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THETA CONSTANT IDENTITIES AT PERIODS OF COVERINGS OF DEGREE 3

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We derive relations between theta functions evaluated at period matrices of cyclic covers of order 3 ramified above $3k$ points.

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1. Introduction

Let R be a Riemann Surface defined by the equation:

$$y^3 = \prod_{i=1}^{i=3m} (z - \lambda_i). \tag{1}$$

We find relations that are satisfied by theta constants with rational characteristics evaluated at τ_R, the period matrix of R. Special type identities for period matrices are known in the case of a general Riemann Surface (Schottky–Jung identities). According to Mumford, for hyperelliptic curves there are vanishing theta constants of even characteristics that characterize the associated period matrix. Special relations among non vanishing theta constants evaluated at period matrices of hyperelliptic curves were obtained by Frobenius.

The original Schottky problem seeks special relations among theta constants that characterize the entire moduli space of algebraic curves of genus g. In this note, we seek special relations that are satisfied by more specialized sets of curves such as cyclic covers of the sphere of degree n. When $n = 2$, cyclic covers constitute the set of hyperelliptic curves. The next case is $n = 3$. Here, we find relations satisfied by theta constants with rational characteristics evaluated at the period matrices of such curves. These identities are consequence of the Thomae formula for cyclic n sheeted covers of the sphere. This formula expresses powers of such theta constants evaluated at the period matrix τ_R by polynomial expressions in the λ_i. A relation between these polynomials products a relation between the corresponding theta constants. Using the representation theory of S_{3m}, the symmetric group of degree $3m$, we find a basis for the space of these polynomials, as a result we obtain relations between the corresponding theta constants.

For the simplest case of 6 branch points, our results overlap those of Matsumoto [7]. In his paper, Matsumoto finds the explicit action of S_6 on theta constants evaluated at τ_R. He obtains the branch points λ_i as quotients of theta constants. He also obtains identities between cubic powers of these constants, which coincide with those in the last section of our note. Using the representation theory of S_6, we see that the space generated by theta constants is 5 dimensional. This seems to be a new result even when $g = 4$.

2. The Thomae Formula for Cyclic Covers and Relations Between Theta Constants

Following Nakayashiki [9], we explain the general Thomae formula for an algebraic curve R satisfying the equation:

$$y^3 = \prod_{i=1}^{i=3m} (z - \lambda_i).$$

Let $f : R \mapsto CP^1$ be the map given by $(z, y) \mapsto z$. Now, let $\{\infty_1, \infty_2, \infty_3\} = f^{-1}\{\infty\}$ and let $Q_i = f^{-1}(\lambda_i)$ be the unique branch point on R that is the pre image of λ_i. Fix a canonical homology basis $a_1, a_2, \ldots, a_{3m-2}, b_1, b_2, \ldots, b_{3m-2}$ on R. Thus, $a_i a_j = 0 = b_i b_j, i \neq j$ and $a_i b_i = -1$. Let $v_1 \cdots v_{3m-2}$ be a basis of normalized holomorphic differentials with respect to the basis $a_1, a_2, \ldots, a_{3m-2}, b_1, \ldots, b_{3m-2}$. Thus, $\int_{a_i} v_j = \delta_{ij}$ and $\int_{b_i} v_j = \tau_{ij}$. The $g \times g$ matrix, $\tau = \tau_{ij}$ is symmetric and $Im\tau$ defines positive definite quadratic form. Fix an ordering of the λ_i. This ordering induces an ordering of the branch points $\{Q_1, Q_2, Q_3, \ldots, Q_{3m}\}$. We abuse notation by identifying Q_i with its branch point image. Thus λ_1 will correspond to Q_1, λ_2 to Q_2, etc.

