{O}_2 \subseteq$. Furthermore

(1) $\mathcal{H}(\Sigma)$ is a coarse moduli space (+) for $H(\Sigma)$

(2) Topological description:

$$\begin{array}{l} \text{geometrically irreducible components of } \mathcal{H}(\Sigma) \\ \xrightarrow{(u,1)} \text{connected components of } \mathcal{H}(\Sigma)(\mathbb{C})^{\text{top}} \\ \xrightarrow{(u,1)} \bar{n}_i(\Sigma) / H_r \end{array}$$

$$H_r = \left\langle (c_1, \dots, c_{r-1}) \mid \begin{array}{l} c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}, \quad i=1, \dots, r-2 \\ c_i c_j = c_j c_i, \quad |j-i| > 2 \\ c_1 \dots c_{r-1} c_{r-1} \dots c_1 = 1 \end{array} \right\rangle$$

$$H_r \times \bar{n}_i(\Sigma) \longrightarrow \bar{n}_i(\Sigma)$$

$$(c_i, g = (g_1, \dots, g_r)) \longmapsto c_i \cdot g = (g_1, \dots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \dots, g_r)$$

Reduced Hurwitz spaces

$$\begin{array}{ccc}
 \mathrm{PGL}_2 \times \mathcal{H}(\underline{c}) & \longrightarrow & \mathcal{H}(\underline{c}) \\
 \psi \downarrow & \wr & \downarrow \psi \\
 \mathrm{PGL}_2 \times \mathcal{U}_r & \longrightarrow & \mathcal{U}_r
 \end{array}$$

Thm: The quotients $\mathcal{H}^{\mathrm{rd}}(\underline{c}) := \mathcal{H}(\underline{c}) / \mathrm{PGL}_2$ and $\mathcal{Y}_r := \mathcal{U}_r / \mathrm{PGL}_2$ exist in the category of affine $\mathbb{Q}_{\underline{c}}$ - and \mathbb{Q} -varieties resp.

$$\begin{array}{ccc}
 \mathcal{H}(\underline{c}) & \xrightarrow{\pi} & \mathcal{H}(\underline{c}) / \mathrm{PGL}_2 =: \mathcal{H}^{\mathrm{rd}}(\underline{c}) \\
 \psi \downarrow & \wr & \downarrow \psi^{\mathrm{rd}} \\
 \mathcal{U}_r & \longrightarrow & \mathcal{U}_r / \mathrm{PGL}_2 =: \mathcal{Y}_r
 \end{array}$$

and the reduced cover $\psi^{\mathrm{rd}}: \mathcal{H}^{\mathrm{rd}}(\underline{c}) \rightarrow \mathcal{Y}_r$ is ramified above the closed subvariety corresponding to PGL_2 -orbits of divisors $\underline{c} \in \mathcal{U}_r$ with non-trivial stabilizers. Furthermore

- (1) $\mathcal{H}^{\mathrm{rd}}(\underline{c})$ is a coarse moduli space (+) for $\mathcal{H}^{\mathrm{rd}}(\underline{c})$
- (2) Geometrically irreducible components of $\mathcal{H}^{\mathrm{rd}}(\underline{c})$
 $\xleftarrow{(U,1)}$ Geometrically irreducible components of $\mathcal{H}(\underline{c})$

Rem: $\dim \mathcal{H}^{\mathrm{rd}}(\underline{c}) = \dim \mathcal{H}(\underline{c}) - 3$
 $\implies \bar{\psi}^{\mathrm{rd}}: \bar{\mathcal{H}}^{\mathrm{rd}}(\underline{c}) \rightarrow \bar{\mathcal{Y}}_r \simeq \mathbb{P}^1$ ramified above $0, 1, \infty$,
 $r = 4$ and the ramification is given by the action of

$$\left. \begin{array}{l}
 \gamma_0 = \omega_1 \omega_2 \text{ mod } \mathbb{H}_8 \\
 \gamma_1 = \omega_1 \omega_2 \omega_3 \text{ mod } \mathbb{H}_8 \\
 \gamma_{\infty} = \omega_2 \text{ mod } \mathbb{H}_8
 \end{array} \right\} \text{ on } \bar{\mathcal{H}}^{\mathrm{rd}}(\underline{c})$$

where $\mathbb{H}_8 = \langle (\omega_1 \omega_2 \omega_3)^2, \omega_1 \omega_3^{-1} \rangle < \mathbb{H}_4$

