OBSTRUCTION IN MODULAR TOWERS

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http://www.math.uci.edu/~mfried/conffiles/
semmen-RedLodgeSlides.pdf

THE MAIN CONJECTURE

Throughout, G_0 is a finite group and $\mathbf{C}$ is an r-tuple of G_0- conjugacy classes. The corresponding modular tower is:

$$\ldots \longrightarrow \mathcal{H}(G_1, \mathbf{C}) \longrightarrow \mathcal{H}(G_0, \mathbf{C}) \longrightarrow U_r$$

Main Conjecture: For every number field k,

$$\exists N, \forall n > N, \mathcal{H}(G_n, \mathbf{C})(k) = \emptyset.$$

Theorem: $\forall$ number field k, $\nexists$ modular tower that has a projective sequence of k-points.

Strategy:

1. Characterize projective systems of components defined over a fixed number field k, i.e. *understand obstruction of components*.

2. For $r = 4$, show genera of (reduced) components in projective systems are unbounded.

OUTLINE OF TALK

Part I

1. Reduce obstruction of Hurwitz space components to an obstruction purely in group cohomology.

2. Describe the canonical sequence of finite groups defining modular towers, and enumerate some of their properties that impinge on obstruction.

Part II

1. Review the concepts of cohomological dimension and duality groups.

2. Record consequences of duality for obstruction in modular towers.

PART I

HURWITZ SPACE CONSTRUCTION

If $|\mathbf{z}| = r$, $\pi_1(\mathbb{P}^1 \setminus \mathbf{z}) \simeq \langle \delta_1, \ldots, \delta_r \mid \delta_1 \cdots \delta_r = 1 \rangle$.
Fix a bouquet $(\delta_1, \ldots, \delta_r)$.
Fix an r-tuple $\mathbf{C}$ of conjugacy classes of G_n.

Definition: The *Nielsen class* $\mathrm{Ni}\,(G_n, \mathbf{C})$ is:

$$\{\varphi : \pi_1(\mathbb{P}^1 \setminus \mathbf{z}) \twoheadrightarrow G_n \mid \exists \sigma \in S_r, \forall i, \varphi(\delta_i) \in C_{\sigma(i)}\}$$

modulo conjugation by $\pi_1(\mathbb{P}^1 \setminus \mathbf{z})$.

Definition: U_r is the configuration space of $\mathbf{z} \subset \mathbb{P}^1_{\mathbb{C}}$ such that $|\mathbf{z}| = r$.

Definition: The *Hurwitz space* $\mathcal{H}\,(G_n, \mathbf{C})$ is the covering space corresponding to

$$\pi_1(U_r, \mathbf{z}) \to \mathrm{Out}(\pi_1(\mathbb{P}^1 \setminus \mathbf{z})) \to \mathrm{Sym}(\mathrm{Ni}\,(G_n, \mathbf{C})).$$

OBSTRUCTION AS EMBEDDING PROBLEM

Let $\psi_n : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathbf{z}) \twoheadrightarrow G_n$ represent an element of $\mathrm{Ni}\,(G_n, \mathbf{C})$ corresponding to $x_n \in \mathcal{H}\,(G_n, \mathbf{C})$. Given an exact sequence

$$1 \longrightarrow M_n \longrightarrow G_{n+1} \xrightarrow{\varphi_n} G_n \longrightarrow 1,$$

let $\phi_n : \mathcal{H}\left(G_{n+1}, \mathbf{C}\right) \to \mathcal{H}\,(G_n, \mathbf{C})$ be given by fiber-wise composition by φ_n. Then, $\phi_n^{-1}(x_n) \neq \emptyset$ iff there is a solution to the embedding problem

$$\begin{array}{ccccccccc} & & & & & & \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \mathbf{z}) & & \\ & & & & & & \downarrow \psi_n & & \\ 1 & \longrightarrow & M_n & \longrightarrow & G_{n+1} & \xrightarrow{\varphi_n} & G_n & \longrightarrow & 1 \end{array}$$

that lies in $\mathrm{Ni}\left(G_{n+1}, \mathbf{C}\right)$.

p-FRATTINI COVERS

Definition: $F \in \mathcal{C}$ is *Frattini* iff

$\forall X \in \mathcal{C}, \forall \phi \in \mathrm{Hom}\,(X, F)\,, \phi$ is an epimorphism .

