On motivic iterated integrals in generic positions (joint work with A. Jafari)

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Furusho, Jafari: Algebraic cycles and motivic generic iterated integrals, arXiv: math. NT/0506370

Furusho, Jafari: Regularization and generalized double shuffle relations for p-adic multiple zeta values, arXiv: math. AG/0510681

references

Bloch, Kriz: Mixed Tate motives, Ann of Math (2), 140, (1994)

Gangl, Goncharov, Levin: Multiple polylogarithms, polygons, trees and algebraic cycles, arXiv: math. NT/0508066

§0. Overview

Levine-Voevodsky side	Bloch-Kriz side
$MTM^{LV}(F)$: tannakian category F : a field with $ch=0$ satisfying Beilinson-Soulé vanishing conjecture	$MTM^{BK}(F)$: Tannakian category F : ANY FIELD!
$\exists$ Hodge realization $\exists$ ℓ-adic étale realization by Levine, Huber	$\exists$ Hodge realization for $F \subset \mathbb{C}$ $\exists$ ℓ-adic étale realization for $ch F \neq \ell$ & $\mu_{\ell^\infty} \not\subset F$
$Ext^p(\mathbb{Q}(0), \mathbb{Q}(r)) \simeq gr^\gamma_r K_{2r-p}(F)_\mathbb{Q}$	$Ext^p(\mathbb{Q}(0), \mathbb{Q}(r)) \simeq gr^\gamma_r K_{2r-p}(F)_\mathbb{Q}$ under $K(\pi,1)$-conjecture
$\exists$ $\pi_1^M(X)$ $X=\mathbb{P}^1-S$ by Deligne, Goncharov	$\exists ?$ π_1^M <u>Th</u> $\exists$ motivic iterated integral in generic position

§1. Review of Bloch-Kriz's Mixed Tate Motive

$$\begin{array}{cc} \square^1_F = \mathbb{P}^1_F \setminus \{1\} , & \square^n_F = (\square^1_F)^n \\ \circlearrowleft & \circlearrowleft \\ (\mathbb{Z}/2) & G_n := S_n \ltimes (\mathbb{Z}/2)^n \end{array}$$

$$\overset{\overset{G_n}{\circlearrowright}}{Z}(\square^n_F, r) = \mathbb{Q}\text{-span of codim } r \text{ irr closed subvar}$$

s.t. intersects all faces properly $\leftarrow$ each coor $= 0$ or ∞

$$\mathcal{N}^p(r) = \mathrm{Alt}\, Z(\square_F^{2r-p}, r)$$

$$\mathcal{N}^\bullet = \bigoplus_r \mathcal{N}^\bullet(r) : \text{DGA with Adams grading}$$

Def $MTM^{BK}(F)$ = category of finite dimensional comodules over gr. com. Hopf alg $H^0B(\mathcal{N})$.

cf

$$\mathrm{Ext}^p_{MTM(F)}(\mathbb{Q}(0), \mathbb{Q}(r)) \overset{\overset{\text{under } K(\pi,1)}{\downarrow}}{\simeq} H^p(\mathcal{N})(r) \overset{\overset{\text{Bloch-Kriz}}{\downarrow}}{\simeq} CH^r(F, 2r-p)_{\mathbb{Q}}$$

$$\underset{\underset{\text{Bloch, Levine}}{\uparrow}}{\simeq} gr^r_r K_{2r-p}(F)_{\mathbb{Q}}$$

§2. Tree

$S \neq \emptyset \subset F$: subset

Def A tree algebra $\mathcal{T}_S^{\bullet} = \bigoplus_r \mathcal{T}_S^{\bullet}(r)$ is

(generic tree algebra $\mathcal{T}_S'^{\bullet} \subset \mathcal{T}_S^{\bullet}$)

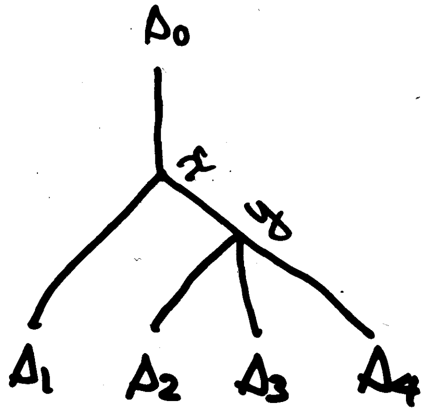
a DGA with Adams grading freely generated by rooted S-decorated trees.

