# MODULAR REPRESENTATIONS FOR MODULAR TOWERS.

*by*

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**Contents**



Section 1 recaps Dèbes' presentation [**De04**] of the universal $p$-Frattini cover as an extension of a finite group by a free pro-$p$ group and gives examples illustrating a common lack of discrete groups to use as a model for this profinite object. Section 2 constructs the quotient groups ${}_p^n\tilde{G}$ instead via modular representation theory, the key tool being a categorical equivalence due to Gruenberg and Roggenkamp; each such group is a canonical extension of the preceeding group ${}_p^{n-1}\tilde{G}$ by that group's $p$-Frattini module. I give several examples of this module in Section 3 and discuss means of reducing calculations to the normalizer of the $p$-Sylow. Section 4 concerns the evolution with $n$ of the dimension and composition series of the $p$-Frattini modules. Only appearances of the trivial simple module in a composition series can obstruct components of Hurwitz spaces; we see this in the final section, as well as how the division of cusps into types forms part of a rubric toward a formula for the genuses of high-level components in a reduced modular tower when $r = 4$. An appendix explicitly displays the functors for the Gruenberg-Roggenkamp equivalence.

Despite relatively few explicit citations herein, the results surveyed have been comprehensively catalogued (and produced) by Fried in his work on modular towers and his series of papers on the subject are a primary source: [**F95**], [**FK97**], [**F02**], [**BF02**], and [**FS04**]. For a reference on modular representation theory, I recommend Benson's text [**Be98**].

Before proceeding, recall some elementary categorical definitions.

***Definition 0.1***. — In any category, for any objects $X$ and $Y$, a morphism $\phi \in \mathrm{Hom}\,(X,Y)$ is epic iff for all objects $Z$ and for all morphisms $\psi_1, \psi_2 \in \mathrm{Hom}\,(Y,Z)$, if $\psi_1 \circ \phi = \psi_2 \circ \phi$ then $\psi_1 = \psi_2$.

This purely categorical definition is synonymous with "surjective" in the categories of abstract groups, profinite groups, and modules. Hence, equivalences between these categories pass along surjectivity of morphisms, as well as the following two properties of objects:

***Definition 0.2***. — An object $P$ of a category $\mathcal{C}$ is projective iff for any objects $X$ and $Y$ of $\mathcal{C}$, any morphism $\psi \in \mathrm{Hom}\,(P,Y)$, and any epic morphism $\phi \in \mathrm{Hom}\,(X,Y)$, there exists a morphism $\pi \in \mathrm{Hom}\,(P,X)$ such that $\phi \circ \pi = \psi$, as illustrated in the following commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\forall\psi"] \arrow[d, "\exists\pi"] & Y \arrow[d, equal] \\ X \arrow[r, two heads, "\forall\phi"] & Y \end{tikzcd}$$

An object $F$ of $\mathcal{C}$ is Frattini iff every morphism to $F$ is epic, i.e. for any object $X$ of $\mathcal{C}$ and any morphism $\phi \in \mathrm{Hom}\,(X,F)$, $\phi$ is epic.

Given an object $X$ of a category $\mathcal{C}$, a cover of $X$ is defined to be an epic morphism in $\mathrm{Hom}\,(Y,X)$ for some object $Y$. The collection of covers of $X$ comprise the class of objects of a category whose morphisms are as follows — given two covers, $\phi_1 \in \mathrm{Hom}\,(Y,X)$ and $\phi_2 \in \mathrm{Hom}\,(Z,X)$, $\mathrm{Hom}\,(\phi_1,\phi_2)$ is defined to be the set of morphisms $\psi$ in $\mathrm{Hom}\,(Y,Z)$ such that $\phi_2 \circ \psi = \phi_1$. We also sometimes consider subcategories where we restrict the covers under consideration, but in these cases the set of morphisms between two objects remains the same as in the full category of covers, i.e. these subcategories are full in the technical sense. In the categories of covers we will consider, epic morphisms will always turn out to be surjective.

**Conventions.** $G$ is always a finite group and $k$ is always a field. The cyclic group of order $n$ is $C_n$, the dihedral group of order $2n$ is $D_n$, the alternating group on $n$ letters is $A_n$, and the symmetric group on $n$ letters is $S_n$. The conjugate $gag^{-1}$ of one element $a$ of $G$ by another element $g$ is denoted by ${}^{g}a$. All modules are finitely generated left-modules.

## 1. The universal $p$-Frattini cover

Fix a finite group $G$ and consider the category of covers of $G$ within the category of profinite groups; call this category of covers $\mathcal{C}(G)$. A projective Frattini object in this category is called the **universal Frattini cover** of $G$, as is its domain, which is given the notation $\tilde{G}$. One proof of the existence of this object comes from a Zorn's lemma construction: projective profinite groups are precisely those isomorphic to closed subgroups of free profinite groups [**FJ86**], so take a minimal closed subgroup mapping onto $G$ in any epimorphism onto $G$ with domain a free profinite group. The kernel of the universal Frattini cover is (pro-)nilpotent by the Frattini Argument from which its name derives. Hence, it is the product of its $p$-Sylows; being closed subgroups of a projective profinite group, they will have to be projective as well, and projective pro-$p$ groups must be free as pro-$p$ groups [**FJ86**].

Now consider ${}_p\tilde{G}$, the quotient of $\tilde{G}$ by the $p'$-Hall subgroup of the kernel of $\tilde{G}$, i.e. the product of all of the $s$-Sylows of the kernel, where $s$ denotes a rational prime distinct from $p$. This quotient profinite group is called the **universal $p$-Frattini cover** of $G$, as is the natural map to $G$ which it enherits. This map is also characterized by being the projective Frattini object in the subcategory $\mathcal{C}_{p^\infty}(G)$ of $\mathcal{C}(G)$ whose objects are precisely those objects of $\mathcal{C}(G)$ with kernel a pro-$p$ group. The kernel of the universal $p$-Frattini cover is a free pro-$p$ group called $\mathbf{ker_0}$.

The easiest example is when $G$ is a $p$-group; then, ${}_p\tilde{G}$ is a free pro-$p$ group with the same minimal number of (topological) generators as $G$. As a consequence of Schur-Zassenhaus, if $G$ merely has a normal $p$-Sylow $P$, then $G \simeq P \rtimes H$, where $H \simeq G/P$; we say $G$ is **$p$-split**. When $G$ is $p$-split, ${}_p\tilde{G} \simeq \hat{F}_n(p) \rtimes H$, where $n$ is the minimal number of generators of the $p$-Sylow $P$ of $G$ and $\hat{F}_n(p)$ is the pro-$p$ completion of the free group on $n$ generators. The rank (minimal number of topological generators) of $\ker_0$ is $1 + (n-1)|P|$, by the Schreier formula.

