PAC Fields, Hilbertian Fields and Fried-Völklein Conjecture

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PAC Fields

Pseudo Algebrically Closed Fields

Let K be a field. If each nonempty variety defined over K has a K-rational point, then K is called psudo algebraically closed field or **PAC field**.

1. Let K be a PAC field and V a variety defined over K. Then the set $V(K)$ is dense in V in Zariski K-topology. In particular K is infinite.

2. Let L be an algebraic extension of an infinite field K. Suppose every plane curve defined over K has an L-rational point. Then L is PAC.

3. **(Ershov):** Infinite algebraic extensions of finite fields are PAC fields.

4. **(Ax-Roquette):** Algebraic extension of a PAC field is a PAC field.

Minimal PAC Fields

The minimal PAC fields are PAC fields whose proper subfields are not PAC fields.

Example: Let $K = \mathbb{F}_p$ be a finite prime field. Let q be a prime number and $\mathbb{F}_p^{q^\infty} = \bigcup_{i=1}^{\infty} \mathbb{F}_{p^{q^i}}$. Therefore $\mathbb{F}_p^{q^\infty}$ is an infinte algebraic extension of $\mathbb{F}_p$ and therefore is a PAC field. It is known that

$$Gal(\mathbb{F}_p^{q^\infty}/\mathbb{F}_p) \cong \mathbb{Z}_q = \varprojlim \mathbb{Z}/q^i\mathbb{Z}.$$

Therefore each proper subfield of $\mathbb{F}_p^{q^\infty}$ is equivalent with a proper closed subgroup of $\mathbb{Z}_q$. If $\mathbb{F}_p^{q^\infty}$ has an infinite proper subfield then $\mathbb{Z}_q$ has a proper closed subgroup of infinite index but $\mathbb{Z}_q$ does not have a proper closed subgroup of infinite index. Thus the only proper subfields of $\mathbb{F}_p^{q^\infty}$ are finite fields which are not PAC. Therefore $\mathbb{F}_p^{q^\infty}$ is a minimal PAC field.

5. Every ultraproduct of PAC fields are PAC field.

6. **(Ax):** Every nonprincipal ultraproduct of distinct finite fields is a PAC field.

Valuation on PAC Fields

1. Let (K, v) be a valued field, and let K' be an algebraic extension of K. Then v has an extension v' to K'.

2. Let (K, v) be a valued field. Then (K, v) is Henselian if and only if v has a unique extension to the algebraic closure $\tilde{K}$ of K.

3. Every algebraic extension of a Henselian valued field is Henselian. Every separably closed field is Henselian with respect to each one of its valuation.

4. Every valued field (K, v) has a Henselian closure $(K^{\hbar}, v^{\hbar})$. $K^{\hbar}$ is an algebraic separable extension of K and $v^{\hbar}(K^{\hbar}) = v(K)$.

5. **(Prestel)**: Let (K, v) be a valued PAC field and let $\tilde{v}$ be an extension of v to $\tilde{K}$. Then K is $\tilde{v}$-dense in $\tilde{K}$.

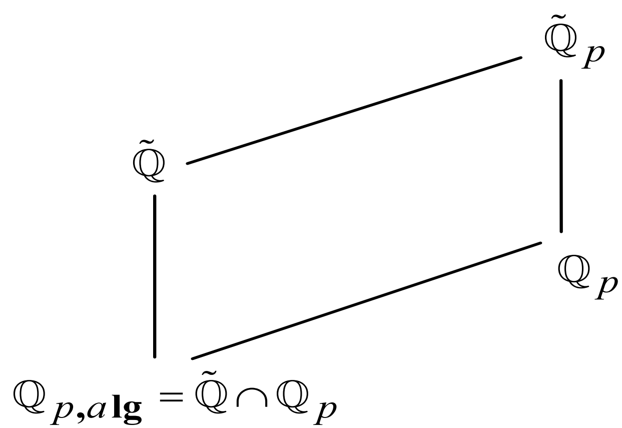
6. **(Frey-Prestel)**: Let (K, v) be a valued PAC field with Henselian closure $(K^{\hbar}, v^{\hbar})$. Then $K^{\hbar} \cong K^{sep}$, the separable closure of K. Moreover, the residue field $K^{\hbar}/v^{\hbar}$ is separably closed and value group $v(K^{\hbar})$ is a divisble group.