(resp. rooted generic S-decorated trees.)

generic

$\Updownarrow$

(non-zero never repeat)



$\Big\downarrow \rho$

$\mathrm{Alt}\left[1-\frac{\Delta_0}{x}, 1-\frac{x}{\Delta_1}, 1-\frac{x}{y}, 1-\frac{y}{\Delta_2}, 1-\frac{y}{\Delta_3}, 1-\frac{y}{\Delta_4}\right]$

cycle map (owes Goncharov)

$\rho : \mathcal{T}_S \longrightarrow$ (algebraic cycles)

by $\overset{a}{\underset{b\neq 0}{|}} \longmapsto 1-\frac{a}{b}$

$\overset{a}{\underset{b=0}{|}} \longmapsto a$

$$\begin{tikzcd} \mathcal{J}_S^{\bullet} \arrow[d, "C"', "\times" marking] \\ {} \end{tikzcd}$$

Prop. $\rho : \mathcal{J}'^{\bullet}_S \to N^{\bullet}$ is a mor of DGA with Adams gr.

· It gives $\rho : H^0B(\mathcal{J}'_S) \longrightarrow H^0B(N)$

Def $a, b, c_1, \cdots, c_n \in S$: in generic position

$T(a, c_1, \cdots, c_n, b) = \sum$  $- \sum$ 

$+ \sum_{P_1 \cup P_2} \Big[\sum$  $\Big| \sum$  $\Big] + \cdots\cdots\cdots$

$$\begin{tikzcd} T(a, c_1, \cdots, c_n, b) \arrow[d, mapsto] & \in H^0B(\mathcal{J}'_S) \arrow[d, "\rho"] \\ \mathbb{I}^{M}(a, c_1, \cdots, c_n, b) & \in H^0B(N) \end{tikzcd}$$

motivic iterated integral in generic position

§3. Motivic iterated integral

Def [BGSV]

① FMHTS is a triple $\mathbb{H}=(H, v, \hat{v})$

with H : MHTS, $v : \mathbb{Q}(r) \to gr_{-2r}H$: frame,

$\hat{v} : gr_{-2s}H \to \mathbb{Q}(s)$: coframe for some r and s.

② $\mathbb{H} \sim \mathbb{H}' \Leftrightarrow {}^{\exists}n, {}^{\exists}H(n) \to H'$: mor of MHTS

compatible with frame and coframe.

• $F \subset \mathbb{C}$, $S \subset F$: finite, $a, b, c_i \in S$.

Def $\mathrm{It}^{\gamma}\int_a^b \frac{dt}{t-c_1} \circ \cdots \circ \frac{dt}{t-c_n}$

$$:= \int_{0 \le t_1 \le \cdots \le t_n \le 1} \frac{d\gamma(t_1)}{\gamma(t_1)-c_1} \wedge \cdots \wedge \frac{d\gamma(t_n)}{\gamma(t_n)-c_n}$$

$$\pi^{Be}(\mathbb{A}^1 - S : a, b) \times \mathbb{C} \simeq \pi^{DR}(\mathbb{A}^1 - S : a, b) \times \mathbb{C} = \mathbb{C}\langle\langle X_\Delta \rangle\rangle_{\Delta \in S}$$

$$\gamma \longmapsto \sum_k \sum_{b_i} \mathrm{It}^{\gamma}\int_a^b \frac{dt}{t-b_1} \circ \cdots \circ \frac{dt}{t-b_{k}} X_{b_1} \cdots X_{b_k}$$