Fine moduli space $U_{r, \mathbb{Z}}$

$$U_{r, \mathbb{Z}} = \mathbb{P}_{\mathbb{Z}}^r \mid \Delta(c_0, \dots, c_r) = \text{disc}(c_0 T^r + \dots + c_r) \neq 0$$

$$\hookrightarrow \mathbb{P}_{\mathbb{Z}}^r$$

$$D_{\text{univ}} = c_0 T^r + \dots + c_r = 0$$

$\hookrightarrow \mathbb{P}_{\mathbb{Z}}^r \times U_{r, \mathbb{Z}}$ universal smooth divisor of degree r

S scheme

$$U_r(S) = \{ D \hookrightarrow \mathbb{P}_S^r \mid D/S \text{ smooth, étale, finite of } \} \\ \text{constant degree } r$$

Prop: The functor isomorphism

$$\begin{array}{ccc}
 U_r(S) & \xrightarrow[\sim]{F_r(S)} & U_{r, \mathbb{Z}}(S) \\
 \begin{array}{ccc}
 D_S \hookrightarrow \mathbb{P}_S^r & \longrightarrow & S \\
 \downarrow \hookrightarrow & \downarrow \hookrightarrow & \downarrow \varphi \\
 D_{\text{univ}} \hookrightarrow \mathbb{P}_{\mathbb{Z}}^r \times U_{r, \mathbb{Z}} & \longrightarrow & U_{r, \mathbb{Z}}
 \end{array} & \longleftarrow & S \xrightarrow{\varphi} U_{r, \mathbb{Z}}
 \end{array}$$

makes $U_{r, \mathbb{Z}}$ a fine moduli space for $U_r \longrightarrow \text{Sch}$

Coarse moduli space $\mathcal{H}_{r, G, Z}$

S scheme

$\mathcal{H}_{r, G}(S) = G$ -isomorphism classes of tamely ramified G -covers of \mathbb{P}^1_S with group G and degree r ramification divisor

$\Psi: \mathcal{H}_{r, G} \longrightarrow U_r$ ramification divisor functor morphism
 $\swarrow \quad \searrow$
 Sch

Tamely ramified G -cover of \mathbb{P}^1_S : pair (f, d) with

cover

- $f: X \longrightarrow \mathbb{P}^1_S$ finite, flat, surjective, separable morphism with X/S smooth, proper with 1-dimensional connected geometric fibers

tamely ramified

- $\exists D \subset \mathbb{P}^1_S \in U_r(S)$ s.t.

(i) $d|_{f^{-1}(\mathbb{P}^1_S \setminus D)} = f^{-1}(\mathbb{P}^1_S \setminus D) \longrightarrow \mathbb{P}^1_S \setminus D$ étale

(ii) For any geometric point $s \in D$, the ramification index of f along D at s is ≥ 1 and prime to the residue characteristic at s .

Galois

- $\text{Aut}(f)$ acts transitively on the geometric fibers

G -

- $d: \text{Aut}(f) \xrightarrow{\sim} G$ group isomorphism

Thm (Newers 98): There exists a smooth scheme $\mathcal{H}_{r, G, \mathbb{Z}} / \mathbb{Z}$ of finite type and a finite étale morphism $\Psi_{r, \mathbb{Z}}: \mathcal{H}_{r, G, \mathbb{Z}} \rightarrow U_{r, \mathbb{Z}}$ s.t.

(1) $\mathcal{H}_{r, G, \mathbb{Z}}$ is a coarse moduli space (+) for $H_{r, G}$
 $\Psi_{r, \mathbb{Z}}$ is a coarse moduli "morphism" (+) for Ψ_r

(2) $\Psi_{r, \mathbb{Z}}: \mathcal{H}_{r, G, \mathbb{Z}} \rightarrow U_{r, \mathbb{Z}}$ has ^{in the obvious meaning} good reduction at all primes $p \nmid |G|$.