***Example 1.1***. — The alternating group on four elements is isomorphic to $V_4 \rtimes C_3$, where a given generator $g$ of $C_3$ acts on the Klein four-group $V_4$ by cyclically permuting the three non-trivial elements. There will be a choice of two (topological) generators $a$ and $b$ of $\hat{F}_2(2)$ such that conjugation by $g$ on $\hat{F}_2(2)$ (in ${}_2\tilde{A}_4 \simeq \hat{F}_2(2) \rtimes C_3$) is given by ${}^g a = b$ and ${}^g b = b^{-1}a^{-1}$. Clearly, $a$ and $b$ generate a discrete, dense free subgroup $F_2$ of $\hat{F}_2(2)$ which is stabilized by $C_3$. We get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_2 & \longrightarrow & F_2 \rtimes C_3 & \longrightarrow & C_3 & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 1 & \longrightarrow & \hat{F}_2(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & C_3 & \longrightarrow & 1 \end{array}$$

By the Schreier formula, $\ker_0$ has rank 5 and its intersection with $F_2$ is a free group $F_5$ of rank 5, normal inside of $F_2$. There is another commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F_5 & \longrightarrow & F_2 \rtimes C_3 & \longrightarrow & A_4 & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & A_4 & \longrightarrow & 1
\end{array}$$

In general, the approach we've been following so far fails to provide detailed information about the universal $p$-Frattini cover, the preceeding example being a rare counterexample describable by a discrete analogue. Even $p$-split groups can often not be described this way. One reason to expect this failure is the non-constructiveness of using Zorn's lemma to create the universal cover. Consider two examples illustrating the limitations.

***Example 1.2.*** — Our first example comes from Holt and Plesken [**HP89**]. Embedding $A_4$ into $A_5$ leads to an embedding of ${}_2\tilde{A}_4$ into ${}_2\tilde{A}_5$ and the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_4 & \longrightarrow & A_4 & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_5 & \longrightarrow & A_5 & \longrightarrow & 1
\end{array}$$

The leftmost vertical map is an isomorphism. However, there is NO group $\Gamma$ which can fit into a commutative diagram of exact sequences of the following form, where the vertical maps are dense monomorphisms:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F_5 & \longrightarrow & \Gamma & \longrightarrow & A_5 & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
1 & \longrightarrow & \hat{F}_5(2) & \longrightarrow & {}_2\tilde{A}_5 & \longrightarrow & A_5 & \longrightarrow & 1
\end{array}$$

The proof examines the character of the 2-adic Frattini lattice of $\mathrm{SL}_2(\mathbb{F}_5)$ and is beyond the scope of these limited notes.

***Example 1.3.*** — A result of Dyer and Scott [**DS75**] says that for any automorphism $\sigma$ of prime order $s$ acting on a discrete free group $F$, there is a basis $X$ of $F$ such that, for every $x$ in $X$, one of the following holds:

- **i) :** $\sigma(x) = x$
- **ii) :** $x$ belongs to a subset of $X$ containing exactly $s$ elements which are cyclically permuted by $\sigma$
- **iii) :** $x$ belongs to a subset $\{x_1, \ldots, x_{s-1}\}$ of $X$ such that $\sigma(x_j) = x_{j+1}$ when $j < s - 1$, while $\sigma(x_{s-1}) = x_{s-1}^{-1} \ldots x_1^{-1}$.

As a corollary, the induced action of $\sigma$ on the free abelian group (and hence $\mathbb{Z}\langle\sigma\rangle$-module) $F/(F,F)$ would force the latter to be a direct sum of copies of the trivial module, the group ring $\mathbb{Z}\langle\sigma\rangle$, and the augmentation ideal of the group ring.

Now let $G = \mathbb{F}_8 \rtimes \mathbb{F}_8^*$, where $\mathbb{F}_8$ denotes the additive group of the field and $\mathbb{F}_8^*$ the multiplicative group, while the action of the latter on the former is by multiplication.

Then, $G \simeq (C_2 \times C_2 \times C_2) \rtimes C_7$, where a generator $g$ of $C_7$ cyclically permutes the non-trivial elements of the 2-Sylow of $G$. The universal 2-Frattini cover ${}_2\tilde{G}$ is isomorphic to $\hat{F}_3(2) \rtimes C_7$, but there is no non-trivial action of $C_7$ on the discrete free group $F_3$.

Furthermore, $\ker_0$ will be a free pro-2 group of rank 17. Conjugation by a lift of $g$ in ${}_2\tilde{G}$ produces a natural $\mathbb{Z}_2 C_7$-lattice structure on $\ker_0 / (\ker_0, \ker_0)$, whose fixed points under the action of $C_7$ form a sublattice of rank 2. Suppose there was a group $\Gamma$ that fit into a commutative diagram of exact sequences of the following form, where the vertical maps are dense monomorphisms:

$$\begin{tikzcd} 1 \arrow[r] & F_{17} \arrow[r] \arrow[d] & \Gamma \rtimes C_7 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \ker_0 \arrow[r] & {}_2\tilde{G} \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

Then $F_{17}/(F_{17}, F_{17})$ will be a $\mathbb{Z}C_7$-lattice, with a dense monomorphism into $\ker_0 / (\ker_0, \ker_0)$; the fixed points of the action of $C_7$ on $F_{17}/(F_{17}, F_{17})$ will thus form a sublattice of rank 2. However, the result of Dyer-Scott would force the fixed point sublattice to have rank at least 5, a contradiction. In fact, by examining the character of the 2-adic Frattini lattice of $G$, we can rule out any group from appearing in the middle term of the above exact sequence, not merely a semidirect product $\Gamma \rtimes C_7$.

## 2. The $p$-Frattini module

Modular represenation theory is the right context to produce a canonical sequence of finite groups whose projective limit is the universal $p$-Frattini cover. This approach is entirely constructive and the modular tower of Hurwitz spaces is defined using this sequence of groups.

Let $R$ be a commutative ring with 1. Every group ring $RG$ has a one-dimensional trivial simple module, a copy of $R$ on which every element of $G$ acts as the identity; we denote it by $\mathbf{1}_{RG}$, omitting the subscript when the context is obvious. The kernel of the canonical morphism from $RG$ to $\mathbf{1}_{RG}$, sending the identity of $G$ to 1, is called the augmentation ideal and is denoted by $\omega_{RG}$. We often omit the subscript on both of these objects when the context is obvious.