7. Let (K, v) be a Henselian valued field and K is not separably closed. Then K is not a PAC field.

- **The fields $\mathbb{Q}_{ab}$ and $\mathbb{Q}_{nil}$ are not PAC fields**. Let $\mathbb{Q}_{p,alg} = \tilde{\mathbb{Q}} \cap \mathbb{Q}_p$.

Since $\mathbb{Q}_p$ is complete therefore it is Henselian and every valuation on $\mathbb{Q}_p$ has a unique extension to its closure $\tilde{\mathbb{Q}}_p$. On the other hand

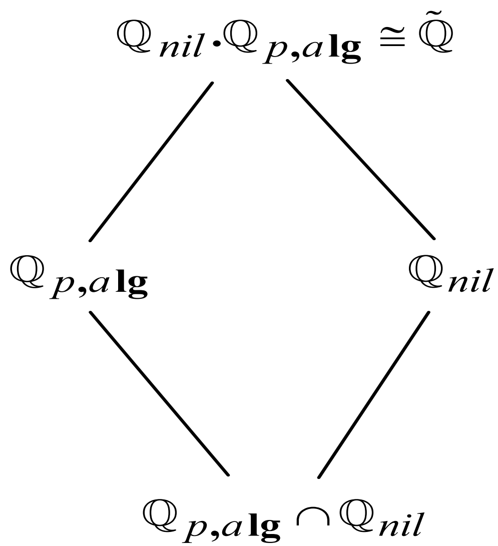
$$Gal(\tilde{\mathbb{Q}}_p/\mathbb{Q}_p) \cong Gal(\tilde{\mathbb{Q}}/\mathbb{Q}_{p,alg})$$



Therefore every valuation on $\mathbb{Q}_{p,alg}$ has also a unique extension to $\tilde{\mathbb{Q}}$ and thus $\mathbb{Q}_{p,alg}$ is Henselian. Let assume $\mathbb{Q}_{nil}$ is PAC field.

$\mathbb{Q}_{p,alg}\mathbb{Q}_{nil}$ is an algebraic extension of $\mathbb{Q}_{p,alg}$. Since $\mathbb{Q}_{p,alg}$ is Henselian therefore $\mathbb{Q}_{p,alg}\mathbb{Q}_{nil}$ is Henselian. $\mathbb{Q}_{p,alg}\mathbb{Q}_{nil}$ is an algebraic extension of $\mathbb{Q}_{nil}$ and since $\mathbb{Q}_{nil}$ is PAC therefore $\mathbb{Q}_{p,alg}\mathbb{Q}_{nil}$ is PAC. $\mathbb{Q}_{p,alg}\mathbb{Q}_{nil}$ is PAC and Henselian therefore it is separably closed. It concludes that $\mathbb{Q}_{p,alg}\mathbb{Q}_{nil} \cong \tilde{\mathbb{Q}}$. Now we have

$$Gal(\mathbb{Q}_{p,alg}) \sim Gal(\tilde{\mathbb{Q}}/\mathbb{Q}_{p,alg}) \cong Gal(\mathbb{Q}_{nil}/(\mathbb{Q}_{p,alg} \cap \mathbb{Q}_{nil}))$$

