Generalized Fuchsian groups and the p-reduction theory of elements in Hurwitz spaces

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A. Hurwitz, 1859-1919

§1 Hurwitz groups

Definition. • *A* **Hurwitz group** $H(\underline{m})$ *(of signature* $\underline{m}$*) is a group with presentation*

$$H(\underline{m}) = \langle\, z_1, \ldots, z_r \mid z_1^{m_1} = \cdots = z_r^{m_r} = \ldots \\ z_1 \cdot z_2 \cdots z_r = 1 \,\rangle$$

where $\underline{m} := (m_1, \ldots, m_r)$ *is a finite sequence of integers satisfying* $m_i \geq 2$*.*

• *Let*

$$\mu(\underline{m}) := \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Then

$$\begin{aligned} H(\underline{m}) \text{ is finite} \quad &\Leftrightarrow \mu(\underline{m}) < 2, \\ H(\underline{m}) \text{ is Bieberbach} \quad &\Leftrightarrow \mu(\underline{m}) = 2, \\ H(\underline{m}) \text{ is Fuchsian} \quad &\Leftrightarrow \mu(\underline{m}) > 2. \end{aligned}$$

§2 Hurwitz spaces

Definition. • *Let* $H(\underline{m})$ *be a Hurwitz group, let* G *be a finite group and let* $\mathcal{C}_1, \dots, \mathcal{C}_r$ *be* G*-conjugacy classes with* $g \in \mathcal{C}_i$ *of order* m_i. *Let* $\underline{\mathcal{C}} := (\mathcal{C}_1, \dots, \mathcal{C}_r)$ *denote the ordered sequence of* G*-conjugacy classes. The set of group homomorphisms*

$$\mathcal{H}(G, \underline{\mathcal{C}}) := \{\, \phi \colon H(\underline{m}) \to G \mid \phi \text{ surjective and } \phi(z_i) \in \mathcal{C}_i \,\}$$

is called the **Hurwitz space** *of* $(G, \underline{\mathcal{C}})$. *Put*

$$[\phi] := (\phi(z_1), \dots, \phi(z_r)).$$

• *Composition with inner automorphisms of* G *yields a left action of* $\mathrm{Inn}(G)$ *on* $\mathcal{H}(G, \underline{\mathcal{C}})$. *The orbit space*

$$\mathcal{H}^{\mathrm{inn}}(G, \underline{\mathcal{C}}) := \mathrm{Inn}(G) \setminus \mathcal{H}(G, \underline{\mathcal{C}})$$

is called the **inner Hurwitz space** *of* $(G, \underline{\mathcal{C}})$.

§3.1 Circular braid groups

Definition. • *The (abstract)* **circular braid group** $\boldsymbol{\Omega_r}$ *is the group generated by elements* $\boldsymbol{Q_1, \ldots, Q_r}$ *with subject to the relations*

$$\begin{aligned} Q_iQ_j &= Q_jQ_i \quad \text{for } i - j \not\equiv \pm 1 \mod r, \\ Q_iQ_jQ_i &= Q_jQ_iQ_j \quad \text{for } i - j \equiv \pm 1 \mod r. \end{aligned}$$

• $\boldsymbol{s \colon Q_i \to Q_{i+1}}$ *is an automorphism of* Ω_r. *The group*

$$\tilde{\Omega}_r = \langle s \rangle \ltimes \Omega_r$$

will be called the **extended circular braid group**.

• It is still an open problem whether the abstract circular braid group coincides with the geometric circular braid group.

§3.2 The action of the circular braid group

- Let G be a group and

$$\boldsymbol{S_r(G) := \{ (g_1, \ldots, g_r) \in G^r \mid g_1 \cdots g_r = 1 \}}.$$

The assignment

$$\begin{aligned}
(g_1, \ldots, g_r).Q_i &:= \\
&(g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_r) \\
(g_1, \ldots, g_r).Q_r &:= (g_1^{-1} g_r g_1, g_2, \ldots, g_{r-1}, g_1), \\
(g_1, \ldots, g_r).s &:= (g_2, \ldots, g_r, g_1).
\end{aligned}$$

defines a right action of $\tilde{\Omega}_r$ on $S_r(G)$ which commutes with the action of $\mathrm{Inn}(G)$.