Let $\mathcal{C}_{RG}(G)$ represent the category of covers of $G$ (in the category of groups) whose kernels are abelian groups with a specified $R$-module structure. note that these kernels are naturally $RG$-modules with the action of an element $g \in G$ given via conjugation by any preimage of $g$ in the domain of the cover. For any $RG$-module $M$, let $\mathcal{C}_{RG}(M)$ be the category of covers of $M$ (in the category of $RG$-modules).

***Fact 2.1* (Gruenberg-Roggenkamp, [GR77]).** — *There is an equivalence of categories between $\mathcal{C}_{RG}(G)$ and $\mathcal{C}_{RG}(\omega_{RG})$ under which corresponding objects have isomorphic kernels.*

**Note:** When $R$ is $\mathbb{Z}$ or $\mathbb{F}_p$, the group structure of the kernel determines its $R$-module structure. If $R$ is $\hat{\mathbb{Z}}$ (or $\mathbb{Z}_p$) and the kernel is a finitely generated $R$-module, the domain of the cover is naturally a profinite group; conversely, if the domain of the cover is given a profinite group structure, the kernel will inherit a canonical $\hat{\mathbb{Z}}$-module structure. Finally, note that the finite-index subgroups of any finitely (topologically) generated profinite group are closed (cf Nikolov-Segal [**NS03**]), so when $R$ is $\hat{\mathbb{Z}}$ and the kernel is a finitely generated $R$-module, the *group structure* of the domain will determine the topology.

***Remark 2.2***. — For any homomorphism $\varphi : H \to G$, there is a covariant functor $\mathrm{res}_\varphi$ from $\mathcal{C}_{RG}(G)$ to $\mathcal{C}_{RH}(H)$ given by taking the fibre product with $\varphi$. The inclusion $\omega_{RH} \subseteq \omega_{RG}$ is an $RH$-module homomorphism and this defines a covariant functor $\mathrm{res}_\varphi$ from $\mathcal{C}_{RG}(\omega_{RG})$ to $\mathcal{C}_{RH}(\omega_{RH})$ by taking the fibre product with the inclusion. These two functors commute with the Gruenberg-Roggenkamp categorical equivalence.

For every finitely generated $kG$-module $M$, there will exist a projective Frattini object in $\mathcal{C}_{kG}(M)$. The domain of such an object will be a projective $kG$-module denoted by $\mathbb{P}_{kG}(M)$; the kernel of a projective Frattini object in $\mathcal{C}_{kG}(M)$ is denoted by $\Omega_{kG}M$. The process of assigning such a kernel to a module is called the Heller operator (denoted by $\Omega_{kG}$, of course), and iterations of it are defined inductively: $\Omega^{n+1}_{kG}M := \Omega_{kG}(\Omega^n_{kG}M)$.

By Gruenberg-Roggenkamp's categorical equivalence, there will be a projective Frattini object in $\mathcal{C}_{\mathbb{F}_pG}(G)$; the domain of this object is denoted by ${}^1_p\tilde{G}$ and is called the **universal elementary abelian $p$-Frattini cover** of $G$. The sequence of finite groups used in the definition of a modular tower are defined inductively from this: ${}^{n+1}_{\phantom{n+}p}\tilde{G} := {}^1_p\left(\widetilde{{}^n_p\tilde{G}}\right)$.

These groups can also be defined inductively as quotients of the entire universal $p$-Frattini cover. The Frattini subgroup $\Phi(P)$ of a pro-$p$ group $P$ is defined to be $\overline{P^p(P,P)}$, the closure of the subgroup generated by the $p$-th powers and commutators of elements of $P$. Iteratively defining $\Phi^{n+1}(P) := \Phi(\Phi^n(P))$ yields the Frattini series, a descending series of closed subgroups of $P$. The intersection of the members of the Frattini series is trivial since this holds true in any finite $p$-group. Now define iteratively $\ker_{n+1} := \Phi(\ker_n)$, beginning with the kernel $\ker_0$ of the map from ${}_p\tilde{G}$ down to $G$.

***Theorem 2.3***. — *For every natural number $n$, ${}_p\tilde{G}/\ker_n \simeq {}^n_p\tilde{G}$, and so ${}_p\tilde{G} \simeq \varprojlim {}^n_p\tilde{G}$.*

*Proof*. — The second isomorphism follows from the first because $\cap_{n\to\infty} \ker_n = 1$. (By convention, ${}^0_p\tilde{G} = G$.) Note that if $H \twoheadrightarrow G$ is Frattini with $p$-group kernel, what we call a $p$-Frattini cover, then ${}_p\tilde{H} \simeq {}_p\tilde{G}$. The first isomorphism will thus be proven by induction once it is shown that ${}_p\tilde{G}/\ker_1 \simeq {}^1_p\tilde{G}$. The universal defining property of ${}^1_p\tilde{G}$

forces it to have an epimorphism onto ${}_p\tilde{G}$; vice-versa, ${}_p\tilde{G}$ must have an epimorphism onto ${}_p^1\tilde{G}$, and the kernel of this must contain $\ker_1$. □

One can specify the isomorphism class of the kernel (the ***p*-Frattini module**) of the universal elementary abelian $p$-Frattini cover of $G$ precisely in terms of the modular representation theory of $G$:

***Theorem 2.4* (Gaschütz, [Ga54]).** — *The p-Frattini module of G is isomorphic to* $\Omega^2_{\mathbb{F}_pG}\mathbf{1}$.

*Proof.* — Since projective $kG$-modules are precisely those isomorphic to a direct summand of a free $kG$-module, there is a projective $\mathbb{F}_pG$-module $P$ such that $\mathbb{F}_pG \simeq P \oplus \mathbb{P}_{\mathbb{F}_pG}(\mathbf{1})$ and hence $\omega_{\mathbb{F}_pG} \simeq P \oplus \Omega_{\mathbb{F}_pG}\mathbf{1}$. Thus, $\mathbb{P}_{\mathbb{F}_pG}(\omega_{\mathbb{F}_pG}) \simeq P \oplus \mathbb{P}_{\mathbb{F}_pG}(\Omega_{\mathbb{F}_pG}\mathbf{1})$ and the result follows from the equivalence of Gruenberg and Roggenkamp. □

***Remark 2.5.*** — A minor corollary of the theorem is that the $p$-Frattini module has dimension congruent to 1 modulo the order of the $p$-Sylow $P$ of $G$, since projective $\mathbb{F}_pP$-modules must be free.