Let ϕ be the automorphism of order 3 defined by $(z, y) \mapsto (z, \omega y)$ with $\omega^3 = 1, \omega = e^{\frac{2\pi i}{3}}$. For divisors α and β on R, define $\alpha \equiv \beta$ if there exists a meromorphic function $g : R \mapsto CP^1$ with divisor $\mathrm{div}(g) = \alpha - \beta$. The group $\mathrm{Div}^0/\equiv$ is $Jac(R)$, the Jacobian of R. (Div^0 — divisors of degree 0.) Let ψ be the mapping $\psi : \mathrm{Div} \mapsto \mathrm{Div}/\equiv$. Then, the following lemma is true:

Lemma 2.1. *Let P_1, $P_2 \in R, P_1 \neq P_2$ and*

$$D_i = P_i + \phi(P_i) + \phi^2(P_i), \quad i = 1, 2$$

then $\psi(D_1) \equiv \psi(D_2)$.

Proof. For P_1, P_2 as above, define $f_1(P) = (f(P) - f(P_1)/f(P) - f(P_2))$, then $\mathrm{div}(f_1) = D_1 - D_2$. □

Define $D = \psi(P_i + \phi(P_i) + \phi^2(P_i))$ to be the equivalence class in the Jacobian.

Lemma 2.2. *Let K be the canonical divisor of the a holomorphic differential. Define Δ to be a divisor such that $2\Delta = K$, then the following holds*:

(1) $D \equiv 3Q_i \equiv \infty^1 + \infty^2 + \infty^3$,
(2) $K \equiv (2m-2)D$,
(3) $\sum_1^{3m} Q_i \equiv mD$.

Proof. The first item follows exactly as in the proof of the previous lemma. To show the rest, note that $z(dz/w^2)$ is a holomorphic differential with divisor Q_1^{6m-6}. □

Now let $\Lambda = \Lambda_0, \Lambda_1, \Lambda_2$ be a partition of $\{1, 2, 3, 4, 5, \ldots 3m\}$ with each $|\Lambda_i| = m$. For each subset S of $\{1, 2 \ldots 3m\}$ we set,

$$X_S = \sum_{Q_j \in S} Q_j.$$

We are interested in the following divisor e_Λ associated with the partition:

$$e_\Lambda = X_{\Lambda_0} + 2X_{\Lambda_1} - D - \Delta.$$

Choose a base point P_0 on R and for each P consider the mapping $\Phi_{P_0}(P) = (\int_{P_0}^P v_1 \ldots \int_{P_0}^P v_{3m-2})$. Using the definition of divisors, we see that Φ_{P_0} extends to the period map $\Phi_{P_0} : \mathrm{Div}(R) \mapsto \mathbb{C}^{3m-2}$. The final definition will be of theta constants with characteristics:

Definition 2.3. Let $\mathbb{H}_g$ be the collection of $g \times g$ symmetric matrices, τ such that the imaginary part of τ forms a positive definite form. For $\left[\begin{smallmatrix}\varepsilon \\ \varepsilon'\end{smallmatrix}\right]$, $\varepsilon, \varepsilon'$, real vectors g vectors and $\tau \in \mathbb{H}_g$, we define theta constant $\Theta\left[\begin{smallmatrix}\varepsilon \\ \varepsilon'\end{smallmatrix}\right](\tau)$ with characteristics $\left[\begin{smallmatrix}\varepsilon \\ \varepsilon'\end{smallmatrix}\right]$ as an infinite series given by:

$$\Theta \begin{bmatrix}\varepsilon \\ \varepsilon'\end{bmatrix} (\tau) = \sum_{l \varepsilon \mathbb{Z}^{2g}} \exp 2\pi i \left\{ \frac{1}{2} \left(l + \frac{\varepsilon}{2}\right)^t \tau \left(l + \frac{\varepsilon}{2}\right) + \left(l + \frac{\varepsilon}{2}\right)^t \frac{\varepsilon'}{2} \right\}.$$

These series are uniformly and absolutely convergent on compact subsets of $\mathbb{H}_g$. For each $w \in \mathbb{C}^{3m-2}$, associate a unique vector $2g$ vector $\left[\begin{smallmatrix}w_1 \\ w_2\end{smallmatrix}\right]$ and w_1, w_2 are unique vectors real g dimensional vectors such that: $w = w_1 + \tau w_2$. Composing with the map p_{P_0}, we associate theta constants with characteristics to divisors. Nakayashiki [9] proves the following theorem of Bershadsky and Radul [2]:

Theorem 2.4. *Let $\theta[e_\Lambda](\tau)$ be a theta constant associated with Λ through the period map p_{P_0}. Then, e_Λ is a point of order 6 on the Jacobian and*

$$\theta[e_\Lambda]^6 (\tau_R) = C_\Lambda (\det A)^3 ((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))^3 (\Lambda_0\Lambda_1)(\Lambda_1\Lambda_2)(\Lambda_0\Lambda_2). \quad (2)$$

Here, A is the matrix of certain holomorphic 1 forms integrated with respect to a_i and C_Λ is a constant depending only on the partition. Moreover,

$$(\Lambda_i\Lambda_i) = \prod_{k<l} (\lambda_{i_k} - \lambda_{i_l}), \quad (\Lambda_i\Lambda_j) = \prod_{k=1,l=1}^{m} (\lambda_{i_k} - \lambda_{j_l}),$$

where

$$\Lambda_i = \{i_1 < \cdots < i_m\}, \quad \Lambda_j = \{j_1 < \cdots < j_m\}.$$

As explained in the introduction, one of the problems in the theory of complex algebraic curves is to understand the set of period matrices associated with certain families of curves. For example, we can seek certain algebraic relations satisfied by theta constants evaluated at period matrices of curves belonging to such families. Because theta constants are modular forms for subgroups of $Sp(g, \mathbf{Z})$ results of this type may have number theoretic significance.

We apply Theorem 2.4 to generate special relations between theta functions with characteristics e_Λ, evaluated at τ_R. For each partition, Λ, we denote the polynomial on the right-hand side of the equation above by p_Λ. To obtain identities, we expand the polynomials and search for identities between them. The key observation on the polynomials: First, choose $\Lambda = \{\{1, 2, \ldots, m\}, \{m+1, \ldots, 2m\}, \{2m, \ldots, 3m\}\}$. Then, by the definition of p_Λ we have:

$$p_\Lambda = \left(\prod_{i<j,j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j,i=m+1,j=m+2}^{j=2m,i=2m-1} (\lambda_i - \lambda_j) \prod_{i<j,j=2m+1}^{j=3m} (\lambda_i - \lambda_j)\right)^3$$

$$\times \prod_{i=1,j=m+1}^{i=m,j=2m} (\lambda_i - \lambda_j) \prod_{i=m+1,j=2m+1}^{i=2m,j=3m} (\lambda_i - \lambda_j) \prod_{i=1,j=2m+1}^{i=m,j=3m} (\lambda_i - \lambda_j). \quad (3)$$

Now, write

$$\left(\prod_{i<j,j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j,i=m+1,j=m+2}^{j=2m,i=2m-1} (\lambda_i - \lambda_j) \prod_{i<j,j=2m+1}^{j=3m} (\lambda_i - \lambda_j)\right)^3$$

$$= \left(\prod_{i<j,j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j,i=m+1,j=m+2}^{j=2m,i=2m-1} (\lambda_i - \lambda_j) \prod_{i<j,j=2m+1}^{j=3m} (\lambda_i - \lambda_j)\right)^2$$

$$\times \left(\prod_{i<j,j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j,i=m+1,j=m+2}^{j=2m,i=2m-1} (\lambda_i - \lambda_j) \prod_{i<j,i=2m,j=2m+1}^{j=3m} (\lambda_i - \lambda_j)\right).$$

We see that we can rewrite p_Λ for this partition as:

$$p_\Lambda = \left(\prod_{i<j,j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j,i=m+1,j=m+2}^{j=2m,i=2m-1} (\lambda_i - \lambda_j) \prod_{i<j,j=2m+1}^{j=3m} (\lambda_i - \lambda_j)\right)^2$$

$$\times \operatorname{disc}(\lambda_1 \cdots \lambda_{3m}),$$

where

$$\operatorname{disc}(\lambda_1 \cdots \lambda_{3m}) = \prod_{i \neq j, 1 \leq i,j \leq 3m} (\lambda_i - \lambda_j).$$

