

# Three modular curve-like properties of Modular Towers (MTs)

Part I. Types of Cusps on curve components

Part II. Compare modular curve cusps with cusp types on all MT levels [Fr05c, §3.2]

Part III. Where the Main Conjecture stands with  $r = 4$

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## Dihedral Analogy

Modular curve towers for a prime  $p$  are to MTs for  $p$  as the dihedral group  $D_p$  is to all  $p$ -perfect finite groups. For  $p$ -perfect  $G$ ,  $p'$  conjugacy classes  $\mathbf{C} = \{C_1, \dots, C_r\}$ , have string of tower levels:

$$(TS) \quad \dots \rightarrow \mathcal{H}_{k+1}^{\text{in,rd}} \rightarrow H_k^{\text{in,rd}} \rightarrow \dots \rightarrow \mathbb{P}_j^1 \setminus \{\infty\} \stackrel{\text{def}}{=} U_j.$$

With  $r = 4$ , use these inputs for conclusions:

1. Frattini Principles, FP1 and FP2.
2. Notions of  $j$ -line covers, Riemann-Hurwitz, reduced Hurwitz spaces.

## Known MT Properties for $r = 4$

- (Proven)  $\mathbb{P}_j^1$  covers: All levels are curves, moduli spaces covering the  $j$ -line  $\mathbb{P}_j^1$  ramified at three ( $j = 0, 1, \infty$ ) points, and upper half plane quotients by a finite index subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$ .
- (Nearly Proven) Main Conj ( $K$  number field): Let

$$(\mathrm{TS}) \quad \cdots \rightarrow \bar{\mathcal{H}}'_{k+1} \rightarrow \bar{H}'_k \rightarrow \cdots \rightarrow \mathbb{P}_j^1$$

be a projective sequence of (compactified) components on (TS) over  $K$  (a  $\mathrm{PSC}_K$ ). Then, excluding **cusps**, level  $k \gg 0$  has no  $K$  points.

## Part I. Types of Cusps on curve components

Absolute *Nielsen classes*  $\text{Ni}(G_k, \mathbf{C})^{\text{abs}}$  :

$$\{\mathbf{g} = (g_1, \dots, g_4) \in \mathbf{C} \pmod{N_{S_n}(G_k)}\}$$

(for *inner* classes  $\pmod{G_k}$ ) with

- $\text{Cond}^1$  – Generation:  $\langle g_1, g_2, g_3, g_4 \rangle = G_0$ ;
- $\text{Cond}^2$  – Product-one:  $g_1 g_2 g_3 g_4 = 1$ .

*Twist* action of  $H_4 = \langle q_1, q_2, q_3 \rangle$  generators on  $\mathbf{g} \in \text{Ni}(G_k, \mathbf{C})^{\text{abs}}$ . Ex.:  $q_2 : \mathbf{g} \mapsto (g_1, g_2 g_3 g_2^{-1}, g_2, g_4)$ .

*Cusps*:  $\text{Cu}_4 \stackrel{\text{def}}{=} \langle q_1 q_3^{-1}, (q_1 q_2 q_3)^2, q_2 \rangle$  orbits.

Let  $\mathcal{Q}'' = \langle q_1 q_3^{-1}, (q_1 q_2 q_3)^2 \rangle$ .

Why  $\bar{M}_4 \stackrel{\text{def}}{=} H_4/Q''$  is  $\text{PSL}_2(\mathbb{Z})$ !

- $q_2 \mapsto \gamma_\infty$ ;
- $q_1q_2q_3$  (shift)  $\mapsto \gamma_1$  (order 2).
- $q_1q_2 \mapsto \gamma_0$  has order 3, from braid relation  $q_1q_2q_1 = q_2q_1q_2 \pmod{Cu_4}$  and Hurwitz relation  $1 = q_1q_2q_3q_3q_2q_1$ :

$$= q_1q_2q_1q_1q_2q_1 = q_1q_2q_1q_2q_1q_2 = (q_1q_2)^3.$$

## Example of computing component genera

From a  $\text{PSC}_K$ , in (CS), what to compute:

- Nature of cusps and their widths (length of  $\text{Cu}_4 \bmod Q''$  orbits).
- How they fall in  $\bar{M}_4$  orbits and of what genera (Riemann-Hurwitz).