By dimension-shifting,

$$\begin{aligned} \mathrm{H}^2(G, \Omega^2_{\mathbb{F}_pG}\mathbf{1}) &\simeq \mathrm{Ext}^2(\mathbf{1}_{\mathbb{F}_pG}, \Omega^2_{\mathbb{F}_pG}\mathbf{1}) \\ &\simeq \mathrm{Ext}^1(\mathbf{1}_{\mathbb{F}_pG}, \Omega^1_{\mathbb{F}_pG}\mathbf{1}) \\ &\simeq \mathrm{Hom}\left(\mathbf{1}_{\mathbb{F}_pG}, \mathbf{1}_{\mathbb{F}_pG}\right) \\ &\simeq \mathbb{F}_p \end{aligned}$$

and so there is a unique group providing a non-split extension of $G$ by its $p$-Frattini module. This must be ${}_p^1\tilde{G}$.

***Example 2.6.*** — The modular curve $X_1(p^{n+1})$ is a quotient of the reduced Hurwitz space associated to $D_{p^{n+1}}$ with $r = 4$ and each conjugacy class the set of involutions. Assume that $p$ is odd. Let's see that ${}_p^1\widetilde{D_{p^n}} \simeq D_{p^{n+1}}$ when $n \geq 1$.

There are two simple $\mathbb{F}_pD_{p^n}$-modules, the trivial module $\mathbf{1}$ and the sign module $\mathrm{Sgn}_p$, which consists of a copy of $\mathbb{F}_p$ with the involutions of $D_{p^n}$ acting as multiplication by $-1$ and the other elements acting trivially. The restriction of any simple module $S$ to a 2-Sylow $H$ is projective and so $S{\downarrow_{\mathbb{F}_pH}}{\uparrow^{\mathbb{F}_pD_{p^n}}} \simeq \mathbb{P}_{\mathbb{F}_pD_{p^n}}(S)$. The theory for the $p$-split case readily shows that $\mathbb{P}_{\mathbb{F}_pD_{p^n}}(\mathrm{Sgn}_p) \simeq \mathbb{P}_{\mathbb{F}_pD_{p^n}}(\Omega_{\mathbb{F}_pD_{p^n}}\mathbf{1})$. Conclude from counting dimensions that the $p$-Frattini module for $D_{p^n}$ is one-dimensional (and, in fact, $\mathrm{Sgn}_p$); the dihedral groups are a model for the very restricted class of groups for which this happens (see Fact 4.1).

Now note that the natural map $D_{p^{n+1}} \twoheadrightarrow D_{p^n}$ is Frattini and, since its kernel is one-dimensional, must be the universal elementary abelian $p$-Frattini cover.

In the sequel, to remove the notational heaviness, ${}^n_p\tilde{G}$ will be denoted by $G_n$ and $\Omega^2_{\mathbb{F}_p {}^n_p\tilde{G}}\mathbf{1}$ by $M_n$.

## 3. Restriction to the normalizer of a $p$-Sylow

There are explicit methods for computing the $p$-Frattini module of a $p$-split group (i.e. the normalizer of a $p$-Sylow), e.g. through the use of an expansion of Jennings' theorem [**S04a**]. I omit these here for reasons of brevity, but will show a relationship between the $p$-Frattini module for the normalizer and that for the whole group. We will also see more intricate examples of $p$-Frattini modules.

Recall the concepts of restriction and induction. Fix a subgroup $H$ of $G$. The restriction $M{\downarrow}_{kH}$ of a $kG$-module $M$ to $kH$ simply means: regard $M$ as a $kH$-module via the canonical inclusion of $kH$ in $kG$. Given a $kH$-module $M$, the induced module $M{\uparrow}^{kG}$ is the tensor product $kG \otimes_{kH} M$. Since projective modules are exactly those isomorphic to direct summands of free modules, both the restriction and induction of a projective module are projective.

***Lemma 3.1.*** — *Let $H$ be a subgroup of $G$. The pullback of $H$ in the cover ${}^1_p\tilde{G} \twoheadrightarrow G$ is a projective object in $\mathcal{C}_{\mathbb{F}_pH}(H)$. There is a projective $\mathbb{F}_pH$-module $N$ such that $M_0{\downarrow}_{\mathbb{F}_pH} \simeq N \oplus \Omega^2_{\mathbb{F}_pH}\mathbf{1}$.*

*Proof.* — The pullback of $H$ in the group cover corresponds under the Gruenberg-Roggenkamp equivalence to the pullback of $\omega_{\mathbb{F}_pH}$ in the cover $\mathbb{P}_{\mathbb{F}_pG}(\omega_{\mathbb{F}_pG}) \overset{\varphi}{\twoheadrightarrow} \omega_{\mathbb{F}_pG}$ (cf Remark 2.2). There is a free $\mathbb{F}_pH$-module $N'$ such that $\omega_{\mathbb{F}_pG}{\downarrow}_{\mathbb{F}_pH} \simeq N' \oplus \omega_{\mathbb{F}_pH}$. Since $N'$ is projective, it splits in the cover $\varphi$ (regarded as an $\mathbb{F}_pH$-module homomorphism), and so $\mathbb{P}_{\mathbb{F}_pH}(\omega_{\mathbb{F}_pG}){\downarrow}_{\mathbb{F}_pH}$ is a direct sum of $N'$ and some projective cover of $\omega_{\mathbb{F}_pH}$: this projective cover corresponds to the pullback of $H$. The final statement follows from the decomposition of this projective cover into the direct sum of a projective module $N$ and $\mathbb{P}_{\mathbb{F}_pH}(\omega_{\mathbb{F}_pH})$. □

Remember that a module is indecomposable if it has no non-trivial direct sum decomposition.

***Fact 3.2.*** — *A $kG$-module $M$ is indecomposable and non-projective iff $\Omega_{kG}M$ is.*

Hence, the $p$-Frattini module of $G$ is indecomposable and non-projective when $p$ divides the order of $G$. Together, the next lemma and the fact following it show a dichotomy between level 0 and the higher levels. The notation $N_G(H)$ denotes the subgroup of elements of $G$ that normalize a given subgroup $H$ of $G$.