Def $\mathbb{I}^{H}(a,c_1,\cdots,c_n,b) := \left(\Pi^{DR}(\mathbb{A}^1-S : a,b), 1, X_{c_1}^{\vee}\cdots X_{c_n}^{\vee}\right)$

: FMHTS ass with It $\int_a^b \frac{dt}{t-c_1} \circ \cdots \circ \frac{dt}{t-c_n}$

Th $\mathrm{real}_H\left(\mathbb{I}^{M}(a,c_1,\cdots,c_n,b)\right) = (-1)^n \mathbb{I}^{H}(a,c_1,\cdots,c_n,b)$

Bloch–Kriz's recipe

$\mathcal{N}' \subset \mathcal{N}$: a sub DGA, satisfying some conditions

$D' = \bigoplus D'^{\cdot}(r)$: DGA

$$D'^{p}(r) = \mathrm{Alt} \varinjlim_{S \in \mathrm{Supp}\,\mathcal{N}^p(r)} H_{2r-2p}\left(S \cup J^{2r-p}, J^{2r-p}\right)$$

① $H^0B(D', \mathcal{N}')_{\oplus} \xrightarrow{\text{`}\int \otimes id\text{'}} H^0B(\mathcal{N}')_{\mathbb{C}}$: MHTS

② $$\begin{tikzcd} H^0B(\mathcal{N}') \arrow[r] & H^0B(\mathcal{N}) \arrow[r, "\mathrm{real}_H"] & (\mathrm{FMHTS}) \\ x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & (H^0B(\mathcal{N}'), x, \varepsilon) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

Proof

Take a sub DGA $\mathcal{N}' \subset \mathcal{N}$ gen by '$\mathbb{I}^{M}(a, c_1, \cdots, c_n, b)$'.

and take $x = \mathbb{I}^{M}(a, c_1, \cdots, c_n, b) \in H^0 B(\mathcal{N}')$.

Define

$$f : \Pi^{DR}(\mathbb{A}^1 - S : a, b)(n) \longrightarrow H^0 B(\mathcal{N}')$$

$$X_{b_1} \cdots X_{b_k} \longmapsto (-1)^k \sum_{\substack{0 = i_0 < \cdots < i_k = n+1 \\ \text{s.t. } c_{i_j} = b_j}} \prod_{j=0}^{k} \mathbb{I}^{M}(c_{i_j}, \cdots, c_{i_{j+1}})$$

$$\begin{tikzcd}
 & \Pi^{DR}(\mathbb{A}^1 - S : a, b)(n)_{\mathbb{C}} \arrow[r, "f_{\mathbb{C}}"] & H^0 B(\mathcal{N}')_{\mathbb{C}} & \bigcirc \\
\gamma \in \arrow[rrru, mapsto, bend left] & \Pi^{Be}(\mathbb{A}^1 - S : a, b)(n) \arrow[u, "\text{period}"] & H^0 B(D', \mathcal{N}') \arrow[u, "\int \otimes \text{id}"] & \overset{\exists ?}{Z_\gamma} \arrow[u, mapsto]
\end{tikzcd}$$

$$Z_\gamma(a,c_1,\dots,c_n,b) = 1\otimes\mathbb{I}^{M}(a,c_1,\dots,c_n,b)$$

$$+\sum_{\substack{P_1\cup P_2\cup\cdots\\ \text{adm decomp}}}\xi_\gamma(P_1)\left[\rho\circ\lambda^{\Sigma_1}(P_2)\,\middle|\,\rho\circ\lambda^{\Sigma_2}(P_3)\,\middle|\,\cdots\cdots\right]$$