In particular, for all primes $p \nmid |G|$

geometrically irreducible components of $\mathcal{H}_{r, G, \mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p$
 $\xleftrightarrow{(1,1)}$ geometrically irreducible components of $\mathcal{H}_{r, G, \mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$

Rem:

(1) $\mathcal{H}_{r, G, \mathbb{Z}} / \mathbb{Z}$ fine moduli space for $H_{r, G}$
 $\iff Z(G) = \{0\}$

(2) Similar statement for $\Psi_r^{rd}: H^{rd}(\mathcal{E}) \rightarrow J_r$
 $\parallel \qquad \parallel$
 $H(\mathcal{E}) / \text{PGL}_2 \qquad U_r / \text{PGL}_2$

Modular compactification I

S scheme

$U_r^\circ(S) := r$ -marked genus 0 S -curve

$\bar{U}_r(S) := r$ -marked genus 0 stable S -curves

$$U_r \hookrightarrow U_r^\circ \hookrightarrow \bar{U}_r$$

Prop: There exist a smooth, integral, proper scheme $\bar{U}_{r,z}/\mathbb{Z}$ and an open immersion $U_{r,z} \hookrightarrow \bar{U}_{r,z}$ s.t.

- (1) $\bar{U}_{r,z}$ is a coarse moduli space for $\bar{U}_r \rightarrow \text{Sch}$
 $U_{r,z} \hookrightarrow \bar{U}_{r,z}$ is a coarse moduli morphism for $U_r \hookrightarrow \bar{U}_r$
- (2) $\Delta_{r,z} = \bar{U}_{r,z} \setminus U_{r,z}$ is a normal crossing divisor

r marked genus 0 S -curve: 3-tuple (x, D, u) with

- $x: X \rightarrow S$ proper, flat morphism with genus 0 1-dimensional smooth geometric fibers.
- $D \subset X$ s.t. D/S smooth, etale, finite of constant degree r
- $u: X \xrightarrow{\sim} \mathbb{P}_S^1$ S -isomorphism

r marked genus 0 stable S -curve: 3-tuple (x, D, λ) with

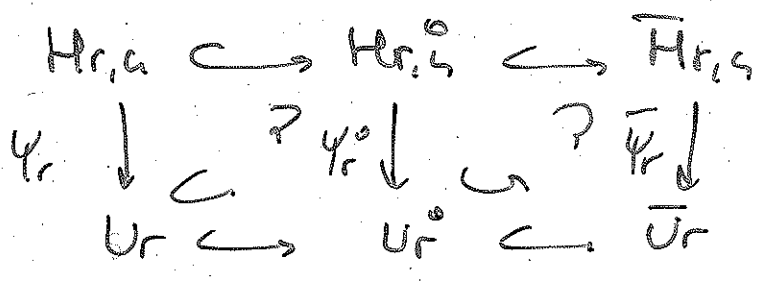
- $x: X \rightarrow S$ proper, flat morphism with genus 0 1-dimensional geometric fibers whose only possible singularities are ordinary double pts
- $D \subset X^{\text{smooth}}$ s.t. D/S smooth, etale, finite of constant degree r
- $|C \cap (\text{supp}(D) \cup X^{\text{sing}})| \geq 3$ for each irr. component C of X
- $\lambda: X \rightarrow \mathbb{P}_S^1$ S -morphism "projective frame"
 (keeping track of u along deformation)

Modular compactification 2

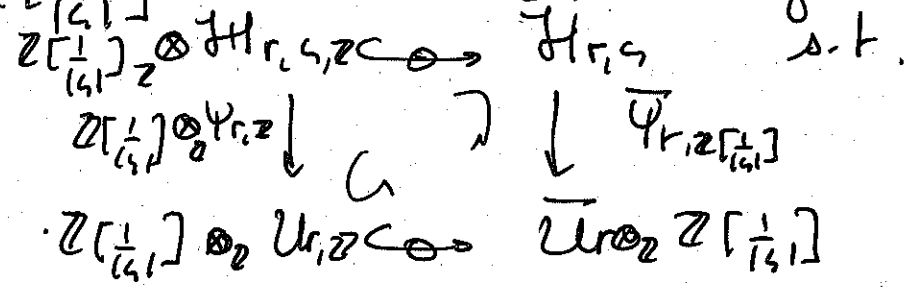
S scheme

$H_{r,c}^0(S) := G$ -isomorphism classes of tame G -covers of π -marked genus 0 S -curves with group G