***Lemma 3.3.*** — *$M_0$ is isomorphic to a direct summand of $\left(\Omega^2_{\mathbb{F}_pN_G(P)}\mathbf{1}\right){\uparrow}^{\mathbb{F}_pG}$, where $P$ is a $p$-Sylow of $G$.*

*Proof.* — Every $\mathbb{F}_pG$-module $M$ is a direct summand of $M{\downarrow_{\mathbb{F}_pN_G(P)}}{\uparrow^{\mathbb{F}_pG}}$ by mapping $m \in M$ to the element

$$\frac{1}{(G:N_G(P))} \sum_{gN_G(P) \subseteq G} g \otimes g^{-1}m$$

of $\mathbb{F}_pG \otimes_{\mathbb{F}_pN_G(P)} M$; the number $(G : N_G(P))$ is the index of $N_G(P)$ in $G$, i.e. $|G|/|N_G(P)|$. Now, by Lemma 3.1, $M_0{\downarrow_{\mathbb{F}_pN_G(P)}}{\uparrow^{\mathbb{F}_pG}}$ is isomorphic to a direct sum of $\left(\Omega^2_{\mathbb{F}_pN_G(P)}\mathbf{1}\right){\uparrow^{\mathbb{F}_pG}}$ and some projective $\mathbb{F}_pG$-module. Since $M_0$ is indecomposable and non-projective, it must be a direct summand of $\left(\Omega^2_{\mathbb{F}_pN_G(P)}\mathbf{1}\right){\uparrow^{\mathbb{F}_pG}}$. □

Those versed in Green's correspondence will note that it commutes with the Heller operator, and recognize the previous lemma as a special case.

***Fact 3.4*** **(Corollary 3.6, [S04b]).** — *Let $n \geq 1$. Regard $M_{n-1}$ as a subgroup of $G_n$. Let $H$ be any subgroup of $G_n$ containing $M_{n-1}$. Then $M_n{\downarrow_{\mathbb{F}_pH}}$ is isomorphic to the $p$-Frattini module of $H$.*

In particular, if $n \geq 1$ and $P_n$ is the $p$-Sylow of $G_n$, then $M_n{\downarrow_{\mathbb{F}_pN_{G_n}(P_n)}} \simeq \Omega^2_{\mathbb{F}_pN_{G_n}(P_n)}\mathbf{1}$.

The next three examples consider $A_5$ for the three rational primes dividing its order. Recall that, for every finite group $G$ with a split BN-pair over the prime $p$ (and in particular for a Chevalley group over a finite field of characteristic $p$), there is a projective simple module called the Steinberg module. When $G$ is $\mathrm{PSL}_2(\mathbb{F}_q)$ or $\mathrm{SL}_2(\mathbb{F}_q)$, this is the quotient of a permutation module by the one-dimensional submodule of elements fixed by $G$, the $G$-set defining the permutation module being the projective line $\mathbb{P}^1(\mathbb{F}_q)$ with the natural action of $G$.

***Example 3.5.*** — Let $p = 2$. There are three isomorphism classes of simple $\mathbb{F}_2A_5$-modules, $\mathbf{1}$, a four-dimensional simple module $T$, and the Steinberg module (via the isomorphism of $A_5$ with $\mathrm{SL}_2(\mathbb{F}_4)$). The simple module $T$ is just the natural module for $\mathrm{SL}_2(\mathbb{F}_4)$, a copy of $\mathbb{F}_4^2$, but regarded as a vector space over $\mathbb{F}_2$.

The normalizer of the 2-Sylow of $A_5$ is isomorphic to $A_4$, a 2-split group. As noted in Example 1.1, the kernel of the universal 2-Frattini cover of $A_4$ will have rank 5, and so the 2-Frattini module will have dimension 5. The 2-Frattini module $M_0$ for $A_5$ also has dimension 5 and so $M_0{\downarrow_{\mathbb{F}_2A_4}} \simeq \Omega^2_{\mathbb{F}_2A_4}\mathbf{1}$; on the other hand, inducing $\Omega^2_{\mathbb{F}_2A_4}\mathbf{1}$ up to $A_5$ produces a module with dimension 25. The 2-Frattini module $M_0$ can also be (spuriously) described as a quotient of a permutation module by $\mathbf{1}$: $\mathbf{1}_{\mathbb{F}_2D_5}{\uparrow^{\mathbb{F}_2A_5}}$ has a basis labelled by the 5-Sylows of $A_5$, and the action of $A_5$ on this basis is given by conjugation of the 5-Sylows — $M_0$ is isomorphic to the quotient of this module by the submodule generated by a vector equalling the sum of the basis vectors, a vector which is clearly stabilized by $A_5$. It turns out that $M_0$ has one simple submodule, a copy of $T$, and its quotient by this submodule is isomorphic to $\mathbf{1}$.

***Example 3.6.*** — Let $p = 3$. There are three isomorphism classes of simple $\mathbb{F}_3A_5$-modules: $\mathbf{1}$, a four-dimensional module $S$, and a six-dimensional module, $T$. The normalizer of the 5-Sylow of $A_5$ is isomorphic to $D_5$ and $T$ is isomorphic to $N{\uparrow}^{\mathbb{F}_3A_5}$, where $N$ is a copy of $\mathbb{F}_3$ on which the involutions of $D_5$ act as multiplication by $-1$ and the other elements act trivially. The permutation module $\mathbf{1}_{\mathbb{F}_3A_4}{\uparrow}^{\mathbb{F}_3A_5}$ has a basis labelled by the 2-Sylows of $A_5$, and the action of $A_5$ on this basis is given by conjugation of the 2-Sylows — $S$ is isomorphic to the quotient of this module by the submodule generated by a vector equalling the sum of the basis vectors, a vector which is clearly stabilized by $A_5$.

The normalizer of the 3-Sylow of $A_5$ is isomorphic to $D_3$ and its 3-Frattini module is the sign module $\mathrm{Sgn}_3$ of Example 2.6. The induced module $\mathrm{Sgn}_3{\uparrow}^{\mathbb{F}_3A_5}$ is ten-dimensional and is isomorphic to $S \oplus T$. Since $T$ is projective, $M_0$ must be isomorphic to $S$.

***Example 3.7.*** — Let $p = 5$. There are three isomorphism classes of simple $\mathbb{F}_5A_5$-modules, $\mathbf{1}$, the Steinberg module (via the isomorphism of $A_5$ with $\mathrm{PSL}_2(\mathbb{F}_5)$), and a three-dimensional module $S$. The latter is a subquotient of a permutation module: $\mathbf{1}_{\mathbb{F}_5A_4}{\uparrow}^{\mathbb{F}_5A_5}$ has a basis labelled by the 2-Sylows of $A_5$, and the action of $A_5$ on this basis is given by conjugation of the 2-Sylows. There is a homomorphism $\varphi$ from $\mathbf{1}_{\mathbb{F}_5A_4}{\uparrow}^{\mathbb{F}_5A_5}$ to $\mathbf{1}$ given by taking an element of the former module to the sum of its coefficients (with respect to the basis just described); the sum of the given basis elements generates a submodule $T$ of $\mathbf{1}_{\mathbb{F}_5A_4}{\uparrow}^{\mathbb{F}_5A_5}$ isomorphic to $\mathbf{1}$. The simple module $S$ is the quotient of $\ker(\varphi)$ by $T$.