Observe that this holds for every partition. That is,

$$\begin{aligned}&(\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2)^3(\Lambda_0\Lambda_1)(\Lambda_1\Lambda_2)(\Lambda_0\Lambda_2)\\ &\quad = (\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2)^2(\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2)(\Lambda_0\Lambda_1)(\Lambda_1\Lambda_2)(\Lambda_0\Lambda_2)\\ &\quad = ((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))^2 \operatorname{disc}(\lambda_1,\ldots\lambda_{3m}). \qquad (4)\end{aligned}$$

Since the factor $\operatorname{disc}(\lambda_1,\ldots,\lambda_{3m})$ is independent of the partition $\Lambda_0,\Lambda_1,\Lambda_2$, we conclude that identities between $\pm\theta^3[e_\Lambda]$ are equivalent to identities between the polynomials:

$$((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2)).$$

The group S_{3m} acts naturally on the polynomials $((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))$ by its action on the partitions of $\{1\cdots 3m\}$. Thus, $\operatorname{Span}(((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))$, the vector space spanned by these polynomials, provides us with a representation of S_{3m}. We exhibit a basis for this space of polynomials in the next section.

3. Explicit Basis

In this section, we provide an explicit basis for the space of polynomials defined in Sec. 2. We do this by following the process described in [5] to construct a basis for the irreducible representations of S_n. For the complex numbers, these representations are completely classified. We describe the construction for any representation of the symmetric group and obtain the relevant case for cyclic covers as an immediate corollary. At this point, we must assume that the reader is familiar with some notions from the representation theory of S_n.

Let n be a natural number and let $k_1\cdots k_m$ be a partition of n. That is, $\sum_{i=1}^m k_i = n$ and $k_1\geq k_2\geq k_3\cdots\geq k_m$.

Definition 3.1. A Young diagram associated to a partition consists of m rows such that the ith row has k_i elements of integers.

Definition 3.2. Let Y be a Young diagram. A tableau is obtained by arranging the numbers $\{1\cdots n\}$ within the m rows of Y so that:

- Each row contains exactly k_i elements,
- The numbers distributed in each row are arranged in increasing order.

Assume that $\Lambda=\{\Lambda_0,\ldots,\Lambda_k\}$ are a tableau of n. Choose an ordering for $1,\ldots,n$. For each member of the tableau $\Lambda_i = i_1<\cdots<i_l$ define the polynomials:

$$(\Lambda_i\Lambda_i) = \prod_{i_k<i_l,\{i_k,i_l\}\in\Lambda_i}(\lambda_{i_k}-\lambda_{i_l}.),$$

and,

$$p_\Lambda = \prod_{i=1}^{k}(\Lambda_i\Lambda_i).$$

S_n acts on Λ and, therefore acts on the polynomials p_Λ. We are interested in finding a basis for the linear span of p_Λ. For this purpose, we use a modification of the Garnier relation (7.1) given in [5].[a] Arrange the tableau in columns (i.e. the first column will be elements of Λ_0 the second column elements of Λ_1 etc). Overall, we have k columns for Λ. Let X be a subset of the ith column of Λ and let Y be a subset of the $i+1$th column of Λ. Let $\sigma_1 \cdots \sigma_k$ be coset representatives for $S_X \times S_Y$ in $S_{X\cup Y}$. Then, we have the Garnier relations:

Theorem 3.3. *Let μ_i denote the number of elements in the ith column of Λ. If $|X \cup Y| > \mu_i$, then*

$$\sum_{m=1}^{k} \operatorname{sign} \sigma_m (p_{\sigma_m \Lambda}) = 0.$$

Proof. If $|X \cup Y| > \mu_i$, then by the pigeon hole principle there exists an involution δ under which $\sigma_m\Lambda$ is invariant. Thus,

$$\sum_{m=1}^{k} \text{sign } \sigma_m (p_{\sigma_m \Lambda}) = \sum_{m=1}^{k} \text{sign } \sigma_m (p_{\delta\sigma_m \Lambda})$$

$$= -\sum_{m=1}^{k} \text{sign } \sigma_m\, (p_{\sigma_m \Lambda}) = 0.$$

□

In order to exhibit an explicit basis, we define a standard Young tableau.

