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The Frattini module and p'-automorphisms of free pro-p groups.

By

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Abstract

If a non-trivial subgroup A of the group of continuous automorphisms of a non-cyclic free pro-p group F has finite order, not divisible by p, then the group of fixed points $\mathrm{Fix}_F(A)$ has infinite rank.

The semi-direct product $F \rtimes A$ is the *universal p-Frattini cover* of a finite group G, and so is the projective limit of a sequence of finite groups starting with G, each a canonical group extension of its predecessor by the *Frattini module*. Examining appearances of the trivial simple module $\mathbf{1}$ in the Frattini module's Jordan-Hölder series arose in investigations ([FK97], [BaFr02] and [Sem02]) of *modular towers*. The number of these appearances prevents $\mathrm{Fix}_F(A)$ from having finite rank.

For any group A of automorphisms of a group Γ, the set of fixed points $\mathbf{Fix}_{\mathbf{\Gamma}}(\boldsymbol{A}) := \{g \in \Gamma \mid \alpha(g) = g, \forall \alpha \in A\}$ of Γ under the action of A is a subgroup of Γ. Nielsen [N21] and, for the infinite rank case, Schreier [Schr27] showed that any subgroup of a free discrete group will be free. Tate (cf. [Ser02, I.§4.2, Cor. 3a]) extended this to free pro-p groups. In light of this, it is natural to ask for a free group F, what is the rank of $\mathrm{Fix}_F(A)$?

When F is a free discrete group and A is finite, Dyer and Scott [DS75] demonstrated that $\mathrm{Fix}_F(A)$ is a free factor of F, i.e. F is a free product of $\mathrm{Fix}_F(A)$ and another free subgroup of F, thus bounding the rank of $\mathrm{Fix}_F(A)$ by that of F itself. That this bound would hold for A that are merely finitely generated was a conjecture attributed to Scott; this was proven first by Gersten [Ge87] and later, independently, by Bestvina and Handel [BH92] in a

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program analogizing, to outer automorphisms of free groups, Thurston's classification of mapping classes.

But when F is a free pro-p group, this depends on whether the order of A is divisible by p. When A is a finite p-group, $\mathrm{Fix}_F(A)$ will again be a free factor of F, first shown by Scheiderer [Sche99] for F having finite rank, and extended by Herfort, Ribes, and Zalesskii [HRZ99] to the general case. It is not yet known whether the rank of $\mathrm{Fix}_F(A)$ will be bounded by that of F when A is an arbitrary (even finitely generated) pro-p group. Contrarily, Herfort and Ribes [HR90] showed much earlier that if F is non-cyclic and α is a non-trivial continuous automorphism of F of finite order, not divisible by p, then $\mathrm{Fix}_F(\langle\alpha\rangle)$ has infinite rank; their proof relies on combinatorial group theory for free pro-p groups and Thompson's theorem on the nilpotency of finite groups with fixed-point-free automorphisms of prime order.

Using instead the construction of a free pro-p group by taking the projective limit of a canonical sequence of finite p-groups, and the modular representation theory attached to this sequence, we generalize the result of Herfort and Ribes to any non-trivial finite group A of automorphisms having order prime to p, not merely the cyclic case $\langle\alpha\rangle$.

A note on reading this paper. The first four sections of this paper consist of background material, and state results largely without proof. For results given no explicit reference, the following sources may be consulted. Fried and Jarden [FJ86, Chapters 1, 15, & 20] provide all the coverage we need on profinite groups, and on the universal Frattini cover (§2) as well. For §2 and the first half of §3, also visit Fried's introduction to modular towers [Fr95, Part II, p.126-136]. Benson has written a dense primer [Be98, Chapter 1] on modular representation theory, which can help with §3 and the sometimes folkloric contents of §4.

§1. Free profinite groups.

A **profinite group** is a projective (inverse) limit of finite groups, regarded as topological groups with the discrete topology. A morphism in the category of profinite groups is a continuous group homomorphism. One important lemma [FJ86, Lem. 1.2] is that the projective limit of any *surjective* system (i.e. an inverse system all of whose maps are surjective) of finite sets will surject onto every set in the system.