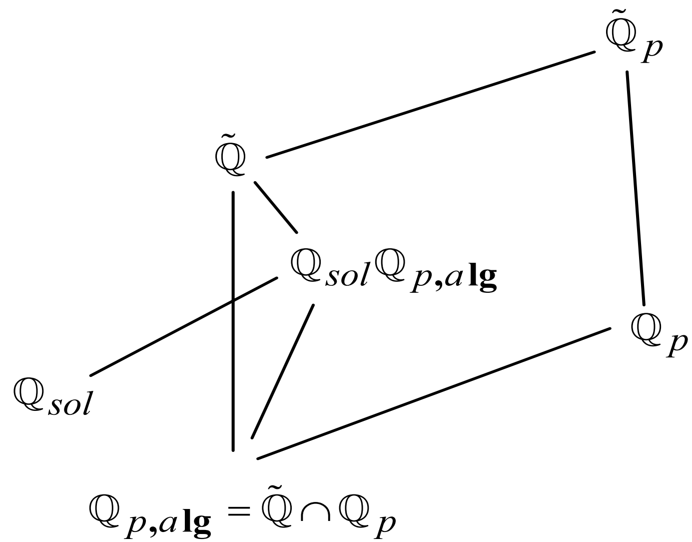


Since $Gal(\mathbb{Q}_{nil}/(\mathbb{Q}_{p,alg} \cap \mathbb{Q}_{nil}))$ is pronilpotent therefore $Gal(\mathbb{Q}_{p,alg})$ is pronilpotent. Thus Galois group of any extension of $\mathbb{Q}_{p,alg}$ is nilpotent.

Let $p = 5$. Using Eisenstein's criterion the polynomial $X^3 + 5$ is irreducible over $\mathbb{Q}_5$ and its discriminant is -27.5^2. Since $-27 \equiv 3 \pmod 5$, -27 is not a quaderatic residue modulo 5 and -27.5^2 is not a squre in $\mathbb{Q}_5$. Therefore $X^3 + 5$ is an irreducible polynomial over $\mathbb{Q}_{5,alg}$ and its discriminant is not a square in $\mathbb{Q}_{5,alg}$. Therefore $Gal(X^3 + 5, \mathbb{Q}_{5,alg})$ is S_3 which is not nilpotent. This proves that $\mathbb{Q}_{5,alg}\mathbb{Q}_{nil} \not\cong \tilde{\mathbb{Q}}$. Therefore $\mathbb{Q}_{nil}$ is not a PAC field. Since $\mathbb{Q}_{ab}$ is an algebraic extension of $\mathbb{Q}_{nil}$, thus $\mathbb{Q}_{ab}$ is not also a PAC field.

- **Is $\mathbb{Q}_{sol}$ a PAC field**?

It is known that the $Gal(\mathbb{Q}_p) = Gal(\tilde{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a prosolvable group. Therefore any finite dimensional Galois extension of $\mathbb{Q}_p$ is solvable. On the other hand there is a one to one correspondence betweeen finite Galois field extension of $\mathbb{Q}_p$ and $\mathbb{Q}_{p,alg}$. Therefore any finite Galois extension of $\mathbb{Q}_{p,alg}$ is solvable. Since $\mathbb{Q}_{sol}$ is compositum of all solvable extensions of $\mathbb{Q}$ then it is concluded that $\mathbb{Q}_{p,alg}\mathbb{Q}_{sol} \cong \tilde{\mathbb{Q}}$.



This shows that $\mathbb{Q}_{sol}^{\hbar} \cong \mathbb{Q}_{p,alg}\mathbb{Q}_{sol} \cong \tilde{\mathbb{Q}}$ and therefore all valuation on $\mathbb{Q}_{sol}$ are non-Henselian. Therefore the statement **"Let (K, v) be a Henselian valued field and K is not separably closed. Then K is not a PAC field"** is failed to prove $\mathbb{Q}_{sol}$ is not a PAC field.

8. **(Ax)**: Let K be a PAC field. Then $Gal(K)$ is projective.

9. The following statements hold for every PAC field K:

   (a) $Gal(K)$ is projective.
   (b) $Br(K)$ is trivial.
   (c) $cd(Gal(K)) \leq 1$.

Comments: The conditions (a) and (c) are equivalent on any field K. It is known that $Gal(\mathbb{Q}_{sol})$ is projective and $Br(\mathbb{Q}_{sol})$ is trivial. Therefore the conditions in (a)-(c) are failed to make a contradiction to prove $\mathbb{Q}_{sol}$ is not a PAC field.