Definition 3.4. A standard tableau is a tableau whose the rows and the columns are arranged in increasing order.

Definition 3.5. Let Λ^1 and Λ^2 be tabloids. Then, we set $\Lambda^1 < \Lambda^2$ if there is an i such that:

- if $j > i$, then j is in the same column of Λ^1, Λ^2,
- i is in a column further left in Λ^1 than Λ^2.

It is easy to see that the ordering defined above imposes total ordering on the tabloids.

Theorem 3.6. *Let $\Lambda^1 \cdots \Lambda^k$ be the collection of standard tableaus for a given partition. Then, $p_{\Lambda^1} \cdots p_{\Lambda^k}$ is a basis for the vector space spanned by Λ.*

Proof. We follow the proof given in [5]. We show that p_{Λ^k} spans any other polynomial corresponding to our partition. Let t be a tableau and suppose by induction that the theorem is proved for each tableau t_1, such that $t_1 < t$. If t is non standard there exists adjacent columns $a_1 < \cdots < a_q < \cdots < a_r$ and

[a] We were not able to find a reference to our approach for constructing Specht modules, though we are confident it's a folklore.

$b_1 < b_2 < \cdots < b_q < \cdots b_s$ such that $a_q > b_q$. Apply the Garnier relations for $X = \{a_1, \ldots, a_r\}, Y = \{b_1, \ldots, b_q\}$. For each σ, a representative of $S_X \times S_Y$ in $S_{X \bigcup Y}$ we have $[t\sigma] < t$ by the definition of the order $<$. The result follows immediately from the induction hypothesis. □

Definition 3.7. For an element k of the tableau t, the hook h_k are the elements of the tableau t to the right of it, including the element itself, and the elements in t below k.

It is well known ([5]) that the number of standard tableaus equals:

$$\frac{n!}{\prod_k h_k}.$$

4. The Ideal of Theta Identities

We apply the theory of the previous paragraph to cyclic covers of order 3. According to the theory, the hooks of the partitions correspond to tableau with 3 rows and m elements in each row. Our first corollary is:

Corollary 4.1. *The dimension of the space of polynomials* p_Λ *(and hence* $\theta^3[e_\Lambda](\tau_R)$ *corresponding to them) is*:

$$\frac{(3m)! \times 2}{(m+2)!(m+1)!m!}.$$

Hence, the corresponding space of $\theta^3[e_\Lambda](\tau_R)$ also has this dimension. We can also give for this space:

Corollary 4.2. *Let* Λ_S *be a standard partition then* $\theta^3[e_{\Lambda_S}](\tau_R)$ *is a basis for the vector space spanned by theta constants,* $\theta^3[e_\Lambda](\tau_R)$ *and* $\Lambda = \Lambda_0, \Lambda_1, \Lambda_2$ *is a partition of* $\{1 \cdots 3k\}$ *such that* $|\Lambda_i| = k$.

5. Example

We consider the case of 6 branch points, so the genus of the surface is 4. We enumerate the 15 partitions and the polynomials corresponding to them:

(1) $\Lambda = \{(1,2),(3,4),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6)$,
(2) $\Lambda = \{(1,2),(3,5),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_6)$,
(3) $\Lambda = \{(1,2),(3,6),(4,5)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_6)(\lambda_4 - \lambda_5)$,
(4) $\Lambda = \{(1,3),(2,4),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_5 - \lambda_6)$,
(5) $\Lambda = \{(1,3),(2,5),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5)(\lambda_4 - \lambda_6)$,
(6) $\Lambda = \{(1,3),(2,6),(4,5)\}\, p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_6)(\lambda_4 - \lambda_5)$,
(7) $\Lambda = \{(1,4),(2,5),(3,6)\}\, p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_6)$,
(8) $\Lambda = \{(1,4),(2,6),(3,5)\}\, p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_6)(\lambda_3 - \lambda_5)$,