§3.3 Reduced Hurwitz spaces

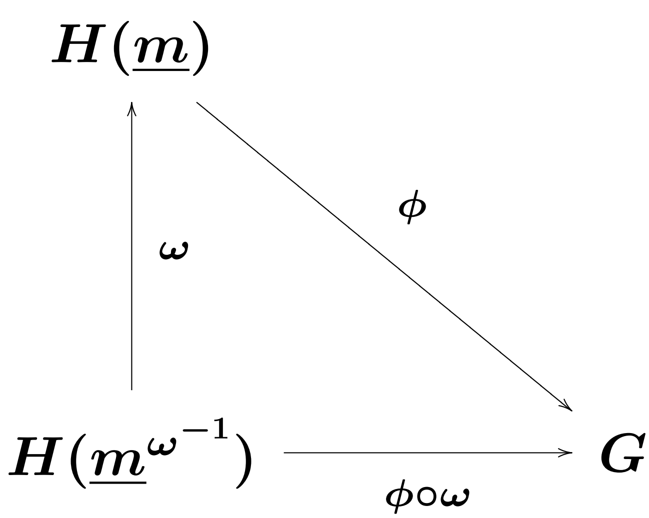
- Put $\mathcal{H}(G, \underline{\mathcal{C}}^*) := \coprod_{\sigma \in S_r} \mathcal{H}(G, \underline{\mathcal{C}}^\sigma)$. Then

$$\mathcal{H}^{\mathrm{red}}(G, \underline{\mathcal{C}}^*) := \mathcal{H}(G, \underline{\mathcal{C}}^*)/\tilde{\Omega}_r,$$

  is called the **reduced Hurwitz space** of $(G, \underline{\mathcal{C}})$.

- $\mathcal{H}^{\mathrm{red}}(G, \underline{\mathcal{C}}^*) = \mathcal{H}^{\mathrm{red,inn}}(G, \underline{\mathcal{C}}^*)$ in M.D.Fried's notation.

- Let $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$, $\omega \in \tilde{\Omega}_r$. Then



§4 Profinite Hurwitz groups

Let p be a prime number which is coprime to $N(\underline{m})\colon = m_1 \cdots m_r$. Let

$$\iota\colon H(\underline{m}) \longrightarrow \hat{H}(\underline{m})$$

denote the profinite completion of the Hurwitz group $H(\underline{m})$.

- Every homomorphism $\phi\colon H(\underline{m}) \to G$ onto a finite group G extends in a unique way to a homomorphism $\hat{\phi}\colon \hat{H}(\underline{m}) \to G$.

- $\hat{H}(\underline{m})$ is a p-perfect group, i.e.,

$$\mathbf{Hom}(\hat{H}(\underline{m}), \mathbb{F}_p) = \mathbf{0}.$$

- [W] The profinite group $\hat{H}(\underline{m})$ is an orientable p-Poincaré duality group of dimension 2.

§5 P-projective profinite groups

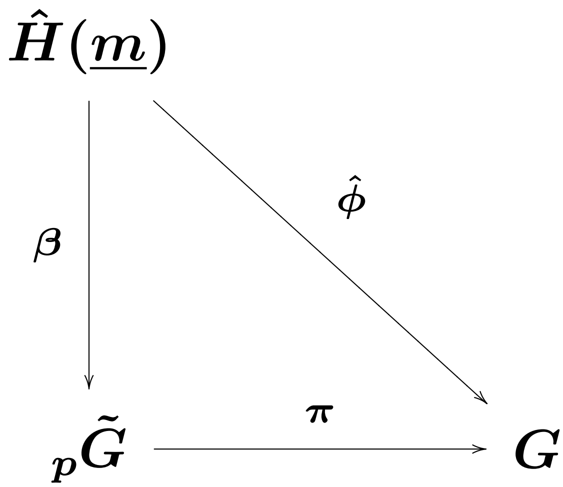
Theorem. *(W.Gaschütz; Cossey, O.H.Kegel & L.Kovacs; M.D.Fried & Ershov; M.D.Fried & M.Jarden; et al.) Every (pro)finite group G has a universal p-Frattini cover $\boldsymbol{\pi} \colon {}_p\tilde{\boldsymbol{G}} \longrightarrow \boldsymbol{G}$. It coincides with the* **minimal p-projective cover***.*

Theorem. *(K.W.Gruenberg) Let $\hat{G}$ be a profinite group. Then*

$$\mathbf{cd}_{\boldsymbol{p}}(\hat{\boldsymbol{G}}) \leq \mathbf{1} \Leftrightarrow \ \hat{\boldsymbol{G}} \text{ is } \boldsymbol{p}\text{-projective.}$$