The normalizer of the 5-Sylow of $A_5$ is isomorphic to $D_5$ and its 5-Frattini module is the sign module $\mathrm{Sgn}_5$ of Example 2.6. The induced module $\mathrm{Sgn}_5{\uparrow}^{\mathbb{F}_5A_5}$ is six-dimensional, so, by Remark 2.5, $M_0$ can be either one-dimensional (and hence $\mathbf{1}$) or the entire induced module; the former can't happen because $M_0{\downarrow}_{\mathbb{F}_5D_5} \supseteq \mathrm{Sgn}_5$, by Lemma 3.1. A simple use of Nakayama's relations (aka Shapiro's lemma) shows that $M_0$ has neither a submodule nor a quotient isomorphic to $\mathbf{1}$. Therefore, $M_0$ has one simple submodule, a copy of $S$, and its quotient by this submodule is also isomorphic to $S$.

## 4. Asymptotics of the $p$-Frattini modules $M_n$

The first recursive formula was hinted at in Fact 3.4. If $M_n$ is regarded as a $p$-group, then its universal $p$-Frattini cover is a free pro-$p$ group of rank equal to the dimension of $M_n$. The Schreier formula takes the form:

$$\dim_{\mathbb{F}_p}(M_{n+1}) = 1 + |M_n| \left[\dim_{\mathbb{F}_p}(M_n) - 1\right].$$

Since $|M_n|$ is equal to $p$ raised to the power of the dimension of $M_n$, this forces the dimension of $M_n$ to rise very rapidly with $n$ via recursive exponentiation, provided $\dim_{\mathbb{F}_p}(M_0) > 1$; but if $\dim_{\mathbb{F}_p}(M_0)$ is 0 or 1 then $\dim_{\mathbb{F}_p}(M_n)$ is the same for all natural numbers $n$. Of course, $\dim_{\mathbb{F}_p}(M_0) = 0$ iff $p$ does not divide the order of $G$, while Griess and Schmid determined precisely the rare circumstance when $\dim_{\mathbb{F}_p}(M_0) = 1$. For the maximal normal $p'$-subgroup (i.e. having order prime to $p$) of $G$, group theorists use the notation $O_{p'}(G)$.

***Fact 4.1* (Griess-Schmid, [GS78]).** — *The $p$-Sylow of $G/O_{p'}(G)$ is non-trivial, cyclic, and normal iff* $\dim_{\mathbb{F}_p}(M_0) = 1$.

The dihedral groups (Example 2.6) provide the natural example of this Fact.

The group $G_n$ does not necessarily act faithfully on the module $M_n$; Griess and Schmid also determined the kernel of this action, the set $\mathrm{Cen}_{G_n}(M_n)$ of elements of $G_n$ that centralize $M_n$. Let $\phi : G \twoheadrightarrow G/O_{p'}(G)$ denote the natural quotient and let $H$ be the maximal normal $p$-subgroup of $G/O_{p'}(G)$; the subgroup $O_{p'p}(G)$ of $G$ is defined to be $\phi^{-1}(H)$.

***Fact 4.2***. — $\mathrm{Cen}_{G_n}(M_n) = \begin{cases} O_{p'p}(G_n) & \textit{if } \dim_{\mathbb{F}_p}(M_n) = 1 \\ O_{p'}(G_n) & \textit{if } \dim_{\mathbb{F}_p}(M_n) \neq 1 \end{cases}$

In some sense, we can reduce to the case where $O_{p'}(G) = 1$. Let $H = G/O_{p'}(G)$. Then $G_n$ is isomorphic to the fibre product over $H$ of ${}_p^n\tilde{H}$ and $G$; the cover $G_n \twoheadrightarrow G$ induces an isomorphism $O_{p'}(G_n) \simeq O_{p'}(G)$ for all $n$.

An interest in the obstruction of Hurwitz space components of a modular tower leads to an interest in the composition series of the $p$-Frattini module, as will be outlined in the next section. The final result here is an asymptotic result on the composition series. The number of times a simple module $S$ appears as a subquotient in a given composition series of a $kG$-module $M$ is an invariant of $M$ denoted by $\#_S(M)$; the density $\varrho_S(M)$ of $S$ in $M$ is defined to be $\#_S(M)/\dim_k(M)$.

***Fact 4.3* (Semmen, [S04b]).** — *If* $\dim_{\mathbb{F}_p}(M_0) > 1$ *then, for any simple* $\mathbb{F}_pG$-*module* $S$, $\lim_{n\to\infty} \varrho_S(M_n) = \varrho_S(\mathbb{F}_pG/O_{p'}(G))$.

The proof of this fact provides a precise recursive formula for $\#_S(M_n)$.

## 5. Modular towers

For any group $G$ and $r$-tuple $\mathbf{C} = (C'_1, \dots, C'_r)$ of conjugacy classes of $G$, the set of inner Nielsen classes $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ is defined to be the set of equivalence classes of $r$-tuples $(g_1, \dots, g_r)$ of $G$ satisfying:

**i) :** $\langle g_1, \dots, g_r\rangle = G$,

**ii) :** $g_1 \dots g_r = 1$, and

**iii) :** there exists $\sigma \in S_r$ such that, for all $i$, $g_{(i)\sigma} \in C'_i$;

two $r$-tuples $(g_1, \dots g_r)$ and $(g'_1, \dots g'_r)$ are equivalent iff there exists $h \in G$ such that $(hg_1h^{-1}, \dots, hg_rh^{-1}) = (g'_1, \dots, g'_r)$. The space $\mathbb{P}^r \setminus D_r$ parametrizes subsets of $\mathbb{P}^1$

of cardinality $r$. The Hurwitz monodromy group $H_r := \pi_1(\mathbb{P}^r \setminus D_r)$ has generators $q_1, \ldots, q_{r-1}$ with a canonical action on $\mathrm{Ni}\,(G, \mathbf{C})^{\mathrm{in}}$:

$$(g_1, \ldots, g_r)q_i = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots g_r).$$

This permutation representation of $H_r$ produces an unramified cover $\mathcal{H}\,(G, \mathbf{C})^{\mathrm{in}} \twoheadrightarrow \mathbb{P}^r \setminus D_r$ with fibre $\mathrm{Ni}\,(G, \mathbf{C})^{\mathrm{in}}$, whose domain is called a Hurwitz space. When $G$ has trivial center (i.e. no non-trivial element of $G$ commutes with all elements of $G$), this is a fine moduli space for equivalence classes of covers $X \twoheadrightarrow \mathbb{P}^1$ together with an identification of the monodromy group with $G$ such that the ramification data is described by an element of $\mathrm{Ni}\,(G, \mathbf{C})^{\mathrm{in}}$ — the equivalence of covers here must be $G$-equivariant.