Hilbertian Fields

Hilbert Sets and Hilbertian Fields

Let $f_1(\mathbf{T},\mathbf{X}),...,f_m(\mathbf{T},\mathbf{X})$ be polynomials in $X_1,...,X_n$ with coefficients in $K(\mathbf{T})$, $T=(T_1,...,T_r)$. Let assume that these polynomials are irreducible in the ring $K(\mathbf{T})[\mathbf{X}]$. For a non-zero plolynomial $g \in K[\mathbf{T}]$ the **Hilbert subset** $H_K(f_1,...,f_m,g)$ of K^r is defined as:

$$H_K(f_1,...,f_m,g)=\{a\in K^r \mid g(a)\neq 0, \quad f_1(a,\mathbf{X}),...,f_m(a,\mathbf{X}), \text{ are irreducible in} K[\mathbf{X}]\}$$

In addition if each f_i is separable in $\mathbf{X}$, $H_K(f_1,...,f_m,g)$ is called a **separable Hilbert subset** of K^r. Let $n=1$, a **(separable) Hilbert set** of K is defined as a (separable) Hilbert subset of K^r for some positive integer r. A field K is called **Hilbertian** if each separable set of K is non-empty.

1. Each separable Hilbertian set of K^r is dense in K^r. Therefore each Hilbertian field is infinite.

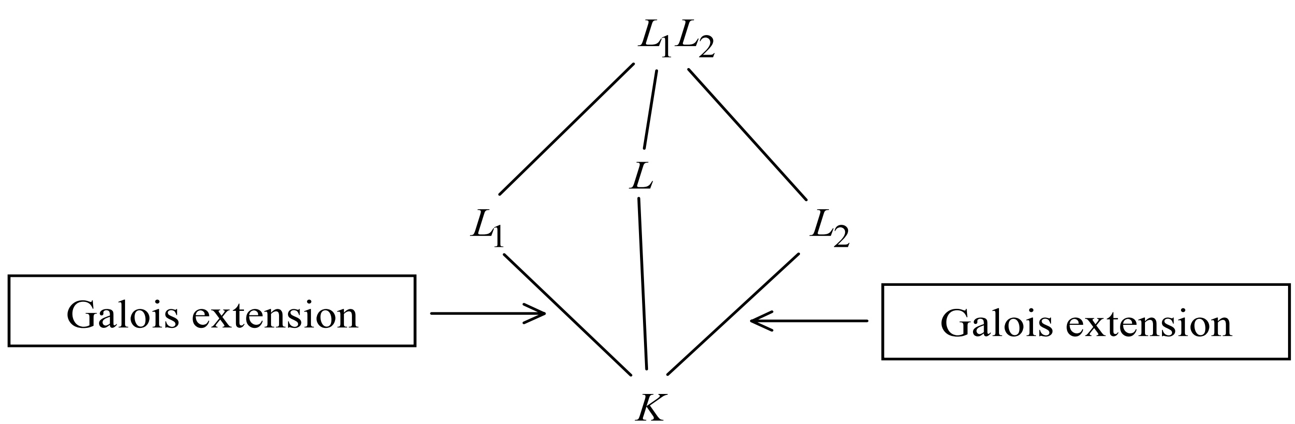
2. Let L be a finite separable extension of K. If K is Hilbertian then L is Hilbertian.

A **global field** K is either a finite extension of $\mathbb{Q}$ (**number field**) or a function field of one variable over a finite field $\mathbb{F}_p$.

3. Suppose K is a global field or a finitely generated transcendental extension of an arbitrary field K_0. Then K is Hilbertian.

4. Let $\aleph$ be a cardinal number and $\{K_\alpha \mid \alpha < \aleph\}$ a transfinite sequence of fields. Suppose that for each $\alpha < \aleph$ the field $K_{\alpha+1}$ is a proper finitely generated regular extension of K_α. Then $K = \bigcup_{\alpha<\aleph} K_\alpha$ is a Hilbertian field.

