Galois Representations with Prescribed Restriction to Inertia

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Basic question:

Let F be a totally real number field. For each prime $\mathfrak{p}$ of F, let $I_{\mathfrak{p}} \subset \mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ be the inertia group. Given a family of continuous complex representations $\rho_{\mathfrak{p}}\colon I_{\mathfrak{p}} \to \mathrm{GL}_2(\mathbf{C})$, when does there exist a geometric object (*e.g.*, a GL_2-abelian variety over F) whose associated Galois representation restricts to each $I_{\mathfrak{p}}$ as $\rho_{\mathfrak{p}}$?

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1. Review of Weil-Deligne representations

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Notation, Weil-Deligne Representations

Let K be a p-adic field, with residue field $\mathbf{F}_q$, and let $G_K = \mathrm{Gal}(\overline{K}/K)$. The Weil group W_K fits into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & \hat{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Local class field theory furnishes an isomorphism $\mathrm{Art} : K^* \tilde{\rightarrow} W_K^{\mathrm{ab}}$, which sends a uniformizer in $\mathcal{O}_K$ to a geometric Frobenius element Φ. We define an absolute value $|\ |$ on W_K^* by $|\sigma| = \left|\mathrm{Art}^{-1}(\sigma)\right|_K$.

A *Weil-Deligne representation* of K is a pair (ρ, N), where

$$\rho \colon W_K \to \mathrm{GL}_n(\mathbf{C})$$

is semisimple and continuous and $\mathrm{N} \in \mathrm{End}\,\mathbf{C}^n$ satisfies

$$\rho(\sigma)\,\mathrm{N}\,\rho(\sigma)^{-1} = |\sigma|\,\mathrm{N},\ \sigma \in W_K$$

(N is the "monodromy operator"; it is necessarily nilpotent.)

In the wild, we normally encounter l-adic Galois representations. The formalism of the Weil-Deligne representation allows us to pass between the l-adic and complex worlds, at least when $l \neq p$:

Theorem. (Grothendieck) *For $l \neq p$, there exists a bijection*

$$\left\{\begin{matrix}\text{WD reps } (\rho, \mathsf{N}) \\ \text{of dimension } n\end{matrix}\right\} \tilde{\to} \left\{\begin{matrix}\text{continuous reps} \\ \rho_l \colon W_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)\end{matrix}\right\}$$

Here one makes use of an isomorphism $\iota \colon \mathbf{C} \tilde{\to} \overline{\mathbf{Q}}_l$, whence (ρ, N) and ρ_l are related by

$$\rho_l(\sigma) = \iota(\rho(\sigma)) \exp(t_l(\sigma)\iota(\mathsf{N})),$$

for σ in an open subgroup of I_K. Here $t_l \colon I_K \to \mathbf{Q}_l$ is an l-adic tame character.

(Thus N measures how far away ρ_l is from being semisimple. Sometimes we refer to the pair (ρ, N) as simply ρ.)

Now let $A/\mathbf{Q}$ be an abelian variety of GL_2-type. This means there is a number field E with $[E : \mathbf{Q}] = \dim A$ and an action $E \hookrightarrow \mathrm{End}_{\mathbf{Q}} A \otimes_{\mathbf{Z}} \mathbf{Q}$. Then for each prime $\lambda | l$ in E, the action of Galois on the λ-adic Tate module

$$T_\lambda A := \lim_{\infty \leftarrow n} A[\lambda^n]$$

gives a representation

$$\begin{aligned} \rho_{A,\lambda} \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}_{E_\lambda} T_\lambda A \quad &\approx \quad \mathrm{GL}_2(E_\lambda) \\ &\hookrightarrow \quad \mathrm{GL}_2(\overline{\mathbf{Q}}_l) \end{aligned}$$

Previous theorem yields a family of Weil-Deligne representations

$$\rho_{A,p} \colon W_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{C})$$

for each prime p. The restrictions to inertia

$$\rho_{A,p}|_{I_{\mathbf{Q}_p}} \colon I_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{C})$$

are trivial at all but finitely many primes. The $\left\{\rho_{A,p}\right\}$ might be called the "bad reduction type" of A.

