Néron models, purity, and Shimura varieties

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ABSTRACT. We show how different purity properties can be used to obtain new examples of Néron models. As a direct application, we use Shimura varieties of Hodge type to get in arbitrary mixed characteristic, the very first examples of general nature of projective Néron models whose generic fibres are not finite schemes over abelian varieties.

Key words: abelian schemes, Mumford–Tate groups, Shimura varieties, and Néron models.

1. NÉRON MODELS

Let D be an integral Dedekind domain. Let K be the field of fractions of D. Let X_K be a smooth and separated K-scheme of finite type.

1.1. DEFINITION. A Néron model of X_K over D is a smooth, separated D-scheme of finite type whose generic fibre is X_K and which has the following universal property:

(NMUP) for each smooth D-scheme Y and each K-morphism $m_K : Y_K \to X_K$, there exists a unique D-morphism $m : Y \to X$ that extends m_K.

1.2. THEOREM (Néron 1964). Each abelian variety over K has a Néron model over D.

1.3. Sentence from the book *Néron models* [BLR, Ch. I, p. 15]. "Although Néron models have been defined within the setting of schemes, their importance seems to be restricted to group schemes or, more generally, to torsors under group schemes ...".

1.4. OPERATIONS with Néron models.

(OP1) the Néron models are stable under *products*;

(OP2) a smooth scheme Z over D which is *finite* over a Néron model X, is itself a Néron model.

1.5. EXAMPLES.

(a) Each smooth, projective curve over D is a Néron model of its generic fibre.

(b) Each étale cover of a Néron model is itself a Néron model.

(c) Starting from the Néron models of abelian varieties, using the operations (OP1) and (OP2), we get Néron models of connected varieties X_K over K which admit *finite morphisms* into abelian varieties (i.e., whose albanese varieties have the same dimension as them).

1.6. CRITERION. Suppose that X is a smooth, separated scheme over D which is of finite type as well as a moduli space of some *class of objects* O. Then X is a Néron model of its generic fibre if and only if the following two properties hold:

(a) *(the purity property)* for each smooth scheme Y over D and each open subscheme U of Y which contains Y_K and which has the property that the codimension of $Y \setminus U$ in Y is at least two, all objects over U extend uniquely to objects over Y.

(b) *(the good reduction property)* for each discrete valuation ring V which is the localization of a smooth D-algebra, all objects over the field of fractions of V extend uniquely to objects over V.

1.6.1. REMARK. If Y is a proper scheme over D, then 1.6 (b) holds. On the other hand, if 1.6 (b) holds, then Y is not necessarily a proper scheme over D.

1.7. QUESTION: What should one search for, in order to construct new classes of Néron models?

FIRST POSSIBLE ANSWER: good classes of objects O whose moduli spaces are smooth, *projective*, and which satisfy the *purity property*.

1.8. QUESTION: What if Sentence 1.3 is partially true?

ANSWER: look for classes of objects O which *involve group schemes*. Milne's insight (1992): use *abelian schemes*.

2. THE PURITY PART (i.e., PURITY RESULTS)

Let Y be a smooth and separated scheme over D. Let U be an open subscheme of Y that contains Y_K and such that the complement $Y \setminus U$ has codimension at least two in Y.

2.1. ZARISKI–NAGATA purity result (ancient). Each étale cover of U extends uniquely to an étale cover of Y.

CLASS OF OBJECTS: étale covers or (even better) finite étale group schemes.

PROBLEM: not much communications between Galois specialists and moduli spaces specialists.

HOPES: the cycle of three conferences that has just started.

HERE IS ANOTHER REASON WHY OUR CONFERENCE IS REALLY GREAT: it brings great communications between the last two types of specialists.

2.2. COLLIOT-THÉLÈNE and SANSUC purity result (1979). Suppose that Y has dimension two. Then each reductive group scheme over U extends uniquely to a reductive group scheme over Y. [Strictly speaking, their works implies the purity result stated)].

CLASS OF OBJECTS: reductive group schemes.

PROBLEMS: (a) dimension two assumption is NEEDED but way too strong for our purposes and (b) no relevant moduli spaces for reductive group schemes.

HOPES: none.

2.3. MORET-BAILLY purity result (1985). Each smooth, projective curve of genus at least two over U extends to a smooth, projective curve over Y.