(9) $\Lambda = \{(1,4),(2,3),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)(\lambda_5 - \lambda_6)$,
(10) $\Lambda = \{(1,5),(2,3),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_5)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_6)$,
(11) $\Lambda = \{(1,5),(2,4),(3,6)\}\, p_\Lambda = (\lambda_1 - \lambda_5)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_6)$,
(12) $\Lambda = \{(1,5),(2,6),(3,4)\}\, p_\Lambda = (\lambda_1 - \lambda_5)(\lambda_2 - \lambda_6)(\lambda_3 - \lambda_4)$,
(13) $\Lambda = \{(1,6),(2,3),(4,5)\}\, p_\Lambda = (\lambda_1 - \lambda_6)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)$,
(14) $\Lambda = \{(1,6),(2,4),(3,5)\}\, p_\Lambda = (\lambda_1 - \lambda_6)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_5)$,
(15) $\Lambda = \{(1,6),(2,5),(3,4)\}\, p_\Lambda = (\lambda_1 - \lambda_6)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_4)$.

In this case, by the dimension formula, the number of basis functions, $\theta^3[e_\Lambda]$ is: $2 \times (6!/4!3!2!) = 5$. The basis for the space spanned by the 15 polynomials are the polynomials that correspond to the standard tableaus:

(1) $\Lambda = \{(1,2),(3,4),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6)$,
(2) $\Lambda = \{(1,2),(3,5),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_6)$,
(3) $\Lambda = \{(1,3),(2,4),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_5 - \lambda_6)$,
(4) $\Lambda = \{(1,3),(2,5),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5)(\lambda_4 - \lambda_6)$,
(5) $\Lambda = \{(1,4),(2,5),(3,6)\}\, p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_6)$.

The remaining 10 polynomials can be rewritten as a linear combinations of the set above applying Garnier's algorithm as in Theorem 3.7. For example:

$$\begin{aligned}
(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_6)(\lambda_4 - \lambda_5) &= -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6) \\
&\quad + (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_6), \\
(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_6)(\lambda_4 - \lambda_5) &= -(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_5 - \lambda_6) \\
&\quad + (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5)(\lambda_4 - \lambda_6), \\
(\lambda_1 - \lambda_6)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_4) &= (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_6) \\
&\quad - (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5)(\lambda_4 - \lambda_6).
\end{aligned}$$

The polynomials can be expressed in a similar way, leading to identities between $\pm\theta^3[e_\Lambda](\tau_R)$. We conclude by noting: in the case of hyperelliptic curves the identities between integral characteristics of theta constants evaluated at period matrices of such curves arise from vanishing properties of theta functions. In our case, it is interesting to know whether an analogous situation arises. The following theorem [6] is the only source of cubic theta identities known to the author:

Theorem 5.1. *Let* $\left[\begin{smallmatrix}\mu \\ \mu'\end{smallmatrix}\right]$ *be an odd integral theta characteristics in genus* $3m-2$ *Then for any* $\tau \in \mathbb{H}_{3m-2}$:

$$\sum_{0 \le \nu_i \le 3} (-1)^{\sum_{i=1}^{3m-2} \mu_i \nu_i} \theta^3 \begin{bmatrix} \mu \\ \mu' + \frac{2\nu}{3} \end{bmatrix} (\tau) = 0$$

where $\mathbb{H}_{3m-2}$ *is the Siegel upper half space of genus* $3m-2$.

It is plausible that the vanishing of theta constants with rational characteristics of order 3 at τ_R will produce a new proof of the special identities obtained in this note using the Thomae formula. Finally, note that for all the identities the coefficients in (4) are ± 1. We conjecture that this is a general phenomenon.

6. Conclusion

There exists a large literature about Schottky–Jung identities and identities for hyperelliptic curves. In this note we obtained special identities for other classes of algebraic curves. We plan to pursue these themes in future notes.

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