Example: Elliptic Curves

If $A = E$ is an elliptic curve and $p > 2$, there are three possibilities for the restriction of the WD representation $(\rho_{A,p}, \mathsf{N})$ to $I_{\mathbf{Q}_p}$.

- E has good reduction at p. Then $\rho_{E,p} = 1 \oplus 1$ and $\mathsf{N} = 0$.

- E has multiplicative reduction at p. Then $\rho_{E,p} = 1 \oplus 1$ and $\mathsf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

- E has additive reduction at p. Then either:

  - E is a twist of a curve of multiplicative reduction by a ramified quadratic character χ, so that $\rho_{E,p} = \chi \oplus \chi$ and $\mathsf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, or

  - E is not such a twist: then there is a quadratic extension $L/\mathbf{Q}_p$, and a character $\theta\colon W_L \to \mathbf{C}^*$, for which $\rho_{E,p}$ is the restriction to $I_{\mathbf{Q}_p}$ of $\mathrm{Ind}_{W_L}^{W_K} \theta$. ($\mathsf{N} = 0$.)

We ask: Suppose we have a family of inertial representations

$$\rho_p \colon I_{\mathbf{Q}_p} \to \mathrm{GL}_2(\mathbf{C})$$

such that

1. Each ρ_p admits an extension to a Weil-Deligne representation of K,

2. All but finitely many ρ_p are trivial,

3. $\prod_p \det \rho_p(\mathrm{Art}(-1)) = 1$.

Then does there exist an abelian variety $A/\mathbf{Q}$ of GL_2 type for which $\rho_{A,p}|_{I_p} \approx \rho_p$?

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Theorem. Such an A exists for all but finitely many families $\{\rho_p\}$ and their twists.

In fact, one can give a formula for the number of *modular* abelian varieties A which match $\{\rho_p\}$ this way. (It is conjectured that all abelian varieties of GL_2 type are modular.)

The Langlands Correspondence for $\mathsf{GL}_n(K)$

Let $\mathsf{Irr}(\mathsf{GL}_n(K))$ denote the set of isomorphism classes of smooth admissible irreducible complex representations of $\mathsf{GL}_n(K)$.

Ex. For $n = 1$, an irreducible representation of $\mathsf{GL}_1(K) = K^*$ is just a homomorphism $K^* \to \mathbf{C}^*$. But we have the isomorphism $\mathsf{Art} : K^* \to W_K^{\mathsf{ab}}$, so $\mathsf{Irr}(\mathsf{GL}_1(K))$ is identified with one-dimensional WD representations of K.

The generalization of this to $n \geq 1$ is embodied in the

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$$\mathsf{rec} : \{n\text{-dim'l WD reps of } K\} \xrightarrow{\sim} \mathsf{Irr}(\mathsf{GL}_n(K))$$

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For $n = 1$, this is local class field theory, established by Hasse (1930).

The existence of rec as above was conjectured by Langlands (1970).

For $n = 2$, the theorem was established by Kutzko (1980).

Proofs for all n were given by Harris/Taylor (1998) (local methods) and Henniart (2000) (global methods).

The Category of Inertial WD representations Let K once again be a local field. An *inertial WD representation* of K is a pair (ρ, N) consisting of a continuous rep. $\rho \colon I_K \to \mathsf{GL}_n(\mathbf{C})$ and an endomorphism $\mathsf{N} \in \mathsf{End}\, \mathbf{C}^n$ such that ρ admits an extension to $\tilde{\rho} \colon W_K \to \mathsf{GL}_n(\mathbf{C})$ which makes $(\tilde{\rho}, \mathsf{N})$ a WD representation.

Ex. For $n = 1$, a WD rep. of W_K is a map $K^* \to \mathbf{C}^*$ (class field theory). The image of I_K in $W_K^{\mathsf{ab}} \to K^*$ is $\mathcal{O}_K^*$, so an inertial WD representation of K is simply a homomorphism $\mathsf{GL}_1(\mathcal{O}_K) \to \mathsf{GL}_1(\mathbf{C})$.

The generalization of this to higher dimension should put into correspondence inertial WD representations and certain irreducible representations of $\mathsf{GL}_n(\mathcal{O}_K)$.