$\overline{H}_{r,c}(S) := G$ -isomorphism classes of tame admissible G -covers of π -marked genus 0 stable S -curves with group G



Thm (Wewers 98): There exists a smooth, proper, integral scheme $\overline{H}_{r,c} / \mathbb{Z}[\frac{1}{c!}]$ and a commutative diagram of morphisms



- $\overline{\Psi}_{r,c}: \overline{H}_{r,c} \rightarrow \overline{U}_r$ is a finite cover tamely ramified along $\Delta_{r,c} \otimes \mathbb{Z}[\frac{1}{c!}]$
- $\overline{H}_{r,c} \setminus \overline{H}_{r,c} \otimes \mathbb{Z}[\frac{1}{c!}]$ is a normal crossing divisor

and

- (1) $\overline{H}_{r,c}$ is a coarse moduli space (+) for $\overline{H}_{r,c}$
 $\overline{\Psi}_{r,c}$ is a coarse moduli morphism (+) for $\overline{\Psi}_r$
- (2) $\overline{H}_{r,c}$ is normal with normal fibers $\overline{H}_{r,c} \otimes_{\mathbb{Z}[\frac{1}{c!}]} \mathbb{Q}$
 and $\overline{H}_{r,c} \otimes_{\mathbb{Z}[\frac{1}{c!}]} \mathbb{F}_p, p \nmid c!$ prime.

Modular towers

data:

$$1 \rightarrow P \rightarrow \tilde{G} \xrightarrow{\tilde{\varphi}} G \rightarrow 1$$

\downarrow \swarrow \searrow \rightarrow \swarrow
 pro-p \checkmark finite
 $\text{rank}(P) < +\infty$

$$P \rightarrow \mathbb{Z}_p$$

$\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_r)$ r -tuple of non trivial p^r conjugacy classes of G

\rightsquigarrow (Reduced) MT

Abelianized (reduced) MT

$$\begin{array}{ccc}
 \underline{H}^{(red)}(\tilde{\varphi}, \underline{\epsilon}) & \longrightarrow & \underline{H}^{(red), ab}(\tilde{\varphi}, \underline{\epsilon}) \\
 (\tilde{G}_{n+1}, \tilde{\epsilon}_{n+1}) \underline{H}_{n+1}^{(red)}(\tilde{\varphi}, \underline{\epsilon}) & \longrightarrow & \underline{H}_{n+1}^{(red), ab}(\tilde{\varphi}, \underline{\epsilon}) \quad (\bar{G}_{n+1}, \bar{\epsilon}_{n+1}) \\
 \downarrow & \searrow & \downarrow \\
 (\tilde{G}_n, \tilde{\epsilon}_n) \underline{H}_n^{(red)}(\tilde{\varphi}, \underline{\epsilon}) & \longrightarrow & \underline{H}_n^{(red), ab}(\tilde{\varphi}, \underline{\epsilon}) \quad (\bar{G}_n, \bar{\epsilon}_n) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

①

$$\begin{array}{ccccccc}
 1 & \rightarrow & P & \rightarrow & \tilde{G} & \xrightarrow{\tilde{\varphi}} & G \rightarrow 1 \\
 & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\
 1 & \rightarrow & P^{ab} & \rightarrow & \tilde{G} & \xrightarrow{\tilde{\varphi}} & G \rightarrow 1
 \end{array}$$

Frattini series:

- P : $P_0 := P, P_1 := P_0^p [P_0, P_0], \dots, P_{n+1} := P_n^p [P_n, P_n], \dots$

- P^{ab} : $(P^{ab})_n = P^n P^{ab}, n \geq 0$

$\rightsquigarrow \tilde{G} = \varprojlim (\tilde{G}_n := \tilde{G}/P_n)$

$\bar{G} = \varprojlim (\bar{G}_n := G/P_n P^{ab})$

Rem: ① $\text{rank}(P) < +\infty \implies |\tilde{C}_n|, |\bar{C}_n| < +\infty, n \geq 0$