Let $\varphi : G_{n+1} \twoheadrightarrow G_n$ be a p-Frattini cover (i.e. a Frattini object in the category of epimorphisms onto G_n with p-group kernel). Then, every *weak solution* to the embedding problem is surjective.

Definition: A conjugacy class C is a p'-c.c. iff

$$\forall g \in C, p \not| \, |\langle g \rangle|.$$

Lemma: φ_n is a p-Frattini cover $\Longrightarrow$

$\forall$ p'-c.c. $C_n \subset G_n$, $\exists!$ p'-c.c. $C_{n+1} \subseteq \varphi_n^{-1}(C_n)$.

$\therefore$ p'-c.c. of G_0 lift uniquely to p'-c.c. of G_n and *elements of the lift have the same order*.

A COHOMOLOGICAL OBSTRUCTION

Definition: Let $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. Then, the *polygonal group* $\Gamma(\mathbf{s})$ is:

$$\langle \sigma_1, \ldots, \sigma_r \mid \sigma_1^{s_1} = \ldots = \sigma_r^{s_r} = \sigma_1 \cdots \sigma_r = 1 \rangle.$$

Assume $G_{n+1} \twoheadrightarrow G_n$ is p-Frattini and every conjugacy class in $\mathbf{C}$ is a p'-c.c.
For each $i = 1, \ldots, r$, let $s_i = |\langle \psi_n(\delta_i) \rangle|$.

1. ψ_n has a solution lying in $\mathrm{Ni}\Big(G_{n+1}, \mathbf{C}\Big)$ iff $\overline{\psi}_n$ has a weak solution:

$$\begin{array}{ccccccccc} &&&&&& \begin{array}{c}\pi_1(\mathbb{P}^1 \setminus \mathbf{z})\\ \left.\begin{array}{c}\downarrow\\ \Gamma(\mathbf{s})\\ \downarrow \overline{\psi}_n\end{array}\right\}\psi_n\end{array} && \\ 1 & \longrightarrow & M_n & \longrightarrow & G_{n+1} & \xrightarrow{\varphi_n} & G_n & \longrightarrow & 1 \end{array}$$

2. If $\alpha_n \in H^2(G_n, M_n)$ represents the exact sequence

$$1 \longrightarrow M_n \longrightarrow G_{n+1} \xrightarrow{\varphi_n} G_n \longrightarrow 1,$$

then $\overline{\psi}_n : \Gamma(\mathbf{s}) \twoheadrightarrow G_n$ has a weak solution iff the inflation $\mathrm{inf}_{\overline{\psi}_n}(\alpha_n)$ of α_n to $\Gamma(\mathbf{s})$ is 0.

3. Consider

$$\mathcal{H}\left(G_{n+1}, \mathbf{C}\right) \xrightarrow{\phi_n} \mathcal{H}\left(G_n, \mathbf{C}\right)$$

and let $\mathcal{O}$ be a connected component of $\mathcal{H}\left(G_n, \mathbf{C}\right)$ such that $\phi_n^{-1}(\mathcal{O}) \neq \emptyset$. Then,

$$H^1(\Gamma(\mathbf{s}), M_n) \hookrightarrow \phi_n^{-1}(\mathcal{O}) \twoheadrightarrow \mathcal{O}$$

is a fibration.

THE GRUENBERG-ROGGENKAMP EQUIVALENCE

Definition: The *augmentation ideal* ω_{RG} is the kernel of the augmentation map, i.e.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{RG} & \longrightarrow & RG & \longrightarrow & R & \longrightarrow & 0 \\ & & & & g & \mapsto & 1 & & \end{array}$$

is exact.