$$\in H^0B(D',N')$$

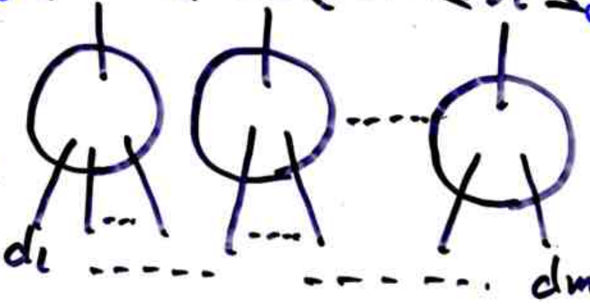
where

top bdry

$$\xi_\gamma(d_0,\dots,d_{m+1}) = \sum_{i=1}^{m}\delta\eta_i^\gamma(d_0,\dots,d_{m+1})\cdot\Gamma\cdot\delta\Gamma^{i-1}$$

$$+\sum_{i=1}^{m}(-1)^i\eta_i^\gamma(d_0,\dots,d_{m+1})\cdot\delta\Gamma^{i}$$

$$\eta_i^\gamma(d_0,\dots,d_{m+1}) = \rho\left(\sum_{\text{tree}} \quad d_0\le s_1\le s_2\le\cdots\le s_i\le d_{m+1}\ \text{on}\ \gamma\right)$$



mixture of alg cycle & top cycle

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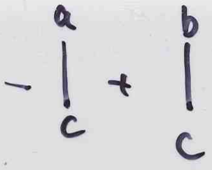
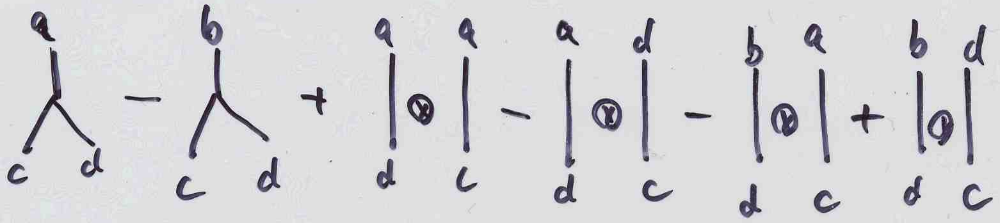
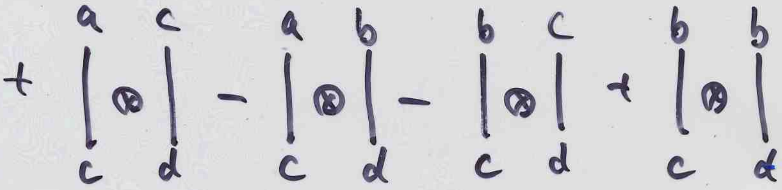
e.g.1 $\int_a^b \frac{dt}{t-c}$

