Infinite global fields and their zeta-functions

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Abstract. The talk starts with exposing my joint paper with Serge Vlăduţ. It has two purposes. First, we start to develop a theory of infinite global fields, i.e., of infinite algebraic extensions either of $\mathbb{Q}$ or of $\mathbb{F}_r(t)$. We produce a series of invariants of such fields, and we introduce and study a kind of zeta-function for them. Second, for sequences of number fields with growing discriminant we prove generalizations of the Odlyzko–Serre bounds and of the Brauer–Siegel theorem, taking into account non-archimedean places. This leads to asymptotic bounds on the ratio $\log hR/\log\sqrt{|D|}$ valid without the standard assumption $n/\log\sqrt{|D|} \to 0$, thus including, in particular, the case of unramified towers. Then we produce examples of class field towers, showing that this assumption is indeed necessary for the Brauer–Siegel theorem to hold. As an easy consequence we ameliorate on existing bounds for regulators.

Then we present a sequel to the paper of Yasutaka Ihara, which puts it in the context of infinite global fields. Let K be a global field, i.e., a finite algebraic extension either of the field $\mathbb{Q}$ of rational numbers, or of the field of rational functions in one variable over a finite field of constants. Let $\zeta_K(s)$ be its zeta-function. Consider its Laurent expansion at $s = 1$ $\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + \ldots$ Ihara introduces and studies the constant $\gamma_K = c_0/c_{-1}$. We study the behaviour of Euler–Kronecker constant γ_K when the discriminant (genus in the function field case) tends to infinity. Results of our papers on infinite global fields easily give us good lower bounds on the ratio $\gamma_K/\log\sqrt{|d_K|}$. Then we produce examples of class field towers, giving the upper bounds for $\liminf \frac{\gamma_K}{\log\sqrt{|d_K|}}$.