## Modular curves $X_0(p^{k+1})$ [Fr05d, Talk #1]

Use  $b \Leftrightarrow \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} \in D_{p^{k+1}}$ . So,

$\mathbf{g} \in \text{Ni}_k \Leftrightarrow (b_1, \dots, b_4) \in (\mathbb{Z}/p^{k+1})^4$ .

Conjugate by power of  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  to assume

$b_1 = 0$  and  $b_2 - b_3 + b_4 = 0$ .

Normalizing: Have  $b_2 - b_3 = ap^u$   $u \geq 0$ ,  
 $a \in \mathbb{Z}/p^{k+1-u}$  and  $(a, p) = 1$ .

For  $\text{Ni}^{\text{abs}}$ , conjugate by  $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$  so  $a = 1$ .

Allows further conjugation by

$$H_u = \{\alpha = 1 + bp^{k+1-u} \in \mathbb{Z}/p^{k+1} \pmod{p^u}, b \in \mathbb{Z}/p^u\}.$$

Take  $c = b_2$ ,  $b_3 = c - p^u$  ( $u$  is a parameter).



## Dihedral group cusp computing cont.

Compute :  $(\mathbf{g})q_2^\ell \Leftrightarrow (b_2, b_3) = (c + \ell p^u, c + ((\ell - 1)p^u)$ .

For  $u = 0$ :  $(b_2, b_3) = (c, c - 1)$  has  $q_2$  orbit of width  $p^{k+1}$  containing  $\mathbf{g} = \mathbf{g}_{\text{H-M}} = (0, 0, 1, 1)$  (unique *Harbater-Mumford rep.*).

Otherwise,  $\langle \mathbf{g} \rangle = D_{p^{k+1}}$  requires  $(c, p) = 1$ . Conjugate by  $H_u$  to assume  $c \in \mathbb{Z}/p^{k+1-u}$  is  $p'$ ,  $u > 0$ : Width =  $|\text{residues mod } p^{k+1-u} \text{ differing by multiples of } p^u|$ .

**Conclude:**  $\varphi(p^{k+1-u})$  Nielsen class elements fall in  $\text{Cu}_4$  orbits of width  $p^{k+1-2u}$  (resp. 1) if  $k+1-2u \geq 0$  (resp.  $k+1-2u < 0$ ).

**Other extreme,  $u = k + 1$ :**  $(b_2, b_3) = (1, 1)$ , the shift of an H-M rep. (orbit width 1).

Part II: Compare modular curve cusps with cusp types on all MT levels [Fr05c, §3.2]

When  $r = 4$ , MT levels ( $k \geq 0$ ) are upper half-plane quotients covering the classical  $j$ -line. Rarely modular curves.

With  $r = 4$ ,  $\mathbf{g} \in \text{Ni}(G, \mathbf{C})^{\text{in}}$ , denote:

$$\langle g_2, g_3 \rangle = H_{2,3}(\mathbf{g}) \text{ and } \langle g_1, g_4 \rangle = H_{1,4}(\mathbf{g}).$$

For  $u \neq k + 1$ , all  $\mathbf{g}$  define  $p$  cusps:  $p \mid \text{ord}(g_2 g_3)$ , and  $p$  divides all inner space cusp widths.

For  $u = k + 1$ :  $(0, c, c, 0) = (b''_1, b''_2, b''_3, b''_4)$  (shift of H-M case) has inner space cusp width ( $= 1$ ) prime to  $p$ . Generalizing property,  $(\mathbf{g})\text{Cu}_4$  is a  $g$ - $p'$  cusp:  $H_{2,3}(\mathbf{g})$  and  $H_{1,4}(\mathbf{g})$  are  $p'$  groups:

Finally:  $o(nly)$ - $p'$  is the phrase for those cusps neither  $p$  nor  $g$ - $p'$ . Modular curves have none.