A profinite group F is **free** on a set S converging to 1 if and only if it satisfies the following three conditions. First, $S \subseteq F$ must converge to 1 in F, i.e. only a finite number of elements of S lie outside any open subgroup of F.

Second, S must generate F. Third, given any map taking S into a profinite group G such that the image converges to 1 in G, there exists a unique extension of said map to a morphism from F to G. The **rank** of F is the cardinality of S.

Projective groups are very close to free groups: a profinite group is projective if and only if it is a closed subgroup of a free profinite group. (Hence, closed subgroups of projective groups are also projective.) The property of being projective is categorical. An object X is projective if, whenever there is a morphism φ from X to an object A and an epimorphism ϕ from another object B to A, there exists a morphism $\hat{\varphi} : X \to B$ such that $\phi \circ \hat{\varphi} = \varphi$. Note that, in the categories of groups and modules, a morphism ϕ is surjective if and only if it is *epic*: a morphism $\phi : B \to A$ is epic if, given any morphisms $\psi_i : A \to Y$ for $i = 1, 2$ with $\psi_1 \circ \phi = \psi_2 \circ \phi$, ψ_1 must equal ψ_2.

For a rational prime p, a **pro-p** group is merely a profinite group all of whose finite quotients (by closed normal subgroups) are p-groups. Pro-p groups form a subcategory with the property [FJ86, Prop. 20.37] that all of the projective objects are free with respect to the subcategory, in the sense that the G's in the above definition of "free" must all be pro-p groups.

Finally, the **Schreier formula** for free groups holds whether the free groups are discrete, profinite or pro-p:

Theorem 1.1 (Nielsen-Schreier [FJ86, Prop. 15.27]). *If a subgroup H of a free group F has finite index, it is free; furthermore, if F has finite rank r, the rank of H is $1 + (r-1)(G : H)$.*

§2. The universal Frattini cover.

The **Frattini subgroup $\Phi(G)$** of a profinite group G is the intersection of all maximal proper closed subgroups of G. Note the analogy to the Jacobson radical of an algebra. The Frattini subgroup of a (pro)finite group is also (pro)nilpotent, a consequence of the *Frattini argument* from whence it gets its name: given a normal subgroup K of G and a p-Sylow P of K, $G = N_G(P) \cdot K$, where $N_G(P)$ is the normalizer of P in G.

The Frattini subgroup of a pro-p group has another characterization [FJ86, Lem. 20.36]: it is the closed normal subgroup generated by the commutators and the p^{th}-powers. Put another way, $G/\Phi(G)$ is the maximal elementary abelian quotient of the pro-p group G. The Frattini series is just the descending sequence of iterations $\Phi^{n+1}(G) = \Phi(\Phi^n(G))$. This series forms a neighborhood basis of 1 in a pro-p group, i.e. the intersection of all of the terms is trivial.

We can also view "Frattinity" categorically. We say an epimorphism $\varphi : X \twoheadrightarrow A$ is a **Frattini cover** if the kernel is in the Frattini subgroup of X.

Equivalently, given any epimorphism $\phi : B \twoheadrightarrow A$ and any morphism $\psi : B \to X$ such that $\varphi \circ \psi = \phi$, the morphism ψ must be surjective. Note that, in the category of epimorphisms to A, an object which has the property that all morphisms to the object are epic will be a Frattini cover of A. We might call this a *Frattini object*. The object X is also commonly referred to as the "Frattini cover" of A; we shall do so as well.

Every profinite group G has a universal Frattini cover $\widetilde{G}$, a Frattini cover which is projective as a profinite group. (Simply find an epimorphism φ from a free profinite group onto your given group G — the universal Frattini cover will be a minimal closed subgroup H of the free group such that $\varphi(H) = G$.) Since the kernel K of our epimorphism from $\widetilde{G}$ to G is pronilpotent, i.e. a direct product of its p-Sylows (maximal pro-p subgroups), $\widetilde{G}$ will be the fibre product over G of the **universal p-Frattini covers $_p\widetilde{G}$**, where $_p\widetilde{G}$ is just the quotient of $\widetilde{G}$ by the maximal closed subgroup of K having no p-group quotient. The universal p-Frattini cover is also characterized by being the unique p-projective Frattini cover of G, p-projective meaning projective only with respect to covers (epimorphisms) with pro-p group kernel; being p-projective is equivalent [FJ86, proof of Prop. 20.47] to having free p-Sylows.