Whenever $H_2 \twoheadrightarrow H_1$ is a group epimorphism with $p$-group kernel, every conjugacy class of $H_1$ whose elements have order prime to $p$ has a unique lift to a conjugacy class of $H_2$ whose elements have order prime to $p$. Hence (cf [**De04**, Lifting Lemma 1.1]), if $\mathbf{C}$ is an $r$-tuple of conjugacy classes whose elements have order prime to $p$, there is a canonical modular tower

$$\ldots \longrightarrow \mathcal{H}\left({}^{n+1}_{p}\tilde{G}, \mathbf{C}\right)^{\mathrm{in}} \xrightarrow{\psi_n} \mathcal{H}\left({}^{n}_{p}\tilde{G}, \mathbf{C}\right)^{\mathrm{in}} \longrightarrow \ldots$$

where the map between Hurwitz spaces is induced by applying the epimorphism $\varphi_n : {}^{n+1}_{p}\tilde{G} \twoheadrightarrow {}^{n}_{p}\tilde{G}$ coordinatewise to the inner Nielsen classes.

Recall that a group is $p$-perfect if it has no non-trivial $p$-group quotient.

***Fact 5.1***. — *If ${}^{n}_{p}\tilde{G}$ is p-perfect and has trivial center, then ${}^{n}_{p}\tilde{G}$ is also p-perfect and has trivial center.*

Assuming the conditions of this Fact hold for $G$, this forces all of the Hurwitz spaces of the modular tower to be fine moduli spaces.

Fix a composition series of $M_n$. For any two adjacent entries $N_2 \subset N_1$ of the series, there is a canonical cover

$$H_2 := {}^{n+1}_{p}\tilde{G}/N_2 \twoheadrightarrow H_1 := {}^{n+1}_{p}\tilde{G}/N_1$$

whose kernel will be a simple $\mathbb{F}_p H_1$-module (in fact, a simple $\mathbb{F}_p G$-module). The map $\psi_n$ factors into a sequence of irreducible maps

$$\mathcal{H}\left({}^{n+1}_{p}\tilde{G}, \mathbf{C}\right)^{\mathrm{in}} \to \ldots \to \mathcal{H}\,(H_2, \mathbf{C})^{\mathrm{in}} \to \mathcal{H}\,(H_1, \mathbf{C})^{\mathrm{in}} \to \ldots \to \mathcal{H}\left({}^{n}_{p}\tilde{G}, \mathbf{C}\right)^{\mathrm{in}}$$

Note that even if all of the groups ${}^{n}_{p}\tilde{G}$ have trivial center, many of the intermediate groups will not (cf Fact 4.3).

***Fact 5.2* (Fried-Kopeliovich, [FK97])**. — *If the kernel of $H_2 \twoheadrightarrow H_1$ is isomorphic to* $\mathbf{1}$*, then $\mathcal{H}\,(H_2, \mathbf{C})^{\mathrm{in}} \to \mathcal{H}\,(H_1, \mathbf{C})^{\mathrm{in}}$ is injective. Otherwise, it is surjective.*

Connected components of a Hurwitz space correspond one-to-one to orbits of the action of $H_r$ on the Nielsen classes; a component $\mathcal{O}$ of $\mathcal{H}\left({}^{n}_{p}\tilde{G}, \mathbf{C}\right)^{\mathrm{in}}$ is obstructed if its preimage under $\psi_n$ is empty. Thus, only intermediate epimorphisms $H_2 \twoheadrightarrow H_1$ with kernel isomorphic to $\mathbf{1}$ can produce obstruction. This observation motivated the analysis leading to Fact 4.3.

Two covers $\phi_1 : X_1 \twoheadrightarrow \mathbb{P}^1$ and $\phi_2 : X_2 \twoheadrightarrow \mathbb{P}^1$ are weakly equivalent if there exist isomorphisms $\alpha : X_1 \xrightarrow{\sim} X_2$ and $\beta : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ such that $\phi_2 \circ \alpha = \beta \circ \phi_1$. The reduced Hurwitz space $\mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}}$ is the quotient of the usual Hurwitz space by this expanded equivalence, and inherits a natural surjection onto $(\mathbb{P}^r \setminus D_r)/\mathrm{PGL}_2(\mathbb{C})$.

When $r = 4$, $\left(\mathbb{P}^4 \setminus D_4\right)/\mathrm{PGL}_2(\mathbb{C})$ is naturally isomorphic to $\mathbb{P}^1 \setminus \{\infty\}$ and the surjection of the reduced Hurwitz space is ramified only over 0 and 1; a natural compactification of the reduced Hurwitz space makes it a curve cover of $\mathbb{P}^1$. The generic fibre $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in,rd}}$ of this cover can be naturally identified with a quotient of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ as follows.

The center of $H_4$ is generated by an element $(q_1 q_3^{-1})^2$ of order 2 which acts trivially on the set of inner Nielsen classes. The quotient $M_4 := H_4/\langle (q_1 q_3^{-1})^2\rangle$ acts faithfully on the class of all inner Nielsen classes for all finite groups $G$, and is isomorphic to a semi-direct product $\mathcal{Q}'' \rtimes \mathrm{PSL}_2(\mathbb{Z})$, where $\mathcal{Q}''$ is generated by the images of $(q_1 q_2 q_3)^2$ and $q_1 q_3^{-1}$, while $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the images of $q_1 q_2$ and $q_1 q_2 q_1$, elements of order 3 and 2, respectively. These latter two elements map down to elements $\gamma_0$ and $\gamma_1$, respectively, of $\overline{M_4} := M_4/\mathcal{Q}''$. The image of $q_2$ in $\overline{M_4}$ is the inverse of $\gamma_0\gamma_1$ and is called $\gamma_\infty$.