An inertial Langlands Correspondence?

Let $\mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}_K))$ denote the set of isomorphism classes of irreducible representations of $\mathrm{GL}_n(K)$ which are *smooth*, *i.e.* which factor through $\mathrm{GL}_n(\mathcal{O}_K/\mathfrak{p}^N)$ for some N.

Conjecture: There is a diagram

$$\begin{array}{ccc} \{\text{WD reps of } K\} & \xrightarrow[\sim]{\text{rec}} & \mathrm{Irr}(\mathrm{GL}_n(K)) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow \\ \{\text{IWD reps of } K\} & \xrightarrow{\sim} & \mathrm{Irr}(\mathrm{GL}_n(\mathcal{O}_K)) \end{array}$$

Theorem Conj. is true for $n = 2$.

Proof relies on explicit classification of elements of $\mathrm{Irr}(\mathrm{GL}_2(K))$ and is not very enlightening.

Need to describe map

$$\mathrm{rec}: \mathrm{Irr}(\mathrm{GL}_2(K)) \to \mathrm{Irr}(\mathrm{GL}_2(\mathcal{O}_K)).$$

Let $\Pi\colon \mathrm{GL}_2(K) \to \mathrm{Aut}\, V$ be an element of $\mathrm{Irr}(\mathrm{GL}_2(\mathcal{O}_K))$. Then there is a unique (up to scalar) *new vector* $v \in V$, fixed by the largest subgroup of the form

$$\Gamma_1(\mathfrak{p}_K^n) := \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ (\mathfrak{p}_K^n) \right\} \subset \mathrm{GL}_2(\mathcal{O}_K).$$

Let π be the $\mathrm{GL}_2(\mathcal{O}_K)$-span of v.

Then π is the image of Π in our map. Note that π is necessarily finite dimensional.

Back to the original problem. Let $\{\rho_p\}$ be a collection of inertial WD representations of $\mathbf{Q}_p$ for each prime p, almost all of which are trivial. Then by "inertial Langlands correspondence" get family of finite-dimensional representations

$$\pi_p := \mathrm{rec}(\rho_p)\colon\ \mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{Aut}\, V_p.$$

Tensor these together to form

$$\pi := \otimes_p \pi_p\colon\ \mathrm{GL}_2(\hat{\mathbf{Z}}) \to \mathrm{Aut}\, V,$$

which is once again finite dimensional.

The condition $\prod_p \det \rho_p(\mathrm{Art}(-1)) = 1$ means that the action of $\mathrm{GL}_2(\hat{\mathbf{Z}})$ factors through $G := \mathrm{GL}_2(\hat{\mathbf{Z}})/\{\pm 1\}$.

We are on the hunt for a abelian variety A of GL_2 type whose "bad reduction type" is $\{\rho_p\}$. A supply of GL_2 abelian varieties is given by cuspidal eigenforms

$$f \in S_2(X(\infty), \mathbf{C}) \circlearrowleft \mathrm{GL}_2(\hat{\mathbf{Z}})/\{\pm 1\}$$

Each such f yields an abelian variety quotient A_f of $J(N)$ (Shimura). How to read off the bad reduction type of A_f from f?

Claim. Let f be a weight 2 cuspidal eigenform. The bad reduction type of A_f is $\{\rho_p\}$ iff the $\mathrm{GL}_2(\hat{\mathbf{Z}})$-span of f is $\cong$ to $\pi = \otimes_p \mathrm{rec}(\rho_p)$.

Explanation. The automorphic form attached to f is a square-integrable function

$$\phi_f\colon \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbf{Q}}) \to \mathbf{C}.$$

The *automorphic representation* Π_f associated to f is the $\mathrm{GL}_2(\mathbb{A}_{\mathbf{Q}})$-span of ϕ_f. Then

$$\Pi_f = \otimes_p \Pi_{f,p}$$

and we have

$$\mathrm{rec}(\rho_{A,p}) \cong \Pi_{f,p}.$$

If f starts out life as a newform, then $\phi_f|_{\mathrm{GL}_2(\mathbf{Q}_p)}$ is a new vector for $\Pi_{f,p}$. Let π_f be the $\mathrm{GL}_2(\hat{\mathbf{Z}})$-span of ϕ, with $\phi_f = \otimes_p \pi_{f,p}$. Then by commutativity of the "reciprocity diagram" get

$$\mathrm{rec}(\rho_{A,p}|_{I_{\mathbf{Q}_p}}) \cong \pi_{f,p}.$$

as required.