5. **(Fried)**: Every field K has a regular extension F which is PAC and Hilbertian. (F is a regular extension of K if F/K is separable and K is algebraically closed in F)
6. **Diamond theorem (Haran)**: Let K be a Hilbertian field , L_1 and L_2 Galois extensions of K, and L an intermediate field of $L_1 L_2/K$. Suppose that $L \not\subseteq L_1$ and $L \not\subseteq L_2$.Then L is Hilbertian.

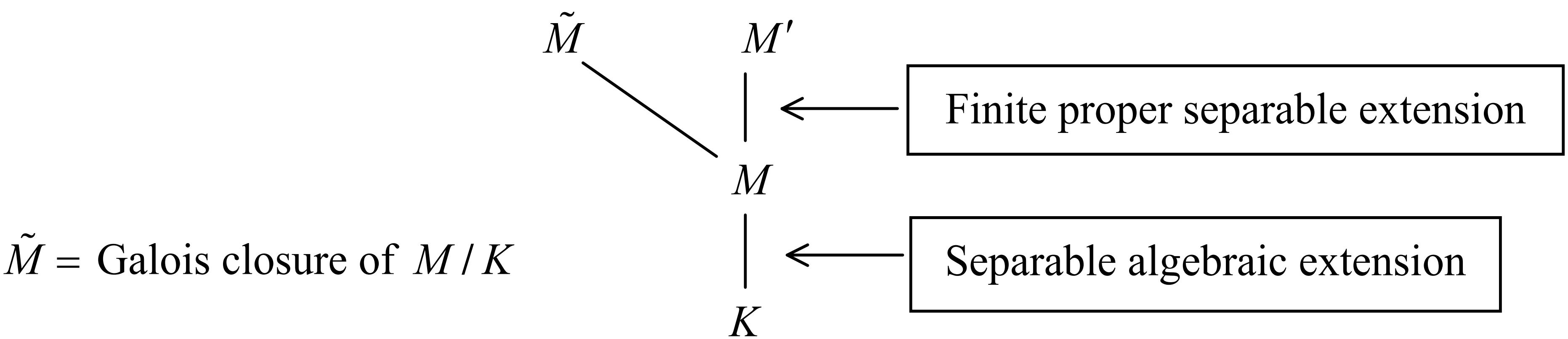


7. **(Haran-Jarden)**: Let K be a Hilbertian field and let N be a Galois extension of K which is not Hilbertian.

Then N is not the compositum of two Galois extensions of K neither of which is contained in the other. In particular, this conclusion holds for separable closure of K i.e., K_s.

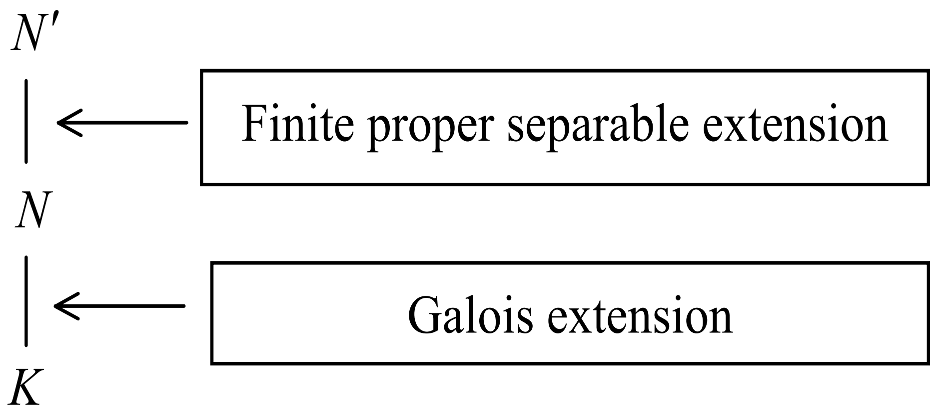
9. Let K be a Hilbertian field.