## Apply R-H to MT components

$\text{Ni}'$  is a  $\bar{M}_4$  orbit on a reduced Nielsen class  $\text{Ni}(G, \mathbf{C})^{\text{abs}} / \mathcal{Q}''$  (or  $\text{Ni}(G, \mathbf{C})^{\text{in}} / \mathcal{Q}''$ ). Denote action of  $(\gamma_0, \gamma_1, \gamma_\infty)$  (p. 9) on  $\text{Ni}'$  by  $(\gamma'_0, \gamma'_1, \gamma'_\infty)$ : Branch cycles for a cover  $\bar{\mathcal{H}}' \rightarrow \mathbb{P}_j^1$ ,

R-H gives genus,  $g_{\bar{\mathcal{H}}'}$ :

$$2(\deg(\bar{\mathcal{H}}' / \mathbb{P}_j^1) + g' - 1) = \text{ind}(\gamma'_0) + \text{ind}(\gamma'_1) + \text{ind}(\gamma'_\infty).$$

To compute genera of components in a MT  
answer these questions

- What are the components  $\bar{\mathcal{H}}'_k$  of  $\bar{\mathcal{H}}_k$  ( $\bar{M}_4$  orbits  $\text{Ni}'_k$  on  $\text{Ni}_k^{\text{rd}}$ )?
- What are the cusp widths (ramification orders over  $\infty$ ; orbit lengths of  $\gamma'_\infty$  on  $\text{Ni}'_k$ )?
- What points ramify in each component over elliptic points  $j = 0$  or  $1$ ; length 3 (resp. 2) orbits of  $\gamma'_0$  (resp.  $\gamma'_1$ ) on  $\text{Ni}'_k$ ?

### Part III. Where is the Main Conjecture with $r = 4$ ?

Let  $B' = \{\mathcal{H}'_k\}_{k=0}^{\infty}$  be an infinite component branch. Possible Main Conj. contradictions:

1.  $g_{\bar{\mathcal{H}}'_k} = 0$  for all  $0 \leq k < \infty$   
( $B'$  has genus 0;  $g_{B'}$  consists of 0's); or
2. For  $k$  large,  $g_{\bar{\mathcal{H}}'_k} = 1$   
( $B'$  has genus 1; almost all of  $g_{B'}$  is 1's).

Reduction to the case the center of  $G$  is  $p'$ : Then, FP1  $\implies$  Every point at level  $k + 1$  over a  $p$  cusp at level  $k$  is ramified (of order  $p$ ).

Example use: From R-H, for  $k \gg 0$ , (2) implies  $\bar{\mathcal{H}}'_{k+1} \rightarrow \bar{\mathcal{H}}'_k$  doesn't ramify. So, FP1 says: For no  $k$  does  $\bar{\mathcal{H}}'_k$  have a  $p$  cusp.



## Possible exceptional cases! [Fr05c, §5]

Assume  $\mathbf{p}'_k \in \bar{\mathcal{H}}'_k$  is a  $p$  cusp (some  $k$ ). Denote:  
 $\deg(\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k) = \nu_k$  and  $|\mathbf{p}'_{k+1} \in \bar{\mathcal{H}}'_{k+1} \text{ over } \mathbf{p}'_k| = u_k$ .

**Theorem 1.** *The Main Conj. is true unless for  $k \gg 0$ ,  $\nu_k = p$ ,  $u_k = 1$  and  $\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k$  is equivalent (as a cover over  $K$ ) to either:*

1.  $(\mathbf{P}^{\text{oly}}\mathbf{M})$  a degree  $p$  polynomial map; or
2.  $(\mathbf{R}^{\text{edi}}\mathbf{M})$  a degree  $p$  rational function ramified precisely over two  $K$  conjugate points.