Two categories of covers:

$$\mathcal{C}_{RG}(\omega_{RG}) = ((RG\text{-module covers } M \twoheadrightarrow \omega_{RG}))$$

$$\mathcal{C}_{RG}(G) = \left(\left(\begin{array}{c}\text{group covers } H \twoheadrightarrow G \\ \text{with } RG\text{-module kernel}\end{array}\right)\right)$$

Theorem (Gruenberg-Roggenkamp):

$$\mathcal{C}_{RG}(\omega_{RG}) \approx \mathcal{C}_{RG}(G)$$

where corresponding objects have isomorphic kernels.

FRIED'S MODULAR TOWERS

${}_p^{n+1}\tilde{G} \overset{\varphi_n}{\twoheadrightarrow} {}_p^n\tilde{G}$ is the projective Frattini object in $\mathcal{C}_{\mathbb{F}_p[{}_p^n\tilde{G}]}({}_p^n\tilde{G})$ [unique up to isomorphism].

1. $\forall g_n \in {}_p^n\tilde{G}, \forall g_{n+1} \in \varphi_n^{-1}(g_n)$,

   $$p \text{ divides } |\langle g_n \rangle| \Longrightarrow |\langle g_{n+1} \rangle| = p \cdot |\langle g_n \rangle|.$$

2. If some conjugacy class in $\mathbf{C}$ is *not* a p'-c.c.,

   $$\forall \text{ number field } k, \exists n, \mathcal{H}\left({}_p^n\tilde{G}, \mathbf{C}_n\right)(k) = \emptyset.$$

   If $\mathbf{C}$ consists of p'-c.c. then ${}_p^n\tilde{G}$ is p- perfect.

3. Definition: $\quad O_{p'}(G) = \max\{H \triangleleft G \mid p \not| \, |H|\}$.

   If G is p-perfect then ${}_p^n\tilde{Q}$ has trivial center:

   Pullback: $$\begin{array}{ccc} {}_p^n\tilde{G} & \twoheadrightarrow & {}_p^n\tilde{Q} \\ \downarrow & \square & \downarrow \\ G & \twoheadrightarrow & Q = G/O_{p'}(G) \end{array}$$

We may assume ${}_p^n\tilde{G}$ is p-perfect and centerless.

A TRICHOTOMY

“Schreier formula:”

$$\dim_{\mathbb{F}_p}(M_{n+1}) = 1 + p^{\dim_{\mathbb{F}_p}(M_n)}\left(\dim_{\mathbb{F}_p}(M_n) - 1\right)$$

1. $p \not| \, |G| \iff \forall n, \dim_{\mathbb{F}_p}(M_n) = 0$
   $\iff \forall n, {}^n_p\tilde{G} = G$

2. Theorem (Griess-Schmid):
   $G/O_{p'}(G) \leq (\mathbb{Z}/p^m\mathbb{Z}) \rtimes \mathbb{F}_p^*$ if and only if $\forall n, \dim_{\mathbb{F}_p}(M_n) = 1$.

   Example: $G = D_{2p} \implies {}^n_p\tilde{G} = D_{2p^{n+1}}$

3. Otherwise, $\dim_{\mathbb{F}_p}(M_0) > 1$.

   Example: $G = (\mathbb{Z}/2\mathbb{Z})^2$:

n	0	1	2
$\dim_{\mathbb{F}_2}(M_n)$	5	129	$1 + 2^{136}$

LIMITATIONS OF THE INVERSE LIMIT

Lemma: $\varprojlim {}_p^n\tilde{G} \simeq {}_p\tilde{G}$, the *universal p-Frattini cover* of G.

The construction of ${}_p\tilde{G}$ uses Zorn's lemma and so is non-constructive, *even when G is a semi-direct product*.

Theorem: Let $G = N \rtimes H$, where $p \not| \, |H|$ and $(|N|, |H|) = 1$. Then, ${}_p\tilde{G} = {}_p\tilde{N} \rtimes H$.