$P \twoheadrightarrow \mathbb{Z}_p \implies \exp(P/P_n), \exp(P^{ab}/P_n P^{ab}) = p^n, n \geq 0$

② finite p -group of exponent p^n

$$\begin{array}{ccccccc} 1 & \rightarrow & P/P_n & \rightarrow & \tilde{C}_n & \rightarrow & C_n \rightarrow 1 \\ & & \downarrow \wr & & \downarrow \wr & & \parallel \\ 1 & \rightarrow & P^{ab}/P_n P^{ab} & \rightarrow & \bar{C}_n & \rightarrow & C_n \rightarrow 1 \end{array}$$

finite commutative p -group of exponent p^n

②

Lemma (Schur-Zassenhaus): $\forall C$ p' -conj. class of G

$\exists! \tilde{C}_n$ p' -conj. class of \tilde{C}_n above C s.t. $o(C) = o(\tilde{C}_n) = o(\bar{C}_n), n \geq 0$
 $\exists! \bar{C}_n$ " " \bar{C}_n

$\tilde{\varphi}, \underline{\varepsilon} = (C_1, \dots, C_r)$ r -tuple of p' -conj. classes of G

$\rightsquigarrow (\tilde{C}_{n+1}, \tilde{\varepsilon}_{n+1}) \twoheadrightarrow (\tilde{C}_n, \tilde{\varepsilon}_n) \rightsquigarrow \text{MT } \underline{\mathcal{H}}^{(rd)}(\tilde{\varphi}, \underline{\varepsilon})$

$\downarrow \wr \quad \downarrow \wr$
 $(\bar{C}_{n+1}, \bar{\varepsilon}_{n+1}) \twoheadrightarrow (\bar{C}_n, \bar{\varepsilon}_n) \rightsquigarrow \text{abelianized MT } \underline{\mathcal{H}}^{(ab, rd)}(\tilde{\varphi}, \underline{\varepsilon})$

Example (Fried's MT):

G p -perfect finite group ($p \nmid |G|, G \not\cong \mathbb{Z}/p$)

$1 \rightarrow P \rightarrow p\tilde{G} \rightarrow G \rightarrow 1$ universal p -Frattini cover of G

$\downarrow \wr \quad \downarrow \wr \quad \parallel$
 $1 \rightarrow P^{ab} \rightarrow p\bar{G} \rightarrow G \rightarrow 1$ universal p -Frattini cover with ab. kernel of G

notation: $\underline{\mathcal{H}}^{(rd)}(p, G, \underline{\varepsilon}), \underline{\mathcal{H}}^{(ab, rd)}(p, G, \underline{\varepsilon})$

$G = D_{2p}$

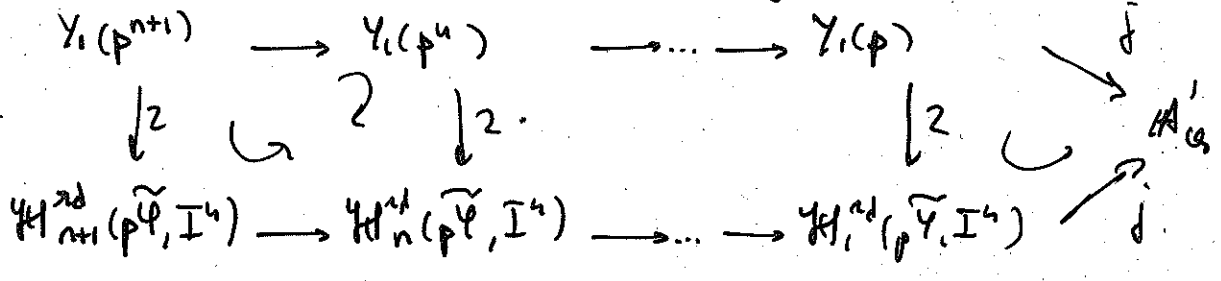
$1 \rightarrow \mathbb{Z}_p \rightarrow D_{2p} \xrightarrow[p\tilde{\varphi} = p\bar{\varphi}]{} D_{2p} \rightarrow 1 \quad \tilde{C}_n = \bar{C}_n = D_{2p^n}, n \geq 0$