CLASS OF OBJECTS: Jacobians.

PROBLEM: their moduli spaces are not projective.

HOPES: not much, as not good moduli subspaces identified.

2.4. FALTINGS–VASIU purity result (1990–2004). Suppose that K has characteristic zero and that for each local ring V of D there exists a prime p such that the following two conditions hold:

(i) V has mixed characteristic $(0, p)$, and

(ii) V has an index of ramification which is at most $\max\{1, p-2\}$.

Then each abelian scheme over U extends uniquely to an abelian scheme over Y.

CLASS OF OBJECTS: (polarized) abelian schemes.

PROBLEM: their moduli spaces are not projective.

HOPES: yes, as they have many moduli subspaces which are projective (Milne's insight).

3. THE GOOD REDUCTION PART (i.e., MUMFORD–TATE GROUPS and MORITA CONJECTURE)

We begin with some notations.

3.1. NOTATIONS.

(a) Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m\mathbb{C}}$ be the two dimensional torus over $\mathbb{R}$ with the property that $\mathbb{S}(\mathbb{R})$ is the multiplicative group of non-zero complex numbers.

(b) Let E be a number field. Let O_E be the ring of integers of E. We fix an embedding $i_E : E \hookrightarrow \mathbb{C}$.

(c) Let A be an abelian variety over E. Let

$$L := H_1(A(\mathbb{C}), \mathbb{Z})$$

be the first Betti homology group of the complex manifold $A(\mathbb{C})$ with coefficients in $\mathbb{Z}$. Let $W := L \otimes_{\mathbb{Z}} \mathbb{Q}$. Let

$$h: \mathbb{S} \to GL_{L\otimes_{\mathbb{Z}}\mathbb{R}}$$

be the homomorphism that defines the Hodge $\mathbb{Z}$–structure on W. Over $\mathbb{C}$ we have the Hodge decomposition

$$L \otimes_{\mathbb{Z}} \mathbb{C} = F^{-1,0} \oplus F^{0,-1}.$$

(d) Let G be the *Mumford–Tate group* of $A_{\mathbb{C}}$. We recall that G is a reductive group over $\mathbb{Q}$ and that G is the smallest subgroup of GL_W with the property that h factors through $G_{\mathbb{R}}$. Let

$$G^{\mathrm{ad}}$$

be the adjoint group of G i.e., the quotient of G by its center.

We recall that a reductive group over a field is an affine, connected group over that field which has no normal subgroup isomorphic to $\mathbb{G}_a^s$ for some $s \in \mathbb{N}$.

3.2. MORITA CONJECTURE (1975). If the $\mathbb{Q}$–rank of G^{ad} is zero (i.e., if the adjoint group G^{ad} has no subgroup isomorphic to $\mathbb{G}_m$), then there exists a finite field extension E_1 of E such that A_{E_1} extends to an abelian scheme over O_{E_1} (i.e., A_{E_1} has good reduction everywhere).

bigskip

3.2.1. REMARK. A Theorem of Borel and Harish-Chandra says that the $\mathbb{Q}$–rank of G^{ad} is 0 if and only if the Shimura variety attached to A (to be detailed in Section 4) is projective tower.

3.2.2. REMARK. The *philosophy* of Morita Conjecture is: a *good* moduli space $\mathcal{M}$ over D is a proper scheme over D if and only if $\mathcal{M}_K$ is a proper scheme over K (here D and K are as in Section 1).

3.3. DEFINITION. We say the abelian variety A has *compact factors*, if for each simple factor H of G^{ad} there exists a simple factor of $H_{\mathbb{R}}$ which is compact.

3.4. ON the CLASSIFICATION of G^{ad}. We consider the case in which G^{ad} is a simple group over $\mathbb{Q}$. According to Satake–Deligne classification, there exist five possible types. In Deligne's notations these types are: A_n, B_n, C_n, $D_n^{\mathbb{H}}$, and $D_n^{\mathbb{R}}$.

TYPE A$_n$. We have a product decomposition $G_{\mathbb{R}}^{\mathrm{ad}} = \prod_{i\in I} PSU(a_i, n+1-a_i)$.

CONDITIONS: for at least one $i \in I$ we have $a_i \in \{1, \ldots, n\}$.

OUR CASES: if one a_i is either zero or $n+1$, then A has compact factors.