§6 Cusp branches

Definition. *One says that the Hurwitz element* $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ *has a* **cusp branch**, *if there exists a mapping* $\beta\colon \hat{H}(\underline{m}) \longrightarrow {}_p\tilde{G}$ *making the diagram*



commute. The mapping β *is necessarily surjective.*

- **Question: Is it possible to characterize Hurwitz elements which have a cusp branch?**

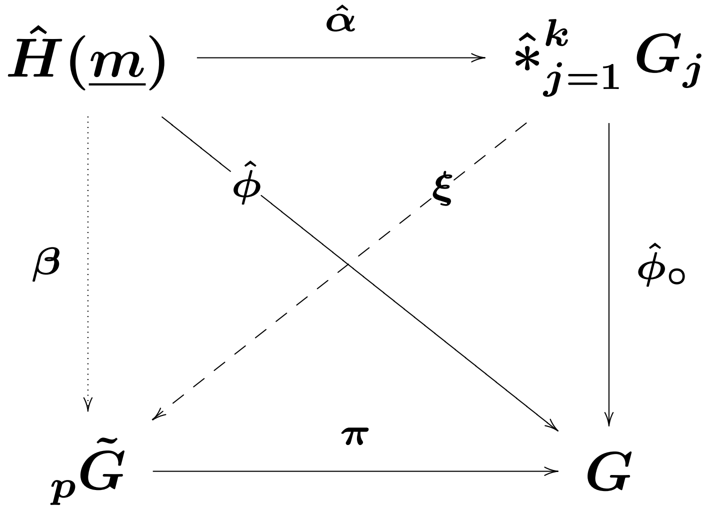
§7.1 Harbater-Mumford elements

Definition. *The element* $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ *is called a* **Harbater-Mumford element**, *if there exists* $\omega \in \tilde{\Omega}_r$ *such that* $[\phi \circ \omega] = (g_1, \ldots, g_r)$ *has the following property: There exists* $i_1, \ldots, i_k$, $k \geq 2$, *such that for all* $j = 1, \ldots, k$

- $G_j := \langle\, g_{i_{j-1}+1}, \ldots, g_{i_j} \,\rangle \leq G$ *is a* p'*-group,*
- $g_{i_{j-1}+1} \cdots g_{i_j} = 1$.

§7.1 Harbater-Mumford elements (cont.)

One has a commutative diagram



- by hypothesis $\hat{\phi} = \hat{\phi}_\circ \circ \hat{\alpha}$,
- ${}_p\tilde{G}$ p-projective $\Rightarrow \xi$ exists,
- $\beta = \xi \circ \hat{\alpha}$.

Thus, ϕ has a cusp branch.

§8.1 Good reduction (g-p'-cusps)

Definition. $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ *is called* **strongly reducible**, *if exists an element* $\boldsymbol{\omega} \in \tilde{\boldsymbol{\Omega}}_{\boldsymbol{r}}$ *such that for* $[\boldsymbol{\phi} \circ \boldsymbol{\omega}] = (\boldsymbol{g_1}, \ldots, \boldsymbol{g_r})$ *there exists* $i_1, \ldots, i_k$, $k \geq 2$, *such that*

- $\boldsymbol{G_j} := \langle\, \boldsymbol{g_{i_{j-1}+1}}, \ldots, \boldsymbol{g_{i_j}} \,\rangle$ *are finite* p'*-groups.*

- Let $\boldsymbol{y_j} := \boldsymbol{g_{i_{j-1}+1} \cdots g_{i_j}} \in G_j$, and put

$$\boldsymbol{K} := \langle\, \boldsymbol{G_j} \mid \boldsymbol{y_1 \cdots y_k = 1} \,\rangle.$$

- Let $\boldsymbol{Y} := \langle\, \boldsymbol{y_1}, \ldots, \boldsymbol{y_k} \,\rangle \leq \boldsymbol{G}$, and let $\boldsymbol{\mathcal{C}'_j}$ denote the Y-conjugacy class of Y containing $\boldsymbol{y_j}$.