For a finite group G, the universal p-Frattini cover $_p\widetilde{G} \overset{\varphi}{\twoheadrightarrow} G$ will be the projective limit of the finite quotients produced by the Frattini series of the kernel $\mathbf{ker_0} = \ker(\varphi)$. Inductively define $\mathbf{ker_{n+1}} = \Phi(\ker_n)$ and $\boldsymbol{G_n} = {_p\widetilde{G}}/\ker_n$. Since each $\ker_n$ is a pro-p group, the quotient $\boldsymbol{M_n} = \ker_n / \ker_{n+1}$ will be an elementary abelian p-group and, in fact, an $\mathbb{F}_pG_n$-module, with the action of an element of G_n induced by conjugation, after lifting to G_{n+1}: $g \cdot m = \hat{g}m\hat{g}^{-1}$ for any $\hat{g}$ such that $g = \hat{g} \cdot \ker_n / \ker_{n+1}$.

§3. The Frattini module.

Assume now that G is finite. In the preceeding discussion, we produced a canonical sequence of finite groups whose projective limit was the universal p-Frattini cover $_p\widetilde{G}$, but only by taking quotients of $_p\widetilde{G}$. This approach depends on knowledge of $_p\widetilde{G}$, currently a mysterious object. Fortunately, we may inductively construct G_{n+1} using the modular representations of G_n.

For $\mathbb{F}_pG$-modules, projectivity has the same categorical definition given in §1. Several properties are analogous to those given earlier for profinite groups. An $\mathbb{F}_pG$-module is projective if and only if it is a direct summand of a free module. It is also projective if and only if [Be98, Cor. 3.6.10] its restriction to a p-Sylow P of G is a free $\mathbb{F}_pP$-module. We will denote the restriction of an $\mathbb{F}_pG$-module M to a subgroup X of G by $\boldsymbol{M{\downarrow}_{\mathbb{F}_pX}}$.

The **projective cover** $\mathbb{P}_{\mathbb{F}_p G}(M)$ of a finitely generated $\mathbb{F}_pG$-module M is the minimal projective $\mathbb{F}_pG$-module which has an epimorphism $\phi : \mathbb{P}_{\mathbb{F}_pG}(M) \twoheadrightarrow M$; the kernel is denoted $\boldsymbol{\Omega M}$ — Ω is known as the **Heller operator**. An $\mathbb{F}_pG$-module S is called **simple** if it has no proper non-trivial $\mathbb{F}_pG$-submodules. There is always a 1-dimensional simple $\mathbb{F}_pG$-module $\mathbf{1}_{\mathbb{F}_pG}$ having trivial G-action; when there will be no ambiguity, the subscript identifying the group ring may be omitted. Gaschütz [Ga54] produced the **Frattini module** by iterating the Heller operator twice on $\mathbf{1}_{\mathbb{F}_pG}$:

Theorem 3.1 (Gaschütz [Fr95, Lem. 2.3, p.128]).

$$\textit{As } \mathbb{F}_pG\textit{-modules}, M_0 \simeq \Omega^2\mathbf{1}_{\mathbb{F}_pG} = \Omega(\Omega\mathbf{1}_{\mathbb{F}_pG}).$$

The Heller operator is the dimension-shift operator on group cohomology [Be98, Prop. 2.5.7], so $H^2(G, M_0)$ will be 1-dimensional [Fr95, Prop. 2.7, p.132] and there will be only one non-split extension of G by M_0, up to isomorphism of groups. This will be G_1.