Lemma: If G is a p-group then ${}_p\tilde{G} \simeq \hat{F}_m(p)$, where m is the minimal number of generators of G.

Example: Let $p = 2$ and $G = \mathbb{F}_8^+ \rtimes \mathbb{F}_8^*$.
Then, ${}_2\tilde{G} \simeq \hat{F}_3(2) \rtimes (\mathbb{Z}/7\mathbb{Z})$.
However, Aut(F_3) has no 7-torsion.

CADORET'S ABELIANIZED VARIANT

${}_p\bar{G} \twoheadrightarrow G$ is the projective Frattini object in $\mathcal{C}_{\mathbb{Z}_p G}(G)$ [unique up to isomorphism]. Let L be its kernel.

Definition: ${}^n_p\bar{G} = {}_p\bar{G}/p^nL$

The argument for Fried's modular towers works also for Cadoret's abelianized variants:
We may assume ${}^n_p\bar{G}$ is p-perfect and centerless, for the purposes of the Main Conjecture.

Note that

$$0 \longrightarrow p^nL/p^{n+1}L \longrightarrow {}^{n+1}_{\ p}\bar{G} \longrightarrow {}^n_p\bar{G} \longrightarrow 1$$

is exact and that $p^nL/p^{n+1}L \simeq M_0$.

SUMMARY OF PART I

1. $\alpha_n : 0 \to M_n \to G_{n+1} \to G_n \to 1$ represents a canonically defined p-Frattini cover.

2. $\mathbf{C}$ consists of p'-c.c.

3. The polygonal group $\Gamma(\mathbf{s})$ is

   $$\langle \sigma_1, \ldots, \sigma_r \mid \sigma_1^{s_1} = \ldots = \sigma_r^{s_r} = \sigma_1 \cdots \sigma_r = 1 \rangle.$$

4. $x_n \in \mathrm{Ni}\,(G_n, \mathbf{C})$ corresponds to $\Gamma(\mathbf{s}) \overset{\bar\psi_n}{\twoheadrightarrow} G_n$.

5. $$\exists x_{n+1} \text{ such that } x_{n+1} \mapsto x_n$$
   $$\Updownarrow$$
   $$0 = \mathrm{inf}_{\bar\psi_n}(\alpha_n) \in H^2(\Gamma(\mathbf{s}), M_n).$$

PART II

Throughout,

1. Γ is a group and R is a commutative ring,

2. $\mathbf{1}_{R\Gamma}$ is the $R\Gamma$-module R on which Γ acts trivially, and

3. unless otherwise stated, every module is a *left* module.

FINITENESS CONDITIONS

Definition: Γ has *cohomological dimension* n *over* R (i.e. $\mathrm{cd}_R(\Gamma) = n$) iff $\exists$ $R\Gamma$-projective resolution of length n of $\mathbf{1}_{R\Gamma}$:

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow \mathbf{1}_{R\Gamma} \longrightarrow 0.$$

Lemma: $\mathrm{cd}_R(\Gamma) = n \iff$
$\forall k > n$, $\forall R\Gamma -$ module M, $H^k(\Gamma, M) = 0$ and
$\exists R\Gamma -$ module M, $H^n(\Gamma, M) \neq 0$.

Definition: Γ is of type $\mathbf{FP}_\infty$ over R iff $\exists$ resolution of $\mathbf{1}_{R\Gamma}$ by finitely generated, projective $R\Gamma$-modules. It is of type $\mathbf{FP}$ if this resolution may be chosen to have finite length.

Lemma: Γ is of type $\mathbf{FP}$ over R iff Γ is of type $\mathbf{FP}_\infty$ over R and $\mathrm{cd}_R(\Gamma) < \infty$.

TOPOLOGICAL INTERPRETATION

Example: If X is a finite n-dimensional CW-complex with contractible universal cover then $\pi_1(X)$ is of type $\mathbf{FP}$ and $\mathrm{cd}_{\mathbb{Z}}(\pi_1(X)) \leq n$.