- Let $\boldsymbol{n_j} := \mathbf{ord}(\boldsymbol{y_j})$, and $\underline{\boldsymbol{n}} := (\boldsymbol{n_1}, \ldots, \boldsymbol{n_k})$. Then

$$\boldsymbol{\psi} \colon \boldsymbol{H}(\underline{\boldsymbol{n}}) \longrightarrow \boldsymbol{Y} \in \mathcal{H}(\mathbf{Y}, \underline{\mathcal{C}}'.)$$

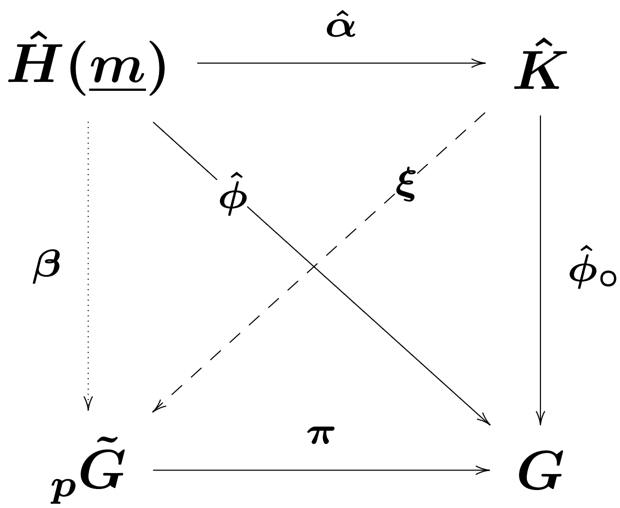
§8.1 Good reduction (cont.)

Theorem. *(M.D.Fried)*

$\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ *has a cusp branch*

$\Leftrightarrow \psi \in \mathcal{H}(Y, \underline{\mathcal{C}}')$ *has a cusp branch.*

Consider



Then:

$$\exists \beta \Leftrightarrow \exists \xi.$$

§8.2 Bad reduction (o-p'-cusps)

Definition. $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ *is called* **weakly reducible**, *if there exists an element* $\omega \in \tilde{\Omega}_r$ *such that for* $[\phi \circ \omega] = (g_1, \ldots, g_r)$ *there exists* $i_1, \ldots, i_k$, $k \geq 2$, *such that*

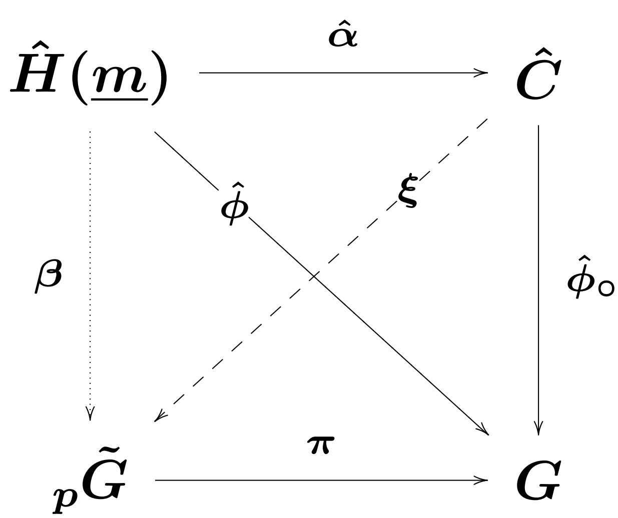
- $y_j := g_{i_{j-1}+1} \cdots g_{i_j}$ *are elements of* p'*-order.*
- Let $s_i := \mathrm{ord}(g_i)$, $t_j := \mathrm{ord}(y_j)$ and put $\underline{s}^* := (s_{i_{j-1}+1}, \ldots, s_{i_j}, t_j)$, $\underline{t} := (t_1, \ldots, t_k)$.
- Let
$$Y_j := \langle\, g_{i_{j-1}+1}, \ldots, g_{i_j} \,\rangle,$$
$$Y^* := \langle y_1, \ldots, y_k \rangle.$$
- Hence for such an element one has $k+1$ Hurwitz elements
$$\psi_j \colon H(\underline{s}^*) \longrightarrow Y_j,\ 1 \leq j \leq k,$$
$$\psi^* \colon H(\underline{t}) \longrightarrow Y^*.$$

§8.2 Bad reduction (cont.)