Since ${}_p\widetilde{G}$ is also the universal p-Frattini cover of G_1, we may use induction to see that $M_n \simeq \Omega^2\mathbf{1}_{\mathbb{F}_pG_n}$ and G_{n+1} will be the unique non-split extension of G_n by M_n.

A block of $\mathbb{F}_pG$ is an indecomposable two-sided ideal direct summand of the ring $\mathbb{F}_pG$. Every indecomposable (i.e. having no proper non-trivial direct summands) $\mathbb{F}_pG$-module M is *contained* in some block B; this means that $B' \cdot M = 0$ for every block $B' \neq B$. The principal block is the one containing $\mathbf{1}_{\mathbb{F}_pG}$ The kernel of a block is simply the set of elements of G that act trivially on all modules contained in the block, i.e. the kernel of the composition $G \hookrightarrow \mathbb{F}_pG \twoheadrightarrow B$ mapping G into the units of the *ring* B.

We now record a few results on the Frattini module. The standard notation for the maximal normal p'-subgroup (i.e. having order prime to p) of G is $\boldsymbol{O_{p'}(G)}$. Using Brauer's identification of $O_{p'}(G)$ as the kernel of the principal block, Griess and Schmid [GS78] proved that $O_{p'}(G)$ was exactly the kernel of the action of G on M_0 whenever M_0 had dimension greater than one. They also identified precisely when the latter happens:

Theorem 3.2 (Griess-Schmid [GS78, Cor. 3, p.264]). *The dimension of M_0 over $\mathbb{F}_p$ is one if and only if G is p-supersolvable with cyclic p-Sylows.*

A group G is p-supersolvable if and only if $G/O_{p'}(G)$ has a normal p-Sylow such that the quotient is abelian of exponent dividing $p-1$; this is quite restrictive.

We denote by $\#_S(M)$ the number of appearances of a simple $\mathbb{F}_pG$-module S in a Jordan-Hölder series of a given $\mathbb{F}_pG$-module M. Define the **density** $\boldsymbol{\rho_S(M)}$ of S in M to be $\#_S(M)/\dim_{\mathbb{F}_p}(M)$.

Theorem 3.3 (Density Theorem [Sem02]). *If $dim_{\mathbb{F}_p}(M_0) \neq 1$ then $\lim_{n\to\infty} \rho_S(M_n) = \rho_S(\mathbb{F}_pG/O_{p'}(G))$, for any simple $\mathbb{F}_pG$-module S.*

The converse will also hold unless $G/O_{p'}(G)$ is a cyclic p-group.

§4. Groups with normal p-Sylow.

Throughout this section, G will be a finite group with normal p-Sylow P and complement A, i.e. $G \simeq P \rtimes A$, the semi-direct product.

In this case, the universal p-Frattini cover has a simpler description. Suppose r is the minimal number of generators of P and F is a free pro-p group having rank r and a given epimorphism $\phi : F \twoheadrightarrow P$. Then A has an embedding into the continuous automorphisms of F such that its action will stabilize $\ker(\phi)$ and its action on the quotient $F/\ker(\phi)$ will correspond to its action on P via the canonical isomorphism. The universal p-Frattini cover of G will be the semi-direct product $F \rtimes A$ defined by this action, cf. [BaFr02, Rem. 5.2] and [R85].

Example. [Fr95, §II.A, p.126] If $G = D_p$ is the dihedral group of order $2p$, then ${}_p\widetilde{G}$ will be the semi-direct product $\mathbb{Z}_p \rtimes C_2$, where conjugation by the non-trivial element of the group C_2 of order 2 inverts the elements of the p-adic integers $\mathbb{Z}_p$; the canonical quotients G_n will be the dihedral groups $D_{p^{n+1}}$ of order $2p^{n+1}$. (These quotient groups appear as the Galois groups for covers of the punctured projective sphere in the Hurwitz space construction of the sequence of modular curves $X_0(p^n)$; replacing $D_{p^{n+1}}$ by G_n leads us to Fried's [Fr95, §III.C, p.144] modular towers when G is centerless and p-perfect. The fact [FK97, Lem. 3.2, p.167] that obstruction of components of the Hurwitz spaces in modular towers can only arise from appearances of $\mathbf{1}_{\mathbb{F}_pG}$ in a Jordan-Hölder series of M_0 motivated the examination of $\rho_{\mathbf{1}}(M_n)$.) The modules M_n in this case are 1-dimensional over $\mathbb{F}_p$, with the p-Sylow of $D_{p^{n+1}}$ acting trivially and the reflection in C_2 acting via mulitplication by -1.