Theorem: $\exists K(\Gamma, 1)$ that is the retract of a finite complex $\Longrightarrow$ Γ is of type $\mathbf{FP}$ over $\mathbb{Z}$.

Definition: The *geometric dimension* of Γ (geom dim(Γ)) is the dimension of a minimal-dimensional $K(\Gamma, 1)$.

Lemma: $\mathrm{cd}_{\mathbb{Z}}(\Gamma) \leq$ geom dim(Γ)

Theorem: Let $n = \begin{cases} 3, & \text{if } \mathrm{cd}_{\mathbb{Z}}(\Gamma) = 2. \\ \mathrm{cd}_{\mathbb{Z}}(\Gamma), & \text{otherwise.} \end{cases}$

Then $\exists$ n-dimensional $K(\Gamma, 1)$. If Γ is finitely presented and of type $\mathbf{FP}$ then this may be chosen to be the retract of a finite complex.

COHOMOLOGY FOR PROFINITE GROUPS

1. Use the completed group algebra $\mathbb{Z}_p[[\Gamma]]$ instead of $\mathbb{Z}_p\Gamma$ (i.e. $R = \mathbb{Z}_p$).

2. Modules are topological and Γ acts continuously on them.

3. The *Pontryagin category* $\mathfrak{B}_p(\Gamma)$ of modules is the union of

   $$\begin{array}{rcl} \mathfrak{D}_p(\Gamma) & = & \text{discrete torsion modules} \\ \mathfrak{C}_p(\Gamma) & = & \text{profinite modules} \end{array}$$

   Pontryagin duality ($\mathrm{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$) is a contravariant functor on $\mathfrak{B}_p(\Gamma)$.

4. A discrete subgroup of a profinite group is "good" iff its cohomology groups are isomorphic to those of its ambient group.

DUALITY GROUPS

Definition: Γ is a *duality group of dimension* n *over* R iff $\mathrm{cd}_R(\Gamma) = n$, Γ is of type $\mathbf{FP}$, and

$$H^k(\Gamma, R\Gamma) = \begin{cases} 0, & k \neq n \\ \text{flat } R\text{-module}, & k = n \end{cases}$$

Definition: If Γ is a duality group of dimension n over R then $D_R(\Gamma) = H^n(\Gamma, R\Gamma)$ is the *dualizing module*; this is a *right* $R\Gamma$-module.

Theorem: If Γ is a duality group of dimension n over R then, $\forall k$, $\exists$ natural isomorphism of functors

$$H^k(\Gamma, *) = H_{n-k}(\Gamma, D_R(\Gamma) \otimes_R *),$$

compatible with the long exact sequences, and where Γ acts diagonally on the tensor product.

POINCARÉ DUALITY GROUPS

Theorem: X is a closed n-dimensional manifold with contractible universal cover $\Longrightarrow \pi_1(X)$ is a duality group of dimension n over $\mathbb{Z}$.

Definition: A duality group Γ over R is a *Poincaré duality group* iff $D_R(\Gamma) \simeq R$ as an R-module.

Definition: A Poincaré duality group Γ over R is *orientable* iff $D_R(\Gamma) \simeq \mathbf{1}_{R\Gamma}$ as a right $R\Gamma$-module.

Corollary: If Γ is an orientable Poincaré duality group of dimension n over R then, $\forall k$, $\exists$ natural isomorphism of functors:

$$H^k(\Gamma, *) = H_{n-k}(\Gamma, *)$$

Remark: It is not known whether existence of an n-dimensional $K(\Gamma, 1)$- manifold characterizes Poincaré duality groups over $\mathbb{Z}$.

BASIC EXAMPLES

Example: A finitely generated free group is a duality group of dimension 1 over $\mathbb{Z}$. It is Poincaré duality iff it is cyclic (in which case it is orientable).

Example: A finitely generated free pro-p group is a duality group of dimension 1 over $\mathbb{Z}_p$ (with respect to profinite cohomology). It is Poincaré duality iff it is cyclic (in which case it is orientable).

More generally, these are the p-projective groups, e.g. ${}_p\tilde{G}$.