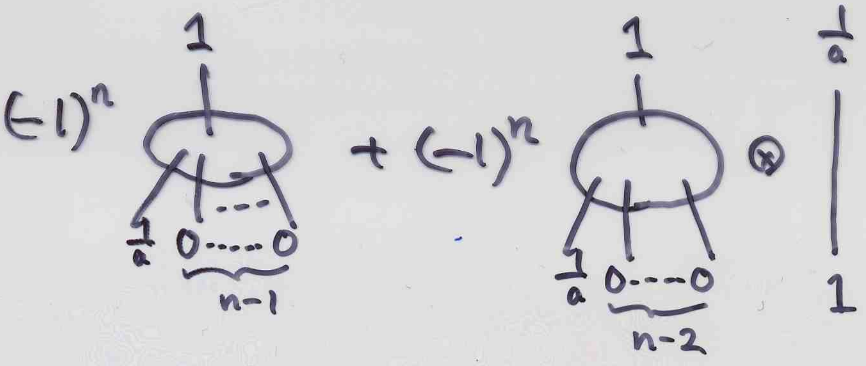


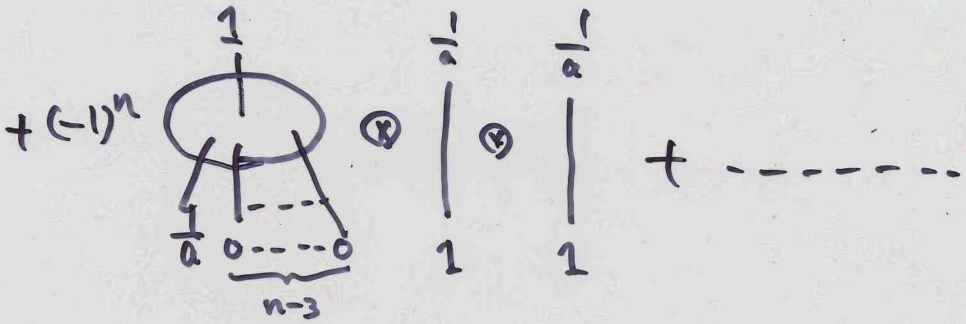
e.g.2 $\int_a^b \frac{dt}{t-c} \circ \frac{dt}{t-d}$ (a, b, c, d : in generic position)





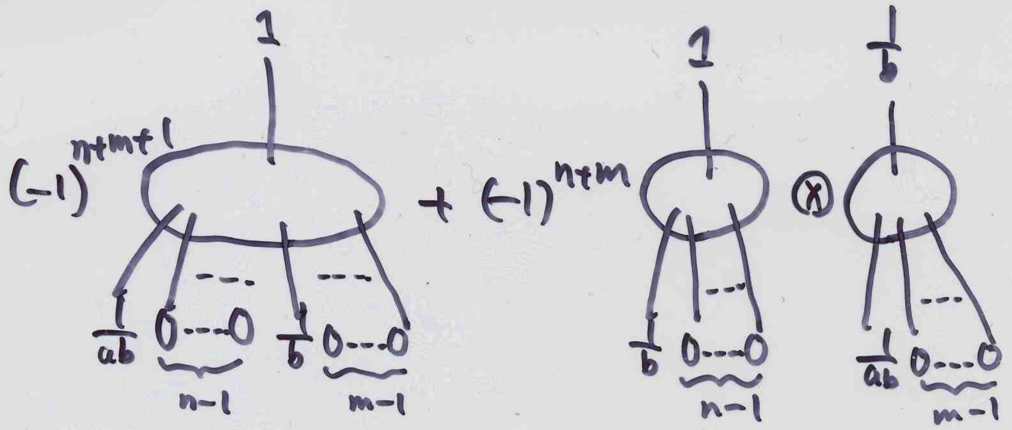
e.g.3 $Li_n(a) = \sum_k \frac{a^k}{k^n}$