So, we are trying to count how many times $\pi = \otimes_p \mathrm{rec}(\rho_p)$ appears as a $\mathrm{GL}_2(\hat{\mathbf{Z}})$-module inside of $S_2(X(\infty), \mathbf{C})$.

The Equivariant Riemann-Roch Formula

Context: Let X be a smooth projective (pro-) curve over an algebraically closed field k admitting a faithful action by a (pro-)finite group G, with quotient Y. Let $\mathcal{L}$ be a G-equivariant invertible sheaf on X, meaning that for each $g \in G$, there is an isomorphism $\phi_g \colon g_*\mathcal{L} \to \mathcal{L}$ satisfying an appropriate cocycle condition. (Such bundles can be identified with classes of G-invariant divisors on X.)

Then G acts on the cohomology groups $H^i(X, \mathcal{L})$ $(i = 0, 1)$. Can define the Euler characteristic

$$\chi(\mathcal{L}) = [H^0(X, \mathcal{L})] - [H^1(X, \mathcal{L})]$$

in the ring $R_k(G)$ of virtual representations of G over k.

Can also define the "equivariant degree" $\deg_{\mathrm{eq}} \mathcal{L} \in R_k(G)$, a virtual representation whose dimension is $\deg \mathcal{L}$.

Theorem (Borne, 2003) The equality

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \mathsf{deg}_{\mathsf{eq}}\,\mathcal{L}$$

holds in $R_k(G)$.

Further, suppose the quotient map $X \to Y$ is ramified over the set Y_{ram}. Then

$$\mathsf{deg}_{\mathsf{eq}}\,\Omega_X = (2g_Y - 2 + \#Y_{\mathsf{ram}})\,[k[G]] - [X_{\mathsf{ram}}],$$

where X_{ram} is the permutation representation on the ramified points of X in $X \to Y$.

We apply the theorem to the data

$$\begin{array}{c} X(\infty) \\ \Big\downarrow {\scriptstyle G:=\mathsf{GL}_2(\hat{\mathbf{Z}})/\pm 1} \\ X(1) \end{array} .$$

This map is ramified over the points 1728, 0, and ∞, with inertia groups $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$, and $\begin{pmatrix} 1 & \hat{\mathbf{Z}} \\ 0 & 1 \end{pmatrix}$, respectively. We are interested in the canonical bundle $\Omega_{X(\infty)}$ because

$$H^0(X(\infty), \Omega_{X(\infty)}) = S_2(X(\infty), \mathbf{C}).$$

The Riemann-Roch theorem gives

$$2\left([S_2(X(\infty))] - \left[H^0(\mathcal{O}_{X(\infty)})\right]\right) = \deg \Omega_{X(\infty)}$$

Meanwhile, a little calculation yields

$$\begin{aligned}\deg \Omega_{X(\infty)} &= \mathbf{C}[G] - \mathbf{C}[G/G_0] \\ &\quad -\mathbf{C}[G/G_{1728}] - \mathbf{C}[G/G_\infty],\end{aligned}$$

where G_0 is the stabilizer of a point in $X(\infty)$ above 0, etc.

Conclusion: If $\rho = \{\rho_p\}$ is a collection of inertial WD representations which are almost all trivial and which satisfy $\prod_p \det \rho_p(\mathrm{rec}(-1)) = 1$, then

$$\#\left\{\begin{array}{c}\text{Modular AVs with} \\ \text{bad reduction type } \{\rho_p\}\end{array}\right\} = \frac{1}{12}\dim V_\rho + o(1),$$

where $V_\rho = \otimes_p \mathrm{rec}(\rho_p)$. (The $o(1)$ is as $\dim V_\rho \to \infty$; up to twisting there are only finitely many ρ with bounded $\dim V_\rho$.)