TYPE B$_n$. We have a product decomposition $G_{\mathbb{R}}^{\mathrm{ad}} = \prod_{i\in I} SO(a_i, 2n+1-a_i)$.

CONDITIONS: for all $i \in I$ we have $a_i \in \{0, 2\}$ and for at least one $i \in I$ we have $a_i = 2$.

OUR CASES: if one a_i is zero, then A has compact factors.

EXAMPLE: Let F be a *totally real* number field whose degree d is at least two. Let $c \in \{1, \ldots, d-1\}$. Let $a_1, a_2 \in F$ be non-zero elements such that the following two conditions hold:

(i) for precisely c embeddings $\iota : F \hookrightarrow \mathbb{R}$, both elements $\iota(a_1)$ and $\iota(a_2)$ are positive, and

(ii) for precisely $d-c$ embeddings $\iota : F \hookrightarrow \mathbb{R}$, both elements $\iota(a_1)$ and $\iota(a_2)$ are negative.

Let Q be the group over F that fixes the quadratic form $a_1x_1^2 + a_2x_2^2 + x_3^3 + \cdots + x_{2n+1}^2$ (on F^{2n+1}). Then if $G = \mathrm{Res}_{F/\mathbb{Q}}Q$ (i.e., if $G(\mathbb{Q}) = Q(F)$), then A has compact factors.

TYPE C$_n$. $\cdots$

TYPE D$_n^{\mathbb{H}}$. $\cdots$

TYPE D$_n^{\mathbb{R}}$. $\cdots$

3.5. BASIC THEOREM (Vasiu, submitted). Suppose that A has compact factors. Then there exists a finite field extension E_1 of E such that A_{E_1} extends to an abelian scheme over O_{E_1}.

3.6. PREVIOUS WORKS.

(a) Morita (1975), Kottwitz (1992), and Paugam (2004).

(b) The first two works pertain to abelian varieties of PEL type. We recall that A is of PEL type, if $A_{\mathbb{C}}$ has a polarization λ such that the derived group of G is also the derived group of the intersection of $GSp(W,\psi)$ with the double centralizer of G in GL_W (here ψ is the alternating form on W defined by λ and PEL stands for polarization, endomorphisms, and level structures).

(c) In the PEL type case, the types of the simple factors of G^{ad} are: (i) A_n type, (ii) totally non-compact C_n type, (iii) totally non-compact $D_n^{\mathbb{H}}$ type, and (iv) totally non-compact, inner $D_4^{\mathbb{R}}$ type (often (iv) is considered as part of (iii)).

(d) Suppose A is such that there exists a prime $p \in \mathbb{N}$ with the property that the group $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is anisotropic (i.e., its $\mathbb{Q}_p$-rank is zero). Then the Morita conjecture holds for A. REMARK: this example is of A_n type.

(e) The work of Paugam pertains to special cases when there exists a good prime $p \in \mathbb{N}$ for which a certain combinatorial condition on the natural action of $Gal(\mathbb{Q}_p)$ on the set of simple factors of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ holds. Such good primes p exist only if A has compact factors and each simple factor H of G^{ad} is "simple enough" (like when $H_{\mathbb{R}}$ has only one simple, non-compact factor). Good primes p do not exist if there exists one simple factor H of G^{ad} such that the group $H_{\mathbb{R}}$: either (i) has more simple, non-compact factors than simple, compact factors or (ii) it is a Weil restriction $\mathrm{Res}_{\tilde{F}/\mathbb{Q}}\tilde{Q}$, where $\tilde{Q}$ is an absolutely simple, adjoint group over some *arithmetically complicated* totally real number field $\tilde{F}$. Thus Paugam's results are particular cases of our Basic Theorem.

3.7. ON the PROOFS.

(a) Morita and Kottwitz: use the Shimura moduli spaces associated to A to show that A can not have semiabelian reductions.

(b) Paugam: relies on Grothendieck's criterion (SGA7) of good and semistable reductions of A.

(c) Vasiu: the starting idea is as in (a). But this idea is supplemented by an *elongation trick*. To explain the trick, we will assume that G^{ad} is a simple group over $\mathbb{Q}$. It is known that there exists a totally real number field F such that

$$G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} Q,$$

with Q as an absolutely simple, adjoint group over F. We decompose $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I} F_i$ as a product of p-adic fields.