In general $p\tilde{G}, p\bar{G}$ are very "mysterious"

Main Conjecture

Prop (Dèbes; Fried 94): $1 \rightarrow \mathbb{Z}_p \rightarrow D_{2p^\infty} \xrightarrow{\rho_{\tilde{\Psi}}} D_{2p} \rightarrow 1$

then we have commutative diagrams of \mathbb{Q} -isomorphisms



Thm (Merel 96): $d \geq 1$ integer $\exists N := N(d)$ s.t.

$$\gamma_1(p^n)^{(d)}(\mathbb{Q}) = \emptyset, \quad n \geq N$$

($X^{(d)} = \underbrace{X \times \dots \times X}_d / \sigma_d$ d th symmetric product)

Conj. (MT Conjecture): $\tilde{\Psi}, \subseteq$ $d \geq 1$ integer $\exists N := N(\tilde{\Psi}, \subseteq, d)$ s.t.
 $\text{Hl}_n^{rd, ab}(\tilde{\Psi}, \subseteq)^{(d)}(\mathbb{Q}) = \emptyset, \quad n \geq N$

Conj. (Strong torsion): $g, d \geq 1$ integers $\exists N := N(g, d)$ s.t. if $\mathcal{A}(g, d, n)$ denotes the set of all abelian varieties A/k with

- $[k:\mathbb{Q}] \leq d$
- $\dim(A) = g$
- $A(k) \supset \mathbb{Z}/n$

then $\mathcal{A}(g, d, n) = \emptyset, \quad n \geq N$

Prop (Ca 05): The strong torsion conjecture implies the MT conjecture.

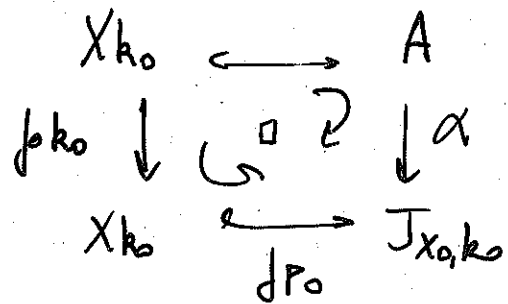
Basic material for a bridge

(reduced) \mathbb{A}^1 \longrightarrow (reduced) abelianized \mathbb{A}^1

$\xrightarrow{\quad}$
 $\xrightarrow{\quad}$
 abelian varieties

Thm 1 (Serre, Lang)

X_0/k smooth, proj, geometrically irreducible curve def $/k$
 $f_0: X \longrightarrow X_0$ etale G -cover def $/k$ with group \mathbb{Z}/n
 k_0/k finite ext. s.t. $X_0(k_0) \neq \emptyset$, $P \in X_0(k_0)$



with $A \in \mathcal{A}(g_{X_0}, [k_0:\mathbb{Q}], n)$

Thm 2 (Weil's pairing) $\text{char}(k) = 0$

Let $\alpha: A \longrightarrow B$ etale degree d isogeny def $/k$
 $\uparrow \quad \uparrow$
 abelian varieties

Then there exists a perfect pairing of T^k -modules

$$\left. \begin{array}{ccc}
 \langle, \rangle: (\ker \alpha)(\bar{k}) \times (\ker \alpha^\vee)(\bar{k}) & \longrightarrow & \mathbb{Z}/d(\mathbb{Z}) \\
 (T, T^\vee) & \longmapsto & \langle T, T^\vee \rangle
 \end{array} \right)$$

Projective systems of rational points

Rem: $\forall \tilde{\varphi}, \underline{c} = (c_1, c_2, c_3, c_4), \forall k/\mathbb{Q}$ n.f.