- Let M be a separable algebraic extension of K and M' a proper finite separable extension of M.



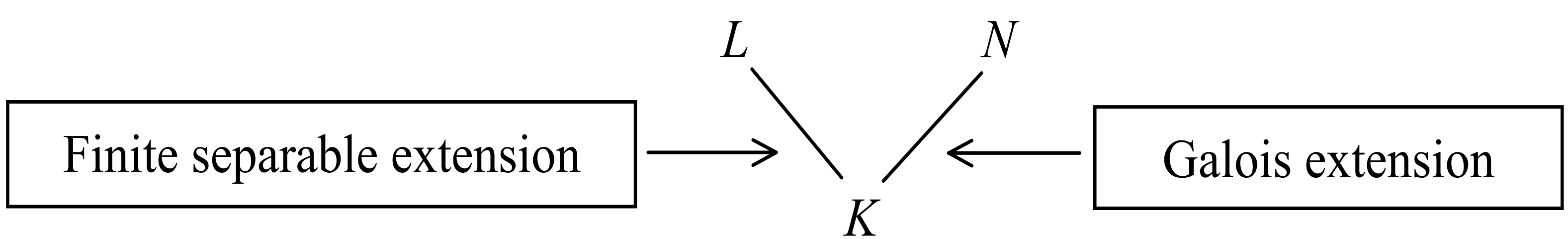
If $M' \not\subseteq \tilde{M}$ then M' is Hilbertian.

- Let N be a Galois extension of K and N' a finite proper separable extension of N.



Then N' is Hilbertian.

- Let N be a Galois extension of K and L a finite separable extension of K. Suppose $L \cap N = K$. Then NL is Hilbertian.



- **$\mathbb{Q}_{sol}$ is not a Hilbertian field.**

Since $\mathbb{Q}_{sol}$ is the compositum of all finite solvable extension of $\mathbb{Q}$ then there is no $a \in \mathbb{Q}_{sol}$ such that $X^2 - a$ is irreducible over $\mathbb{Q}_{sol}$. Therefore if $f(X,T) = X^2 - T$ then $H_{\mathbb{Q}_{sol}}(f)$ is empty. This shows that $\mathbb{Q}_{sol}$ is not a Hilbertian field.

- **Any finite proper extension of $\mathbb{Q}_{sol}$ is Hilbertian.**

Using Weissauer's theorem and since $\mathbb{Q}_{sol}$ is a Galois extension of $\mathbb{Q}$ then any finite proper extension of $\mathbb{Q}_{sol}$ is Hilbertian.

- **$\mathbb{Q}_{sol}$ is not a compositum of two separate Galois extensions of $\mathbb{Q}$.**

Using Haran-Jarden theorem since $\mathbb{Q}_{sol}$ is a Galois extension of $\mathbb{Q}$ which is not Hilbertian then $\mathbb{Q}_{sol}$ is not compositum of two Galois extensions of $\mathbb{Q}$ neither of which is contained in the other.

10. No Henselian field is Hilbertian.

- $\mathbb{Q}_p$, $\mathbb{Q}_{p,alg}$ and **formal power series** $K_0[[X]]$ are complete discrete valued fields and therefore Henselian. Thus they are **not Hilbertian**.

11. **(Kuyk):** Every abelian extension of a Hilbertian field is Hilbertian.

- $\mathbb{Q}_{ab}$ is **Hilbertian**. Abelian closure of any number field is Hilbertian.

A profinite group G is **small** if for each positive integer n the group G has only finitely many open subgroups of index n. (**Example:** $\mathbb{Z}_p$ is small.)

12. Let K be a Hilbertian field. Then $Gal(K)$ is niether prosolvable nor small.

13. Let L be a Galois extension of a Hilbertian field K. Suppose $L \neq K_s$. Then $Gal(L)$ is nither prosolvable nor it is contained in a closed small subgroup of $Gal(K)$.