We pick an $i_0 \in I$ such that F_{i_0} "keeps track" of a compact, simple factor of $G_{\mathbb{R}} = \prod_{j: F \hookrightarrow \mathbb{R}} Q \times_{F\ j} \mathbb{R}$ (i.e., under a suitable identification $\mathrm{Hom}(F, \mathbb{R}) = \mathrm{Hom}(F, \overline{\mathbb{Q}_p}) = \cup_{i \in I} \mathrm{Hom}(F_i, \overline{\mathbb{Q}_p})$, there exists $j_0 \in \mathrm{Hom}(F, \mathbb{R})$ for which the group $Q \times_{F\ j_0} \mathbb{R}$ is compact and to which corresponds an element of the set $\mathrm{Hom}(F_{i_0}, \overline{\mathbb{Q}_p})$). Let F_1 be a totally real number field that contains F and for each we have a similar decomposition $F_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p = (\prod_{i \in I, i \neq i_0} F_i) \times F_{1 i_0}$.

The elongation trick refers to the process of replacing *the role* of (G, Q) with that one of (G_1, Q_{F_1}), where $G_1^{ad} := \mathrm{Res}_{F_1/\mathbb{Q}} Q_{F_1}$. Under this replacement, A gets replacement by another abelian variety $\tilde{A}$ over some finite field extension $\tilde{E}$ of E whose Mumford–Tate group $\tilde{G}$ has the following two properties:

- its adjoint is G^{ad} and
- it is naturally a subgroup of another Mumford–Tate group G_1 whose adjoint is G_1^{ad}.

If $\tilde{A}$ has semiabelian reduction $\tilde{A}_v$ with respect to a prime v of $\tilde{E}$, then the rank r_v of the toric part $\tilde{T}_v$ of $\tilde{A}_v$ obeys the following two rules:

(i) it is either zero or at least equal to $[F_1 : \mathbb{Q}]$ (i.e., the field F_1 acts on the group of characters of $\tilde{T}_v$);

(ii) it depends only on the factor $\prod_{i\in I, i\neq i_0} F_i$ of either $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ or $F_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and thus it is bounded from above in terms of F alone (i.e., independently of F_1).

Thus by taking $[F_1 : F] = [F_{1i_0} : F_{i_0}] >> 0$, one gets that $r_v = 0$. Thus $\tilde{A}_v$ is an abelian variety. This implies that $A_{\tilde{E}}$ has good reduction with respect to v.

4. A NEW and LARGE CLASS of NÉRON MODELS

Let the notations $\mathbb{S}$, E, A, L, W, G, $h : \mathbb{S} \to G_{\mathbb{R}}$ be as in Section 3. Let $g := \dim(A)$. Let $\mathcal{X}$ be the $G(\mathbb{R})$-conjugacy class of h. The pair $(G, \mathcal{X})$ is called a *Shimura pair*.

We review some standard properties of Shimura varieties which allow us to state our new results on Néron models.

4.1. SIEGEL MODULAR PAIR. Suppose A has a principal polarization λ. Let $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$ be the perfect, alternating form on L induced by λ. Let $\mathcal{S}$ be the $GSp(W, \psi)(\mathbb{R})$-conjugacy class of the homomorphism $\mathbb{S} \to GSp(W, \psi)_{\mathbb{R}}$ defined by h; it is a double copy of the Siegel domain of genus g. The pair $(GSp(W, \psi), \mathcal{S})$ is called a Siegel modular pair.

4.2. ON $\mathcal{X}$. Let C_h be the centralizer of Im(h) in $G_{\mathbb{R}}$. We have $\mathcal{X} = G(\mathbb{R})/C_h(\mathbb{R})$. It turns out that the existence of the polarization λ implies that both $\mathcal{X}$ and $\mathcal{S}$ are hermitian symmetric domains.