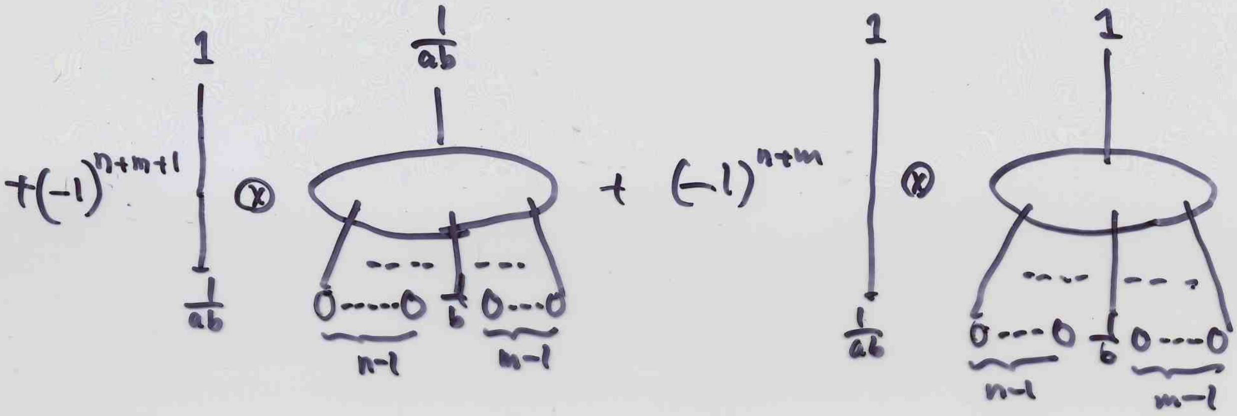


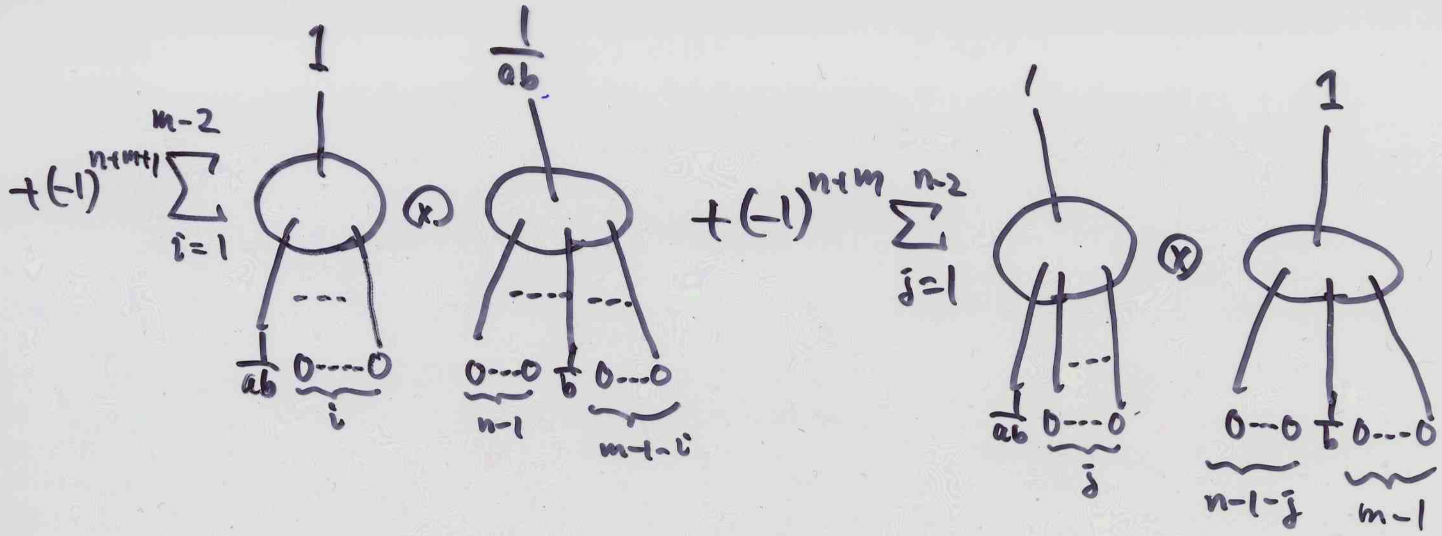


e.g.4

$$\mathrm{Li}_{n.m}(a,b)=\sum_{0<k<l}\frac{a^k b^l}{k^n l^m} \quad \begin{pmatrix} b\neq 1 \\ ab\neq 1 \end{pmatrix}$$







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Th The following holds:

triviality $T(a,b)=1$

$T(a,s_1,\cdots,s_n,a)=0$

shuffle product formula

$$T(a,s_1,\cdots,s_n,b)\cdot T(a,s_{n+1},\cdots,s_{n+m},b)=\sum_{\sigma\in sh} T(a,s_{\sigma(1)},\cdots,s_{\sigma(n+m)},b)$$

path composition formula

$$T(a,s_1,\cdots,s_n,b)=\sum_{k=0}^{n} T(a,s_1,\cdots s_k,c)\cdot T(c,s_{k+1},\cdots,s_n,b)$$

antipode formula

$$T(a,s_1,\cdots,s_n,b)=(-1)^n T(b,s_n,\cdots,s_1,a)$$

coproduct formula

$$\Delta T(a,s_1,\cdots,s_n,b)=\sum T(a,s_{i_1},\cdots,s_{i_k},b)\otimes\prod_{j=0}^{k} T(s_{i_j},\cdots,s_{i_{j+1}})$$