When P is non-cyclic, the canonical quotients G_n and the modules M_n are not so easily described, even, for example, when G is the alternating group A_4 or the Klein 4-group. However, the normality of the p-Sylow still strongly affects the modular representation theory of G; we'll now collect four results we'll need.

First, observe that we can calculate $\dim_{\mathbb{F}_p}(M_0)$ explicitly when G is a p-group P. Since projective $\mathbb{F}_pP$-modules must be free, $\mathbb{P}_{\mathbb{F}_pP}(\mathbf{1}) \simeq \mathbb{F}_pP$, and $\Omega\mathbf{1}_{\mathbb{F}_pP}$ will be the augmentation ideal of $\mathbb{F}_pP$. This is well-known to have the same number of generators r as a module that P has as a group, so $\mathbb{P}_{\mathbb{F}_pP}(\Omega\mathbf{1})$ will be isomorphic to the direct sum of r copies of $\mathbb{F}_pP$. Hence, $\dim_{\mathbb{F}_p}(M_0)$ will equal $\dim_{\mathbb{F}_p}(\mathbb{P}_{\mathbb{F}_pP}(\Omega\mathbf{1}))$ minus the dimension over $\mathbb{F}_p$ of the augmentation ideal, i.e. $1+(r-1)|P|$. Note the similarity to the Schreier formula; for pro-p groups, one can be derived from the other.

Second, examine the density $\rho_{\mathbf{1}}(\mathbb{F}_pG)$. The restriction of the $\mathbb{F}_pG$-module $N = \mathbb{F}_pG \otimes_{\mathbb{F}_pA} \mathbf{1}_{\mathbb{F}_pA}$ to P will be isomorphic to the group ring $\mathbb{F}_pP$, and hence N has no proper non-trivial projective submodule. But the quotient of N by the submodule generated by $\{(g-1)\otimes m \mid g \in P \text{ and } m \in \mathbf{1}_{\mathbb{F}_pA}\}$ is isomorphic to $\mathbf{1}_{\mathbb{F}_pG}$, so N will be the projective cover of $\mathbf{1}_{\mathbb{F}_pG}$.

For any finite group Γ, $\#_{\mathbf{1}}(\mathbb{F}_p\Gamma) = \dim_{\mathbb{F}_p}(\mathbb{P}_{\mathbb{F}_p\Gamma}(\mathbf{1}))$, cf. [Be98, Lem. 1.7.7 & Prop. 3.1.2] Therefore,

$$\rho_{\mathbf{1}}(\mathbb{F}_pG) = \frac{\#_{\mathbf{1}}(\mathbb{F}_pG)}{\dim_{\mathbb{F}_p}(\mathbb{F}_pG)} = \frac{\dim_{\mathbb{F}_p}(N)}{|G|} = \frac{|P|}{|G|} = \frac{1}{|A|}.$$

Third, $\Omega^2\mathbf{1}_{\mathbb{F}_pG}\downarrow_{\mathbb{F}_pP} \simeq \Omega^2\mathbf{1}_{\mathbb{F}_pP}$, so our first observation allows us to explicitly calculate $\dim_{\mathbb{F}_p}(\Omega^2\mathbf{1}_{\mathbb{F}_pG})$ to be $1+(r-1)|P|$. This can be seen through Ribes' result in the second paragraph of this section, since in either case the Frattini module will be isomorphic to $\ker(\phi)/\Phi(\ker(\phi))$. Alternatively, and equivalently, one could use representation theory to prove this. We just saw that $\mathbb{P}_{\mathbb{F}_pP}(\mathbf{1}) \simeq \mathbb{P}_{\mathbb{F}_pG}(\mathbf{1})\downarrow_{\mathbb{F}_pP}$; it turns out that this isomorphism preserves the two modules' radical series, i.e. the product of the module with the successive powers of the Jacobson radical of the group algebra. For example, see [Sem02].