4.3. THE ADÈLIC CONSTRUCTION. Let $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles of $\mathbb{Q}$. For K a compact, open subgroup of $G(\mathbb{A}_f)$, let

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}/K := G(\mathbb{Q})\backslash \mathcal{X} \times G(\mathbb{A}_f)/K;$$

it is a finite disjoint union of quotients of a connected component of the hermitian symmetric $\mathcal{X}$ by arithmetic subgroups of $G(\mathbb{Q})$. A theorem of Baily and Borel says that $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}/K$ has a canonical structure of a normal, quasi-projective complex scheme which is smooth if K is small enough. Thus the projective limit (*i.e., the Shimura tower*)

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}} := \mathrm{proj.lim.}_{K \leqslant G(\mathbb{A}_f),\ compact\ +\ open} \mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}/K$$

of all the normal, quasi-projective complex schemes $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}/K$'s, has a canonical structure of a regular complex scheme. One calls $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}$ as the *complex Shimura variety* attached to the Shimura pair $(G, \mathcal{X})$.

4.3.1. EXAMPLE. If A is an elliptic curve such that $A_{\mathbb{C}}$ does not have complex multiplication, then $G = GL_2 = GSp(W, \psi)$. Moreover, the schemes $\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}}/K$ are finite unions of complex modular curves of different type of levels.

4.4. THE REFLEX FIELD. We have $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ and thus $h : \mathbb{S} \to G_{\mathbb{R}}$ defines a $G(\mathbb{C})$-conjugacy class ν of homomorphisms $\mathbb{G}_m \times \mathbb{G}_m \to G_{\mathbb{C}}$. This conjugacy class is defined over $\overline{\mathbb{Q}}$ and thus we get a $G(\overline{\mathbb{Q}})$-conjugacy class $\nu_{\overline{\mathbb{Q}}}$ of homomorphisms $\mathbb{G}_m \times \mathbb{G}_m \to G_{\overline{\mathbb{Q}}}$. The *reflex field* $E(G, \mathcal{X})$ of $(G, \mathcal{X})$ is the number field fixed by the open subgroup of $Gal(\mathbb{Q})$ that fixes $\nu_{\overline{\mathbb{Q}}}$.

4.4.1. EXAMPLE. If $G = GSp(W, \psi)$, then $E(G, \mathcal{X}) = \mathbb{Q}$. On the other hand, if A has compcat factors, then $E(G, \mathcal{X})$ is a CM field which is different from $\mathbb{Q}$.

4.5. MUMFORD MODULI SCHEMES (Geometric Invariant Theory, 1965). Let $N \geq 3$. Let $\mathcal{A}_{g,1,N}$ be the moduli scheme over $\mathbb{Z}[\frac{1}{N}]$ that parameterizes principally polarized abelian schemes which are of relative dimension g and which are equipped with a level-N symplectic similitude structure. Let

$$K(N) := \{h \in GSp(L, \psi)(\widehat{\mathbb{Z}}) | h \text{ mod } N \text{ is the identity}\}.$$

Let

$$K_A(N) := K(N) \cap G(\mathbb{A}_f).$$

The intersection $K_A(N) \cap G(\mathbb{Q})$ is an *arithmetic subgroup* of $G(\mathbb{Q})$ which is a more general version of the classical Γ_N arithmetic subgroups.

It is known that we can identify

$$\mathrm{Sh}(GSp(W,\psi),\mathcal{S})_{\mathbb{C}}/K(N)=\mathcal{A}_{g,1,N_{\mathbb{C}}}.$$

We have a finite (functorial) morphism of complex schemes

$$f(N)_{\mathbb{C}}:\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}/K_A(N)\to \mathrm{Sh}(GSp(W,\psi),\mathcal{S})_{\mathbb{C}}/K(N)=\mathcal{A}_{g,1,N_{\mathbb{C}}}$$

which is obtained from the embedding

$$\mathcal{X}\times G(\mathbb{A}_f)\hookrightarrow \mathcal{S}\times GSp(W,\psi)(\mathbb{A}_f)$$

between complex spaces via a natural passage to quotients. It is well known that this morphism is in fact defined over $E(G,\mathcal{X})$ (Shimura, Deligne). More precisely, we have a finite (functorial) morphism of $E(G,\mathcal{X})$-schemes

$$f(N):\mathrm{Sh}(G,\mathcal{X})/K_A(N)\to \mathrm{Sh}(GSp(W,\psi),\mathcal{S})_{E(G,\mathcal{X})}/K(N)=\mathcal{A}_{g,1,N_{E(G,\mathcal{X})}};$$

the $E(G,\mathcal{X})$-scheme $\mathrm{Sh}(G,\mathcal{X})/K_A(N)$ is called the *canonical model* of $\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}/K_A(N)$ over $E(G,\mathcal{X})$ and the $\mathbb{Q}$-scheme $\mathrm{Sh}(GSp(W,\psi),\mathcal{S})=\mathcal{A}_{g,1,N}$ is called the *canonical model* of $\mathrm{Sh}(GSp(W,\psi),\mathcal{S})_{\mathbb{C}}/K(N)$ over $\mathbb{Q}$.