Example: The pro-p Poincaré duality groups of dimension 2 over $\mathbb{Z}_p$ (with respect to profinite cohomology) are exactly the Demuškin groups (pro-p one-relator groups).

FINITENESS CONDITIONS AND FINITE-INDEX SUBGROUPS

Let $1 \neq \Upsilon \leq \Gamma$ such that $(\Gamma : \Upsilon) < \infty$.

Definition: Γ has no R-torsion iff $\forall g \in \Gamma$, $|\langle g\rangle|$ is invertible in R if $|\langle g\rangle| < \infty$.

Theorem (Serre): If Γ has no R-torsion then $\mathrm{cd}_R(\Gamma) = \mathrm{cd}_R(\Upsilon)$.

Theorem: Γ is of type $\mathbf{FP}_\infty$ iff Υ is of type $\mathbf{FP}_\infty$.

Theorem: If Γ has no R-torsion then Γ is a duality group over R iff Υ is a duality group over R. Furthermore, $\mathrm{Res}^\Gamma_\Upsilon(D_R(\Gamma)) \simeq D_R(\Upsilon)$.

Theorem: Let $\Delta \lhd \Gamma$ and let both Δ and Γ/Δ be duality groups over R. Then Γ is a duality group over R and

$$\mathrm{cd}_R(\Gamma) = \mathrm{cd}_R(\Delta) + \mathrm{cd}_R(\Gamma/\Delta).$$

THE HURWITZ MONODROMY GROUP

Definition: U_r is the configuration space of $\mathbf{z} \subset \mathbb{P}^1_{\mathbb{C}}$ such that $|\mathbf{z}| = r$. The *Hurwitz monodromy group* H_r is $\pi_1(U_r)$.

For $r \geq 3$,

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H_r \longrightarrow \mathrm{Mod}_r^+ \longrightarrow 1$$

is exact, where Mod_r^+ is the orientation-preserving mapping class group of $\mathbb{P}^1_{\mathbb{C}} \backslash \mathbf{z}$ (with $|\mathbf{z}| = r$).

For $r \geq 4$,

$$1 \longrightarrow F_{r-2} \rtimes \ldots \rtimes F_2 \longrightarrow \mathrm{Mod}_r^+ \longrightarrow S_r \longrightarrow 1$$

is exact.

Corollary: For $p > r \geq 4$, both Mod_r^+ and H_r are (non-Poincaré) duality groups of dimension $r - 3$ over $\mathbb{Z}_p$ and are "good" groups.

THE POLYGONAL GROUP

Theorem (Weigel): For prime $p \nmid \prod_{i=1}^{r} s_i$, $\Gamma(\mathbf{s})$ is an orientable Poincaré duality group of dimension 2 over $\mathbb{Z}_p$ and is a "good" group, provided

$$\sum_{i=1}^{r}\left(1-\frac{1}{s_i}\right) > 2. \qquad (1)$$

Sketch of proof: Inequality (1) is equivalent to: for every Galois cover $X \twoheadrightarrow \mathbb{P}^1_{\mathbb{C}}$ with ramification prescribed by $\mathbf{s}$, the genus of X is greater than 1:

$$\overbrace{\underbrace{\mathbb{H} \longrightarrow X}_{\mathbb{S}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}}^{\Gamma(\mathbf{s})}$$

where $\mathbb{S}$ is an orientable Poincaré duality group of dimension 2 over $\mathbb{Z}$.

$(\Gamma(\mathbf{s}) : \mathbb{S}) < \infty$ and the only elements in $\Gamma(\mathbf{s})$ having finite order are the "elliptic" elements (the conjugates of $\sigma_1, \ldots, \sigma_r$). $\square$

A CONSEQUENCE FOR MODULAR TOWERS

In the following commutative diagram of exact sequences, $\bar{\psi}_n : \Gamma(\mathbf{s}) \twoheadrightarrow G_n$ has a weak solution for the top row iff it has a weak solution for the bottom row:

$$\begin{array}{ccccccccc} 0 & \to & M_n & \to & G_{n+1} & \to & G_n & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (M_n)_{G_n} & \to & {}_p\hat{G}_n & \to & G_n & \to & 1 \end{array}$$

where $(M_n)_{G_n} = M_n/\omega_{\mathbb{F}_p G_n} M_n$.