- \Downarrow (1) $\varprojlim \mathcal{H}_n^{ab, rd}(\tilde{\varphi}, \underline{c})(k) = \emptyset$
 (2) All the geo. ined. components of $\mathcal{H}_n^{ab, rd}(\tilde{\varphi}, \underline{c})$
 have genus ≥ 2 , $n \gg 0$

Then, by Faltings' theorem, the MNT Conj. holds for $\pi = k, d = 1$
 $\text{proj } P \rightarrow \mathbb{Z}_p$ finite

Thm (Proj GOS): $\forall L \rightarrow P \rightarrow \tilde{G} \rightarrow G \rightarrow 1, \forall k/\mathbb{Q}$ n.f.,
 $\forall K/k(T)$ finite ext. π_{reg}/k

- (i) There is no Galois ext. $L/K.T$ with group \tilde{G} and form k
- (ii) There is no unramified Galois ext. $L/K.T$ with group \tilde{G} and form k^{cycl} (cyclotomic closure of k)

Special case ($K = k(T)$)

(i) $\forall (G_{n+1} \rightarrow G_n)_{n \geq 0}$ proj. syst. of finite groups s.t.

$$\tilde{G} = \varprojlim G_n, \forall n \geq 0 \text{ with } \mathcal{H}_n := \prod_{r \geq 3} \mathcal{H}_{G_n, r}$$

Then $\varprojlim \mathcal{H}_n^{rd}(k) = \emptyset$

(ii) $\forall \underline{c} = (c_1, \dots, c_r)$ r -tuple of p' -conj. classes of G
 $\varprojlim \mathcal{H}_n^{rd}(\tilde{\varphi}, \underline{c})(k^{\text{cycl}}) = \emptyset$

Comments

Comment 1/: gap (RIGP/Q) \leftarrow profinite (RIGP/Q)

a) profinite (RIGP/Q) does not hold

counter-examples: \mathbb{Z}_p (Sene)

D_{2p^∞} (Fried)

$p\tilde{G}$

b) Patching methods / $\mathbb{Q}(CT)$

$\implies \forall G$ finite group $\exists K/\mathbb{Q}(CT)$ finite ext reg / \mathbb{Q}
~~Débes - Deschamps~~ s.t. G is the Galois group of a Galois ext.
 of K reg / k .

Comment 2/: Reformulation of the MT Conjecture.

Thm (Fried-Kopeliovich 97, Ca. 05)

MT Conj for Fried's MT (resp. Fried's abelianized MT)

$\iff \forall \pi \geq 3, \forall d \geq 1$ only finitely many of the
 \tilde{G}_n (resp. \bar{G}_n) can be realized as Galois groups
 of a finite ext of $\mathbb{Q}(CT)$ with form of degree $\leq d$
 and less than π ramification points.

\implies consider all MT towers

Conj. (not serious): $1 \rightarrow P \rightarrow p\tilde{G} \xrightarrow{+P} G \rightarrow 1$ G p -perfect
 $\forall n \geq 0 \exists r_n \geq 3 \exists \subseteq = (G_1, \dots, G_{r_n})$ r_n -tuple of
 p^2 -conj. clans of G s.t. $H^n(p.c., \subseteq)(\mathbb{Q}) \neq \emptyset$

Conj. (dihedral/ k): $\forall u \geq 2 \exists K_u/k(CT)$ Galois ext.
 reg / k with group D_{2u} and only order 2 inertia groups.

Dihedral conjecture / \mathbb{Q}^{ab}

Prop: $n \geq 3$. Assume there exists a genus g hyperelliptic curve $C: X \rightarrow \mathbb{P}^1_k$ with a k -rational Weierstrass point $P \in X(k)$. Then

- (1) $J_X(k) \supset \mathbb{Z}/n \implies \exists K/k(S_n)(T)$ Galois ext. reg. / $k(S_n)$ with group D_{2n} and i.c.i. I^{2g+2}
- (2) $J_X(k) \supset \mathbb{Z}/n \implies \exists K/k(T)$ Galois ext. reg. / k with group D_{2n} and i.c.i. I^{2g+2}

Cor: (1) The dihedral conj. holds for $k = \mathbb{Q}^{ab}$

(2) The dihedral conj. holds for $k = \mathbb{Q}$ and $n = 1, \dots, 12, 23, 29$ and 35 .

Rem: $\forall d, g \geq 1$ if $\exists H_n^{abs, rd} (P, D_{2P}, I^{2g+2})^{(cd)}(\mathbb{Q}) = \emptyset, n \gg 0$
 then the strong torsion conj. for jacobian varieties of genus g hyperelliptic curves holds.