By Shimura varieties, one usually means such canonical models $\mathrm{Sh}(G,\mathcal{X})/K_A(N)$ or their projective limit (i.e., the *arithmetic Shimura tower*) $\mathrm{Sh}(G,\mathcal{X})$.

The advantage offered by $\mathrm{Sh}(G, \mathcal{X})$ in comparison with $\mathrm{Sh}(G, \mathcal{X})/K_A(N)$, is that the group $G(\mathbb{A}_f)$ acts *continuously* on $\mathrm{Sh}(G, \mathcal{X})$ and thus in this way one keeps track of all *Hecke operators*.

4.6. GLOBAL MODULI SPACES DEFINED by A. Let

$$D := O_{E(G,\mathcal{X})}[\frac{1}{N}] \text{ and } K := E(G, \mathcal{X}).$$

Let X be the normalization of $\mathcal{A}_{g,1,N}$ in the ring of fractions of $\mathrm{Sh}(G, \mathcal{X})/K_A(N)$; it is a D-scheme. The notation X is non-standard but it relates to Section 1.

One thinks of X as the *global deformation space* of A over D.

4.7. BASIC THEOREM (Vasiu, submitted). If A has compact factors, then X is a projective D-scheme.

4.7.1. REMARK. Theorem 4.7 is equivalent to Theorem 3.7.

4.8. THEOREM (Néron models as smooth, projective, integral models of Shimura varieties). Suppose that the following three conditions hold:

(i) the abelian variety A has compact factors (infinite source of examples);

(ii) the D-scheme X is smooth (this condition automatically holds if $N >> 0$);

(iii) if p is a prime that does not divide N, then each local ring of D of mixed characteristic $(0, p)$ has an index of ramification at most $\max\{1, p-2\}$ (this condition automatically holds if $N >> 0$).

Then X is a Néron model over D of its generic fibre.

PROOF: Let Y be a smooth D-scheme and let $m_K : Y_K \to X_K$ a morphism. The D-scheme X is projective, cf. Theorem 4.7. Thus from the valuative criterion of properness, we get that there exists an open subscheme U of Y which contains Y_K, whose complement in Y has codimension at least two in Y, and which also has the property that m_K extends uniquely to a morphism $m_U : Y \to X$. Let $l_U : Y_U \to \mathcal{A}_{g,1,N}$ be the composite of m_U with the natural morphism $X \to \mathcal{A}_{g,1,N}$. Due to the purity result 2.4, the morphism l_U extends uniquely to a morphism $l : Y \to \mathcal{A}_{g,1,N}$.

As X is finite over $\mathcal{A}_{g,1,N}$ and as Y is a normal scheme, the morphism l factors uniquely through a morphism $m : Y \to X$ that extends m_U and thus that extends m_K. Thus the Néron mapping universal property holds for X. Therefore X is a Néron model of its generic fibre.□

4.9. EXAMPLE. Suppose

$$G^{\mathrm{ad}}_{\mathbb{R}} \tilde{\to} PSU(a,b)_{\mathbb{R}} \times_{\mathbb{R}} PSU(a+b,0)_{\mathbb{R}},$$

with a, $b \in \mathbb{N} \setminus \{1,2\}$. A result of Parthasarathy says that for each connected component $\mathcal{C}$ of $X_{\mathbb{C}} = \mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}}/K_A(N)$, we have $H^{1,0}(\mathcal{C}(\mathbb{C}),\mathbb{C}) = 0$.

This means that the albanese variety of $\mathcal{C}$ is trivial i.e., each morphism from $\mathcal{C}$ to a complex abelian variety is trivial. This implies (as $\mathcal{C}$ is projective) that each morphism from $\mathcal{C}$ to a complex group is trivial.

CONCLUSION: the Néron models constructed by Theorem 4.8 form a completely new and large class of Néron models.

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