Finally, remember Maschke's theorem:

Theorem 4.1 (Maschke [Be98, Cor. 3.6.12]). *If a finite group A has order relatively prime to the characteristic of the field k, then kA is semi-simple.*

In other words, $\mathbb{F}_pA$-modules will have trivial cohomology, since there can be no non-split exact sequences of $\mathbb{F}_pA$-modules.

§5. Automorphisms of free groups.

Note that if a pro-p group is (topologically) finite generated, Serre [Ser75] has shown that all of its subgroups having finite index are open; Anderson [A76, Thm. 3] uses, and extends, the proof. Consequently, any automorphism of a (topologically) finitely generated pro-p group will be continuous.

Theorem 5.1. *If a non-trivial subgroup A of the group of continuous automorphisms of a non-cyclic free pro-p group F has finite order, not divisible by p, then the group of fixed points $Fix_F(A)$ has infinite rank.*

Proof. The intuition is that the Density Theorem will force the rank of $\mathrm{Fix}_{\ker_n}(A)$ to be bounded below by a fixed proportion of the rank of $\ker_n$, but this growth in its rank is too much for the Schreier formula to allow if $\mathrm{Fix}_F(A)$ were to have finite rank. To make this concrete, we need some notation.

Let G be the semi-direct product $F/\Phi(F)\rtimes A$: since $\Phi(F)$ is characteristic in F, A will act canonically on $F/\Phi(F)$ and we can regard $\alpha \in A$ as acting via conjugation — ${}^{\alpha}x := \alpha x \alpha^{-1} = \alpha(x)$. The universal p-Frattini cover ${}_p\widetilde{G}$ of G is just $F\rtimes A$ with $\ker_0 = \Phi(F)$. We define $\ker_n$, G_n, and M_n as before, and for convenience write F_n in place of $F/\ker_n$. Since the p-Sylow $F/\Phi(F)$ of G is non-cyclic, the theorem of Griess-Schmid shows that the condition for the Density Theorem holds. By a theorem of Philip Hall [Ha63, Thm. 12.2.2] , any group, having order prime to p, of automorphisms of a pro-p group P must act faithfully on $P/\Phi(P)$: $\mathrm{Fix}_{F_n}(A)$ will be a proper subgroup of F_n for all n.

Consider the possibility that F has infinite rank, but $\mathrm{Fix}_F(A)$ has not. In this case, take any element x of F not fixed by A and consider the closed subgroup of F generated by the union of the orbit of x under A and a finite set of (topological) generators of $\mathrm{Fix}_F(A)$. As a closed subgroup of a free pro-p group, this will also be free pro-p, but now of finite rank and with a non-trivial quotient of A acting faithfully as continuous automorphisms. But then $\mathrm{Fix}_F(A)$ would be the subgroup of fixed points of A inside this finite rank free pro-p group. So, assume that F and $\mathrm{Fix}_F(A)$ have finite ranks $r > 1$ and s, respectively.

Let us first see that $\mathrm{Fix}_{\ker_n}(A)/\mathrm{Fix}_{\ker_{n+1}}(A) \simeq \mathrm{Fix}_{M_n}(A)$; this is equivalent to $\mathrm{Fix}_{F_n}(A)$ forming a *surjective* system whose projective limit is $\mathrm{Fix}_F(A)$. (Note that $\mathrm{Fix}_{\ker_n}(A)$ is just $\mathrm{Fix}_F(A) \cap \ker_n$.) Given x in $\mathrm{Fix}_{F_n}(A)$, consider an element $\hat{x}$ of the preimage of $\{x\}$ under the natural quotient map $\varphi_n : F_{n+1} \twoheadrightarrow F_n$. The assignment $\alpha \mapsto a_\alpha := {}^{\alpha}\hat{x}\hat{x}^{-1}$ is a 1-cocycle for A with values in M_n: ${}^{\alpha}a_\beta a_\alpha a_{\alpha\beta}^{-1} = 1$, writing M_n multiplicatively. Since $(|A|, p) = 1$, Maschke's theorem applies and $M_n{\downarrow}_{\mathbb{F}_pA}$ has trivial cohomology. Hence, there exists a $\mu \in M_n$ such that $a_\alpha = {}^\alpha\mu\mu^{-1}$ for all α in A. The element $\mu^{-1}\hat{x}$ of $\varphi_n^{-1}(\{x\})$ is then fixed by A.