**Corollary 2.** *If neither  $(P^{\text{oly}}M)$  nor  $(R^{\text{edi}}M)$  hold for the component branch  $B'$ , then high levels of  $B'$  have no  $K$  points.*

*For  $B'$  with full elliptic ramification (includes when  $B'$  has fine reduced moduli) for  $k \gg 0$ , the Main Conj. holds unless  $(R^{\text{edi}}M)$  holds.*

## Part IV. What happens in real cases!

- Main point to finish Main Conjecture for  $r = 4$ :  
Find  $p$  cusps at high levels.
- If the  $\limsup$  of  $\deg(\bar{\mathcal{H}}'_{k+1}/\bar{\mathcal{H}}'_k)$  is *not*  $p$ , one  $p$  cusp guarantees the  $p$  cusp count (at level  $k$ ) is unbounded as  $k \mapsto \infty$ .

The case  $(A_5, \mathbf{C}_{3^4}, p = 2)$  (four 3-cycles):

- Level 0:  $\mathcal{H}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$  has one component, and no  $p$  ( $=2$ ) cusps.
- Apply lift invariant for  $\text{Spin}_5 \rightarrow A_5$  (App<sub>2</sub>): Shows all level 1 comps. have  $p$  ( $= 2$ ) cusps [BFr02, Cor. 8.3] (Fr-Se formula).
- Level 1 [BFr02, Prop. 9.14]: Two components ( $\bar{M}_4$  orbits,  $\text{Ni}_{1,\pm}$ ), distinguished by embedding  $G_1(A_5) \leq A_{40}$  giving  $s_{\text{Spin}_{40}}(\mathbf{g}) = \pm 1$  depending on  $\mathbf{g} \in \text{Ni}_{1,\pm}$ .

On compactification  $\bar{\mathcal{H}}_+$  of  $\mathcal{H}_+(G_1(A_5), \mathbf{C}_{34})^{\text{in,rd}}$ :

- Contains all H-M cusps (FP2  $\implies$   ${}_2\tilde{G}$  is a limit group for a comp. branch over it).
- Has genus 12 and degree 16 over the unique component of  $\bar{\mathcal{H}}(A_5, \mathbf{C}_{34})^{\text{in,rd}}$ .
- Has all the real (and so all the  $\mathbb{Q}$ ) points at level 1 [BFr02, §8.6]. On its compactification  $\bar{\mathcal{H}}_+$ ,  $\bar{\mathcal{H}}_+(\mathbb{R})$  is connected. All except the shift of the H-M cusps are 2 cusps.

On compactification  $\bar{\mathcal{H}}_-$  of  $\mathcal{H}_-(G_1(A_5), \mathbf{C}_{34})^{\text{in,rd}}$ :

- Has genus 9, but no real points.
- Because of the lifting invariant, nothing above it at level 2:  ${}_2\tilde{G}(A_5)$  (the whole 2-Frattini cover of  $A_5$ ) is not a limit group.

Higher ( $A_5, \mathbf{C}_{34}, p = 2$ ) levels: modular curve-like  
cusp properties

Let  $\{\mathcal{H}'_k\}_{k=0}^{\infty}$  be an H-M comp. branch (FP2).

**Proposition 3.** *On all  $\bar{\mathcal{H}}'_k$ ,  $g$ - $p'$  cusps are H-M. It has no  $o$ - $p'$  cusps [Fr05c, Prop. 3.12]. Number of  $p$  cusps on  $\mathcal{H}'_k \mapsto \infty$ .*

**Uses a General Idea:** Let  $B = \{\mathbf{p}_k\}_{k=0}^{\infty}$  be a  $g$ - $p'$  cusp branch. Assume for each  $k \geq k_0$ ,  $\mathbf{p}_k$  braids to a  $p$  cusp  $\mathbf{p}'_k$  with ramification index exactly divisible by  $p$ . Then, FP1 allows, with  $k = k_0 + u$ , inductively braiding  $\mathbf{p}_k$  to a sequence of cusps  $\mathbf{p}'_k(1), \dots, \mathbf{p}'_k(u)$  with  $\mathbf{p}'_k(t)$  having ramification index exactly divisible by  $p^t$ ,  $u = 1, \dots, t$ .