In other words, if $\bar{\alpha}_n \in H^2(G_n, (M_n)_{G_n})$ corresponds to the bottom row and α_n to the top, then

$$\inf_{\bar{\psi}_n}(\alpha_n) = 0 \iff \inf_{\bar{\psi}_n}(\bar{\alpha}_n) = 0.$$

Corollary: If, above a connected component $\mathcal{O}$ of $\mathcal{H}\left({}^n_p\bar{G}, \mathbf{C}\right)$, $\exists$ connected component in $\mathcal{H}\left({}^{n+N}_p\bar{G}, \mathbf{C}\right)$ for a suitably large N, then $\exists$ projective system of connected components above $\mathcal{O}$ in the *abelianized* modular tower.

EXCEPTIONAL POLYGONAL GROUPS

$$\sum_{i=1}^{r} \left(1 - \frac{1}{s_i}\right) \leq 2$$

$\mathbf{s}$	$\Gamma(\mathbf{s})$	Coxeter graph
Spherical triangle groups:		
(n, n)	$\mathbb{Z}/n\mathbb{Z}$	$I_2(n)$
$(2, 2, n)$	D_{2n}	$I_2(n) \sqcup A_1$
$(2, 3, 3)$	A_4	A_3
$(2, 3, 4)$	S_4	B_3
$(2, 3, 5)$	A_5	H_3
Euclidean triangle groups:		
$(2, 3, 6)$	$(\mathbb{Z} \times \mathbb{Z}) \rtimes (\mathbb{Z}/6\mathbb{Z})$	$\tilde{H}_2$
$(2, 4, 4)$	$(\mathbb{Z} \times \mathbb{Z}) \rtimes (\mathbb{Z}/4\mathbb{Z})$	$\tilde{B}_2$
$(3, 3, 3)$	$(\mathbb{Z} \times \mathbb{Z}) \rtimes (\mathbb{Z}/3\mathbb{Z})$	$\tilde{A}_2$
$(2, 2, 2, 2)$	$(\mathbb{Z} \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$	$\tilde{I}_1 \sqcup \tilde{I}_1$

Each group is the “alternating” subgroup of the corresponding Coxeter group.

HURWITZ SPACES FOR $r = 4$

Let U^r be the configuration space of $\vec{z} \in \left(\mathbb{P}^1_{\mathbb{C}}\right)^r$ with distinct coordinates. Since

$$\begin{array}{ccccc} \mathrm{PSL}_2(\mathbb{C}) & \times & \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} & \xrightarrow{\sim} & U^4 \\ (\alpha & , & z) & \mapsto & \alpha(0,1,\infty,z) \end{array}$$

the universal cover of U_4 is the composition:

$$\mathrm{SL}_2(\mathbb{C}) \times \mathbb{H} \overset{(\mathbb{Z}/2\mathbb{Z})\times F_2}{\twoheadrightarrow} U^4 \overset{S_4}{\twoheadrightarrow} U_4.$$

Since the stabilizer in $\mathrm{SL}_2(\mathbb{C})$ of a generic fiber of $U^4 \twoheadrightarrow U_4$ is $(\mathbb{Z}/2\mathbb{Z})^2 \lhd S_4$, reduction modulo $\mathrm{SL}_2(\mathbb{C})$ yields

$$\mathbb{H} \overset{F_2}{\twoheadrightarrow} \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} \overset{S_3}{\twoheadrightarrow} \mathbb{C},$$

a $\mathrm{PSL}_2(\mathbb{Z})$-cover ramified over 0 and 1728, since the final map may be taken to be j.

Thus, reduced Hurwitz spaces for $r = 4$ are quotients of $\mathbb{H}$ by subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.