Since $\mathrm{Fix}_F(A)$ is a closed subgroup of the free pro-p group F, it must also be free, and so we can use the Schreier formula to compute the rank of its subgroups $\mathrm{Fix}_{\ker_{n+1}}(A)$:

$$\begin{aligned}\text{rank of } \mathrm{Fix}_{\ker_{n+1}}(A) &= 1 + (s-1)|\mathrm{Fix}_F(A)/\mathrm{Fix}_{\ker_{n+1}}(A)| \\ &= 1 + (s-1)|\mathrm{Fix}_F(A)/\mathrm{Fix}_{\ker_n}(A)||\mathrm{Fix}_{M_n}(A)| \\ &\leq 1 + (s-1)|F_n||\mathrm{Fix}_{M_n}(A)|.\end{aligned}$$

As noted before, the density $\rho_{\mathbf{1}}(\mathbb{F}_pG)$ is $1/|A|$. By the Density Theorem,

$$\lim_{n\to\infty} \frac{\log_p |\mathrm{Fix}_{M_n}(A)|}{\log_p |M_n|^{1/|A|}} = \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathrm{Fix}_{M_n}(A))}{\frac{1}{|A|}\cdot \dim_{\mathbb{F}_p}(M_n)} = |A| \cdot \rho_{\mathbf{1}}(\mathbb{F}_pG) = 1.$$

Since $|A| > 1$, there must then exist a real number $\varepsilon \in (0,1)$ and a positive integer N such that $|\mathrm{Fix}_{M_n}(A)| < |M_n|^\varepsilon$ for all $n > N$. Now use the identities $\dim_{\mathbb{F}_p}(M_{n+1}) = 1 + (r-1)|F_{n+1}|$ and $|F_{n+1}| = |F_n| \cdot |M_n|$, and the fact that $|F_n|$ and $|M_n|$ increase monotonically and without bound, to get:

$$\begin{aligned}\lim_{n\to\infty} \frac{\text{rank of } \mathrm{Fix}_{\ker_{n+1}}(A)}{\dim_{\mathbb{F}_p}(M_{n+1})} &\leq \lim_{n\to\infty} \frac{1 + (s-1)|F_n||\mathrm{Fix}_{M_n}(A)|}{1 + (r-1)|F_n||M_n|} \\ &= \lim_{n\to\infty} \frac{(s-1)|\mathrm{Fix}_{M_n}(A)| \cdot |M_n|^{-\varepsilon}}{(r-1)|M_n|^{1-\varepsilon}} \\ &= 0.\end{aligned}$$

Another use of the Density Theorem gives a contradiction:

$$\begin{aligned}\lim_{n\to\infty} \frac{\text{rank of } \mathrm{Fix}_{\ker_{n+1}}(A)}{\dim_{\mathbb{F}_p}(M_{n+1})} &\geq \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(\mathrm{Fix}_{M_{n+1}}(A))}{\dim_{\mathbb{F}_p}(M_{n+1})} \\ &= \rho_{\mathbf{1}}(\mathbb{F}_pG) \\ &= \frac{1}{|A|} \\ &> 0. \qquad \square\end{aligned}$$

We can actually do better in the infinite rank case than this proof might indicate. A slight generalization of [BaFr02, Prop. 5.3] to profinite p-Sylows will show that even if A fixes no non-trivial element of $F/\Phi(F)$, a possibility even if F has infinite rank, then the rank of $\mathrm{Fix}_{\Phi(F)/\Phi^2(F)}(A)$ will already be infinite.

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