From their ramification indices over  $j = \infty$ , these give  $u$  different  $p$  cusps at level  $k_0 + u$ .

For  $\text{Ni}(G_k(A_5), \mathbf{C}_{34})$  take  $k_0 = 1$ :  $\mathbf{p}'_k$  is produced as the *near* H-M rep. associated to  $\mathbf{p}_k$  [BFr02, Prop. 6.8].



$A_n$  examples of two braid orbits from lifting inv.

**Example 4 ( $A_n$  and 3-cycles).** For each pair  $(n, r)$  with  $r \geq n$ , there are exactly two braid orbits on  $\text{Ni}(A_n, \mathbf{C}_{3^r})$ . One contains a  $g$ -2' representative and the other is obstructed at level 0. Braid orbit reps for  $n = r = 4$ :

$$\begin{aligned} \mathbf{g}_{4,+} &= ((1\ 3\ 4), (1\ 4\ 3), (1\ 2\ 3), (1\ 3\ 2)), \\ \mathbf{g}_{4,-} &= ((1\ 2\ 3), (1\ 3\ 4), (1\ 2\ 4), (1\ 2\ 4)). \end{aligned}$$

## Nonbraidable, isomorphic $M_{\tilde{g}}$

Suppose two extensions  $M_{g_i} \rightarrow G$ , arise from  $g_i \in \text{Ni}(G, \mathbf{C})$ ,  $i = 1, 2$ . Assume they are isomorphic. Still might not be braidable.

The Nielsen class  $\text{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$  has six braid orbits. Two extensions correspond to the two H-M components called  $\mathcal{H}_1^{+, \beta}$ ,  $\mathcal{H}_1^{+, \beta^{-1}}$ . An *outer* automorphism of  $G_1(A_4)$  takes  $g_1$  to  $g_2$ , giving elements in different braid orbits. These are H-M components, so *FP2* gives isomorphic extensions  $M_{g_i} \rightarrow {}_p\tilde{G}$ ,  $i = 1, 2$  in distinct braid orbits.

## Part V. Generalizing Serre's OIT and the $g$ - $p'$ conjecture

Stay with  $r = 4$  to simplify notation.

1. Why you expect a  $\text{PSC}_K$  for some number field  $K$  only if you have a  $g$ - $p'$  cusp.
2. Generalize in  $(G, \mathbf{C}, p)$  to allow many primes. **Use higher rank MTs:** a group  $H$  ( $\mathbf{C}$  are classes in  $H$ ) acting on either a free group or a lattice  $L$ , and for all allowable  $p$  look at  $(L/pL \times^s H, \mathbf{C}, p)$ .

3. Decide when you can inductively find infinitely many points corresponding to “complex multiplication, ” (i.e. prediction of full Galois image for the fiber over  $j_0 \in U_\infty$ ).
4. Where (when?) are the Hecke operators?

Topics (2) and (3) are in [Fr05c, §6], with extensive examples comparing modular curve to the general case. My NSF proposal outline how topics (1) and (4) work. These will be in my RIMS talk in October.

(Lots of evidence for)  $g$ - $p'$  Conjecture: Each  $\text{PSC}_K$  is defined by a cusp sequence called  $g$ - $p'$ . Their **shifts** often resemble sequences of *width*  $p^{k+1}$  *cusps* on  $\{X_0(p^{k+1})\}_{k=0}^{\infty}$ ; moduli interpretation generalizing Tate elliptic curve.

## App. A<sub>2</sub>: A Formula for Spin-Lift Invariant

For  $g \in A_n$  of odd order, let  $w(g)$  be the sum of  $(l^2 - 1)/8 \pmod 2$  over all disjoint cycle lengths  $l$  in  $g$  ( $l \not\equiv \pm 1 \pmod 8$  contribute).