Theorem. *(M.D.Fried) Assume that ψ^* and ψ_j have cusp branches for all $j \in \{1, .., k\}$. Then ϕ has a branch cusp.*

- $\Leftarrow$ is wrong: Let

$$C :=\langle\, g_1, .., g_r, y_1, .., y_k \mid g_i^{m_i} = y_j^{t_j} = g_{i_{j-1}+1} \cdots g_{i_j} y_j^{-1} = y_1 \cdots y_k = 1\rangle,$$



The existence of ξ implies the existence of β but not vice versa.

§9 M.D.Fried's conjecture

Conjecture 1. *(M.D.Fried[1])* $\phi \in \mathcal{H}(G, \underline{\mathcal{C}})$ **weakly reducible** $\Rightarrow$ ϕ **strongly reducible** *($r = 4$).*

Conjecture 2. ϕ **has a cusp branch** $\Rightarrow$ ϕ **is strongly reducible**.

Conjecture 3. *Let* $\phi\colon \hat{H}(\underline{m}) \longrightarrow {}_p\tilde{G}$ *be a continuous homomorphism of profinite groups such that*

- $\underline{m} = (m_1, m_2, m_3)$, *($\hat{H}(\underline{m})$ is a profinite triangle group),*
- $(p, m_1 m_2 m_3) = 1$.

Then $\mathbf{im}(\phi)$ *is finite.*

[1] During the meeting I learnt through conversations with Mike Fried and Darren Semmen, that Conjecture 3 **cannot** hold in this form. Indeed, on the last day of the meeting Darren and me, we constructed morphism onto profinite Frobenius groups which violate Conjecture 3, and therefore Conjecture 1 and 2 for $r = 3$. However, it might be possible, that some version of these conjectures (with additional hypothesis) is true.)

§10 A test case

Theorem. *(Stallings; Swan) Let* $\tilde{F}$ *be a (discrete) group. Then*

$$\mathrm{vcd}(\tilde{F}) \leq 1 \Leftrightarrow \tilde{F} \text{ is virtually free.}$$

Theorem. *(J-P.Serre) Let* $a, b, c \in \tilde{F}$ *be elements of finite order in the finitely generated virtually free group* $\tilde{F}$ *satisfying* $a \cdot b \cdot c = 1$*. Then* $\langle a, b, c \rangle$ *is finite.*

Theorem. *(W.) Let* $\underline{g} = (g_1, \ldots, g_r)$ *be a sequence of elements of finite order in the finitely generated virtually free group* $\tilde{F}$ *satisfying*

$$g_1 \cdots g_r = 1.$$

Then there exists an element in the extended circular braid group $\omega \in \tilde{\Omega}_r$ *such that*

$$\langle (\underline{g}.\omega)_1, (\underline{g}.\omega)_2 \rangle$$

is a finite group.

§11.1 P-projective profinite groups and $\mathbb{Z}_p$-trees

Theorem. *(W.) Let* $\hat{\boldsymbol{G}}$ *be a* **finitely generated** $\boldsymbol{p}$**-projective virtual pro-**p **group**. *Then* $\hat{\boldsymbol{G}}$ *has an action on a* $\mathbb{Z}_{\boldsymbol{p}}$-**tree** $\hat{\boldsymbol{T}}$ *with the following properties:*

(i) $\hat{G}$ *is* **acting without inversion of edges**,

(ii) every **vertex** *and every* **edge stabilizer** *is a (finite)* $\boldsymbol{p'}$**-group**,

(iii) every (finite) **subgroup of** $\hat{\boldsymbol{G}}$ **of** $\boldsymbol{p'}$**-order** *fixes a* **vertex**,

(iv) the vertex group $\mathbf{V}(\hat{\boldsymbol{T}})$ *and the edge group* $\mathbf{E}(\hat{\boldsymbol{T}})$ *are* p*-projective* $\hat{G}$*-modules.*

- If $\hat{G}$ is a finitely generated virtual pro-p group acting on a $\mathbb{Z}_p$-tree $\hat{T}$ such that (i), (ii) and (iv) are satisfied, then $\hat{G}$ is p-projective.

§11.2 The difference to Bass-Serre

- There are finitely generated p-projective virtual pro-p group $\hat{G}$ that can act on a $\mathbb{Z}_p$-tree $\hat{T}$, such that (i), (ii), (iv) is satisfied, but (iii) fails.

- There are finitely generated p-projective virtual pro-p groups $\hat{G}$ which cannot act with finitely many orbits on vertices and edges on a $\mathbb{Z}_p$-tree $\hat{T}$ satisfying (i) and (ii).