Let K be a field and G a profinite group. Suppose K has Galois extension L with $Gal(L/K) \cong G$. Then, G **occurs** over K and L is a G**-extension of** K.

14. Let L be a Galois extension of a Hilbertian field K. If $Gal(L/K)$ is small then L is Hilbertian.

15. Let K be a Hilbertian field and p a prime number. Then $\mathbb{Z}_p$ occurs over K. (Therefore there is a Galois extension L of K such that $Gal(L/K) \cong \mathbb{Z}_p$).

- For each p, according to the statement in (15), $\mathbb{Q}$ has a Galois extension $\mathbb{L}_p$ with $Gal(\mathbb{L}_p/\mathbb{Q}) \cong \mathbb{Z}_p$. By (14), $\mathbb{L}_p$ is Hilbertian. Since $\mathbb{Z}_p$ has no nontrivial closed finite subgroups then:

$\mathbb{L}_p$ **is a Hilbertian field which is not a proper finite extension of any field**.

Fried-Völklein Conjecture

Let G be a profinite group. The **Borel field** of G, $\mathcal{B}(G)$ is the σ-algebra generated by all closed subsets of G.

1. Every profinte group has a unique Haar measure on $\mathcal{B}(G)$.

For a field K and for a $\sigma \in Gal(K)^e$ let $K_s(\sigma)$ be the fixed field in K_s of the entries of σ by $K_s(\sigma)$.

2. **(Jarden):** Let K be a countable Hilbertian field and e a positive integer. Then $K_s(\sigma)$ is a PAC field for almost all $\sigma \in Gal(K)^e$.

3. **(Fried-Jarden):** Let K be a countable Hilbertian field. Then K has a Galois extension N which is Hilbertian and PAC with $Gal(N/K) \simeq \prod_{k=1}^{\infty} S_k$

Outline of the Proof:

- List all plane curve over K in a sequence $C_1, C_2, C_3, \ldots$.
- Construct a sequence of points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ of $\mathbb{A}^2(K)$, and a linearly disjoint sequence $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \ldots, \ldots$ of Galois extensions of K satisfying:

  1. $Gal(\mathbf{L}_k/\mathbf{K}) \simeq S_k, \;\; k = 1, 2, 3, \ldots$
  2. $\mathbf{p}_i \in C_i(\mathbf{L}_k)$
  3. The points $\mathbf{p}_1, \mathbf{p}_2, \ldots$ are distinct.

- Define $N = \prod_{k=1}^{\infty} \mathbf{L}_k$. Then N is a Galois extension and $Gal(N/K) \simeq \prod_{k=1}^{\infty} S_k$.
- N is finite proper extension of the Galois extension $\prod_{k=2}^{\infty} \mathbf{L}_k$ and thus N is Hilbertian.
- Each plane curve over K has an N-rational point and therefore N is a PAC field.

An **embedding problem** for a profinite group G is a pair

$$(\phi : G \to A, \alpha : B \to A)$$

in which ϕ and α are epimorphisms of profinite groups. The $Ker(\alpha)$ is called **kernel of the problem**. The problem is called **finite** if B is finite. The problem is called solvable if there exist an epimorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \phi$.

5. **(Iwasawa):** Let K be a countable field. Then if every finite embedding problem over K is solvable then $Gal(K)$ is ω-free. ($Gal(K) \simeq \hat{F}_\omega$)
6. **(Fried-Völklein):** Every finite embedding problem over a Hilbertian PAC field is solvable.
7. **(Fried-Völklein):** Let K be a countable Hilbertian PAC field. Then $Gal(K)$ is ω-free.

Fried-Völklein Conjecture

Let K be a countable Hilbertian field. If the absolute Galois group of K, $Gal(K)$, is projective then $Gal(K)$ is ω-free.

(**Class Field Thery**: Absolute Galois group of abelian closure of any number field is projective.)

Shafarevich Conjecture

Let K be abelian closure of a number field. Then the absolute Galois group of K, $Gal(K)$, is ω-free.