**Theorem 5 (Fried-Serre).** *If  $\varphi : X \rightarrow \mathbb{P}^1$  is in Nielsen class  $\text{Ni}(A_n, \mathbf{C}_{3n-1})^{\text{abs}}$ , then  $\deg(\varphi) = n$ ,  $X$  has genus 0, and  $s(\varphi) = (-1)^{n-1}$ .*

*Generally, for any genus 0 Nielsen class of odd order elements, and representing  $\mathbf{g} = (g_1, \dots, g_r)$ ,  $s(\mathbf{g})$  is constant, equal to  $(-1)^{\sum_{i=1}^r w(g_i)}$ .*

**Meaning:** Let  $\hat{X} \rightarrow \mathbb{P}_z^1$  be Galois closure of  $\varphi$ . Then,  $s(\varphi) = 1 \implies \exists \mu : Y \rightarrow \hat{X}$  unramified, so  $\varphi \circ \mu$  is Galois with group  $G \times_{A_n} \text{Spin}_n$ .

**Exercise:** Genus 0 assumption doesn't apply to

$$\mathbf{g}_1 = ((1\ 2\ 3)^{(3)}, (1\ 4\ 5)^{(3)}), \text{ or to}$$

$$\mathbf{g}_2 = ((1\ 2\ 3)^{(3)}, (1\ 3\ 4), (1\ 4\ 5), (1\ 5\ 3)),$$

but you can easily compute  $s(\mathbf{g}_i)$ ,  $i = 1, 2$ .

## App. B<sub>2</sub>: **sh**-incidence Matrix for $(A_4, \mathbf{C}_{\pm 3^2})$

**Goal:** There are two components  $\bar{\mathcal{H}}_{\pm}$ . Want their branch cycle description  $(\gamma_0^{\pm}, \gamma_1^{\pm}, \gamma_{\infty}^{\pm})$  as  $j$ -line covers.

Let  $O$  be all the reduced Nielsen class reps. in a cusp orbit. Then  $(O)\mathbf{sh}$  is collection of shifts of all elements in  $O$ . If  $O_1, \dots, O_t$  is a complete list of cusp sets, then the  $(i, j)$  entry of the **sh**-incidence matrix is  $|O_i \cap (O_j)\mathbf{sh}|$ .



## Listing cusp sets and blocks for $(A_4, \mathbf{C}_{\pm 3^2})$

There are six easily computed cusp sets on  $(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$  listed in [Fr05c, §6.3.1]:

- $O_{1,1}$ : cusp orbit of an H-M rep.  $g_{1,1}$  with 3rd and 4th entries  $((1\ 3\ 4), (4\ 3\ 1))$ ;
- $O_{3,1}$ : cusp orbit of another H-M rep.,  $(g_{1,1})q_3$ ;
- $O_{1,4}$ : cusp orbit of

$$g_{1,4} = ((1\ 2\ 3), (1\ 2\ 4), (1\ 2\ 3), (1\ 2\ 4)),$$

- $O_{1,5}$ : cusp orbit of  $(g_{1,4})q_3$ , etc.

As cusp orbits and **sh** of them are easy to compute, easily get the  $6 \times 6$  **sh**-incidence matrix blocks.

Orbit	$O_{1,1}$	$O_{1,3}$	$O_{3,1}$
$O_{1,1}$	1	1	2
$O_{1,3}$	1	0	1
$O_{3,1}$	2	1	0
Orbit	$O_{1,4}$	$O_{3,4}$	$O_{3,5}$
$O_{1,4}$	2	1	1
$O_{3,4}$	1	0	0
$O_{3,5}$	1	0	0

**Lemma 6.** *In general, **sh**-incidence matrix is same as matrix from replacing  $\mathbf{sh} = \gamma_1$  by  $\gamma_0$ . Only possible elements fixed by either lie in  $\gamma_\infty$  orbits  $O$  with  $|O \cap (O)\mathbf{sh}| \neq 0|$ .*

*On  $\text{Ni}_0^+$  (resp.  $\text{Ni}_0^-$ ),  $\gamma_1$  fixes 1 (resp. no) element(s), while  $\gamma_0$  fixes none.*