- **Question:** Assume that the finitely generated p-projective virtual pro-p group $\hat{G}$ is acting on a $\mathbb{Z}_p$-tree such that (i), (ii), (iii) and (iv) are satisfied. Let $g \in \hat{G}$ be an element of p'-order. Is it true that $\hat{T}^g$ is connected?

§11.3 Boolean sets

- A **boolean (or profinite) set** is a compact totally disconnected Hausdorff space. Let $\mathbf{bool}$ denote the category of boolean sets.

- Let $\mathbf{ab_p}$ denote the category of abelian pro-p groups. The forgetful functor $\mathbf{for}\colon \mathbf{ab_p} \to \mathbf{bool}$ has a left adjoint
$$\mathbb{Z}_{\boldsymbol{p}}[\![\_]\!]\colon \mathbf{bool} \longrightarrow \mathbf{ab_p}.$$

§11.4 What is a $\mathbb{Z}_p$-tree?

- A **profinite graph** $\hat{\Gamma}$ is the collection of a boolean set $\mathfrak{V}(\hat{\Gamma})$, the **vertices**, a boolean set $\mathfrak{E}(\hat{\Gamma})$, the **edges**, an **origin mapping** $o\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$, a **terminus mapping** $t\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{V}(\hat{\Gamma})$ and an **inversion mapping** $\bar{\_}\colon \mathfrak{E}(\hat{\Gamma}) \to \mathfrak{E}(\hat{\Gamma})$ satisfying the usual identities. (All mappings are mappings in $\mathbf{bool}$).

- One puts

$$\begin{aligned} \mathbf{V}(\hat{\boldsymbol{\Gamma}})\colon =&\mathbb{Z}_{\boldsymbol{p}}[\![\mathfrak{V}(\hat{\boldsymbol{\Gamma}})]\!],\\ \mathbf{E}(\hat{\boldsymbol{\Gamma}})\colon =&\mathbb{Z}_{\boldsymbol{p}}[\![\mathfrak{E}(\hat{\boldsymbol{\Gamma}}]\!]/\langle\, \mathbf{e}+\bar{\mathbf{e}} \mid \mathbf{e}\in \mathfrak{E}(\hat{\boldsymbol{\Gamma}})\,\rangle. \end{aligned}$$

  Then $\partial\colon \mathbf{E}(\hat{\Gamma}) \to \mathbf{V}(\hat{\Gamma})$, $\partial(\mathbf{e})\colon = t(\mathbf{e}) - o(\mathbf{e})$, is a morphism of abelian pro-p groups.

- The **profinite graph** $\hat{\boldsymbol{\Gamma}}$ is called $\mathbb{Z}_{\boldsymbol{p}}$-**connected**, if $\mathbf{coker}(\boldsymbol{\partial}) \simeq \mathbb{Z}_{\boldsymbol{p}}$, and a $\mathbb{Z}_{\boldsymbol{p}}$-**tree**, if it is **connected** and $\mathbf{ker}(\boldsymbol{\partial}) = \mathbf{0}$.

§12 Generalized Fuchsian groups

Definition. *A* **geodesic space** (X, d) *with a non-empty subset* $\mathfrak{P}$ *of closed subspaces satisfying*

- $(P, d) \simeq \mathbb{H}^2$ *or* $\mathbb{R}^2$ *for* $P \in \mathfrak{P}$,
- *for* $P, Q \in \mathfrak{P}$, $P \neq Q$, *one has* $|P \cap Q| \leq 1$,
- *for all* $x \in X$, $|\{P \in \mathfrak{P} \mid x \in P\}| < \infty$,

will be called a **geodesic plane arrangement**.

Definition. *A (discrete) group* G *is called a* **generalized Fuchsian group**, *if it has a* **faithful, discontinuous and co-compact action** *on a* **contractible geodesic plane arrangement** $(X, d, \mathfrak{P})$, *such that* G_P *acts co-compactly on* P *for all* $P \in \mathfrak{P}$.

§12 Generalized Fuchsian groups (cont.)

Proposition. *The group K of §8.1 is (in general) a generalized Fuchsian group, C of §8.2 is not.*

Proposition. *Let G be a generalized Fuchsian group acting on the geodesic plane arrangement $(X, d, \mathfrak{P})$. Then X is an $E_{\mathfrak{F}}G$ of G. In particular, $\mathrm{cd}_{\mathfrak{F}}(G) = 2$.*