Henselian valued fields of char 0

- (1) $\forall k$ ample field of char 0 $\forall n \geq 2 \exists E_n/k$ elliptic curve s.t. $E_n(\bar{k}) \supset \mathbb{Z}/n$
- (2) $\forall k$ h.v.f. of char 0 $\exists E/k$ elliptic curve s.t. $E(k^{cycl})$ tors is infinite
- (3) $\forall k$ h.v.f. of char 0 $\exists E/k$ elliptic curve s.t. the k -isogeny class of E contains infinitely many k -isomorphism classes of elliptic curves.

Theorems for abelian varieties

Thm (Faltings' isogeny) k/\mathbb{Q} n.f. A/k abelian variety

Then the k -isogeny class of A contains only finitely many k -isomorphism classes

Thm (Ribet 81) k/\mathbb{Q} n.f. A/k abelian variety

Then $|A(k^{\text{cycl}})|_{\text{tors}} < +\infty$

Thm (Coxeter-Tate 66) $\forall k$ complete valued field, $\forall n \geq 1$

$\exists E_n/k$ elliptic curve s.t. $E_n(k) \cong \mathbb{Z}/n$

Thm (Mazur 55) $\forall k$ h.v. field $\forall A/k$ abelian variety

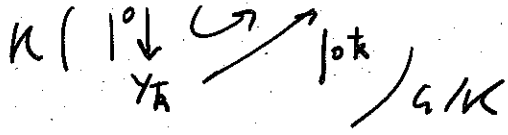
$|A(k)_{\text{tors}}| < +\infty$

Lifting lemmas

lemma 1/: $\forall G, \subseteq = (G_1, \dots, G_r), \forall f^{rd} \in \mathcal{H}^{rd}(\subseteq)(k) \exists p \in \mathcal{H}(\subseteq)(\bar{k})$
 s.t. $\pi(p) = f^{rd}$ and $[k(p):k] \leq c(r)\pi!$

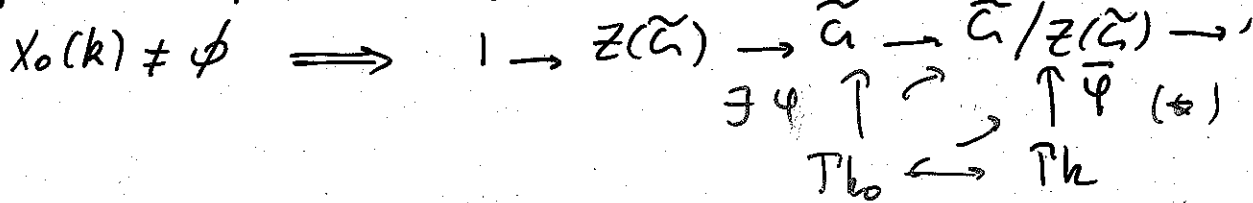
lemma 2/: $\forall ((G_{n+1}, \subseteq_{n+1}) \rightarrow (G_n, \subseteq_n))_{n \geq 0}, \forall f^{rd} \in \varinjlim \mathcal{H}^{rd}(\subseteq_n)(k)$
 $\exists k_0/k$ finite ext. (depending on f^{rd}), $\exists p \in \varinjlim \mathcal{H}(\subseteq_n)(k_0)$
 s.t. $\pi(p) = f^{rd}$

lemma 1/: $K \triangleleft G$ finite groups
 X/k smooth, geo. med., proj. curve
 $f_0: Y \rightarrow X$ G -cov. def / k with group G/K
 $Z \rightarrow X_{\bar{k}}$ G -cov. def / \bar{k} with group G



f has form k as G -cov $\implies f^0$ has form k as G -cov

lemma 2/ (profinite coh. obstruction): X_0/k smooth, geo. med.,
 proj. curve, $(X_n \xrightarrow{f_n} X_0_{\bar{k}})_{n \geq 0}$ proj. syst. of G -covers with
 group G_n def / \bar{k} with f.o.m. k , $\tilde{G}_i = \varprojlim G_n$



k_0/k f.o.d. for $(f_n)_{n \geq 0}$ as proj. syst. of G -covers