The stabilizer in $\mathrm{PGL}_2(\mathbb{C})$ of a point $z$ of $\mathbb{P}^4 \setminus D_4$ lying over a point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a Klein 4-group whose action on the fibre over $z$ in $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ is the same as the action of $\mathcal{Q}''$ on the set of inner Nielsen classes; the set of reduced inner Nielsen classes is the quotient. Inertia generators for the cover $\mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}} \twoheadrightarrow \mathbb{P}^1$ over 0, 1, and $\infty$ are exactly the images of $\gamma_0$, $\gamma_1$, and $\gamma_\infty$, respectively, in the monodromy group.

Fried conjectures [**De04**, Remark 2.5] that only a finite number of components in a reduced modular tower for $r = 4$ will have genus 0 or 1. The Riemann-Hurwitz formula computes these genuses from the ramification indices determined by the actions of $\gamma_0$, $\gamma_1$, and $\gamma_\infty$ on the set of reduced inner Nielsen classes. The first two inertia generators have prime order (3 and 2, respectively), so the number of their fixed points (together with the degree of the cover) will determine their contribution to the Riemann-Hurwitz formula; §8 of [**FS04**] aims at eliminating these fixed points.

However, $\gamma_\infty$ has infinite order in $\overline{M_4}$, so there is no *a priori* constraint on the ramification it determines. The points lying over $\infty$ Fried calls *cusps*, and these naturally correspond to the orbits of $\langle\gamma_\infty\rangle$ acting on the reduced inner Nielsen classes. The cusp width (i.e. ramification index) of the cusp represented by an element $\mathbf{g} = (g_1, g_2, g_3, g_4)$ of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ equals the order of $g_2 g_3$ (denoted by $(\mathbf{g})\mathrm{mp}$) divided by 1, 2, or 4, this divisor being the cardinality of the intersection $\mathbf{g}\langle q_2\rangle \cap \mathbf{g}\mathcal{Q}''$. The middle product $(\mathbf{g})\mathrm{mp}$ is an invariant of the cusp independent of the representative $\mathbf{g}$ chosen.

The cusps are broken into three types. The non-$p'$ cusps are those for which $p$ divides the middle product $(\mathbf{g})$mp, while the $p'$ cusps are those for which $p$ doesn't divide the middle product. The third type, the $g$-$p'$ cusps, form a subset of the second and consists of those cusps for which a (equivalently, all) representative(s) $\mathbf{g} = (g_1, g_2, g_3, g_4)$ satisfy: $\langle g_1, g_4\rangle$ and $\langle g_2, g_3\rangle$ are groups of order prime to $p$.

***Lemma 5.3.*** — *Let $\varphi_n : {}_p^{n+1}\tilde{G} \twoheadrightarrow {}_p^{n}\tilde{G}$ be the canonical cover. Let $g$ be an element of ${}_p^{n+1}\tilde{G}$. If $p$ divides the order of $\varphi_n(g)$ then the order of $g$ is $p$ times the order of $\varphi_n(g)$.*

*Proof.* — It suffices to prove this when $\varphi_n(g)$ has order $p$. Then $\langle\varphi_n(g)\rangle$ is a copy of the cyclic group $C_p$ inside ${}_p^{n}\tilde{G}$. The cover $\varphi_n^{-1}(C_p) \twoheadrightarrow C_p$ is projective by Lemma 3.1, and so must map surjectively onto the Frattini cover $C_{p^2} \twoheadrightarrow C_p$. Every element of $C_{p^2}$ mapping onto $\phi_n(g)$ has order $p^2$ and $g$ must map to one of these elements and hence have order at least $p^2$. Since $g^p \in M_n$, $g$ has order at most $p^2$. □

This, combined with the Branch Cycle Argument [**De04**, 1.5], is what lies behind Theorem 2.6 in Dèbes' lecture notes [**De04**]. It also shows that the cusp width of a non-$p'$ cusp will always be a factor of $p$ greater than the cusp width of its image in the level below: Fried's Frattini Cusp Principle 1 [**F04**].

Every $g$-$p'$ cusp has a $g$-$p'$ cusp lying over it. Furthermore, for any representative $(g_1, g_2, g_3, g_4)$ of such a cusp, the restriction $M'$ of the $p$-Frattini module to $\langle g_2, g_3\rangle$ will be semi-simple and its isomorphism class will be determined by the numbers $\#_S(M_n)$, for which we have recursive formulae (cf Fact 4.3). Analysis of the cusps immediately lying over this cusp relies only on studying the split extension $M'\rtimes\langle g_2, g_3\rangle$, as does classifying the cusp type of perturbations $(g_1, ag_2a^{-1}, bg_3b^{-1}, g_4)$, where $a$ and $b$ are elements of $M'$.

What are the types of the cusps that lie over a $p'$ cusp which is not $g$-$p'$? For this, we require more detailed information about the appearances of $\mathbf{1}$ in the composition series of the $p$-Frattini module; some of the most important appearances are those that can occur at the top of a composition series. To end, let me remind you of the definition of the elementary abelian $p$-Schur multiplier of $G$: this is the quotient of $M_0$ by the smallest submodule $N$ such that all elements of $G$ act trivially on $M_0/N$.

## A
## The Gruenberg-Roggenkamp equivalence

For every object $f : H \twoheadrightarrow G$ of $\mathcal{C}_{RG}(G)$, choose a transversal $t$, a function $t : G \to H$ such that $f \circ t$ is the identity on $G$ and $t(1) = 1$.[1] Similarly, for every

[1] Set theorists might object to making a choice on a proper class. To do this rigorously, create a new category whose objects are all ordered pairs with the first coordinate an object from $\mathcal{C}_{RG}(G)$ and

object $c : N \twoheadrightarrow \omega_{RG}$ of $\mathcal{C}_{RG}(\omega_{RG})$, choose an $R$-module splitting of $c$, an $R$-module homomorphism $s : \omega_{RG} \to N$ such that $c \circ s$ is the identity on $\omega_{RG}$.

Define a functor $\Phi : \mathcal{C}_{RG}(G) \to \mathcal{C}_{RG}(\omega_{RG})$ as follows. Let $f : H \twoheadrightarrow G$ be a cover of $G$ with transversal $t$. Let $K$ be the kernel of $f$, an $R$-module. Let $N$ be the $R$-module $K \times \omega_{RG}$; this becomes an $RG$-module when the action of an element $h \in G$ is specified as follows:

$$h\left(k, \sum_{g\in G} a_g g\right) := \left({}^{t(h)}k + \sum_{g\in G} a_g \left(t(h)t(g)t(hg)^{-1}\right), \sum_{g\in G} a_g hg\right).$$

Projection onto the second coordinate is a cover $\Phi(f)$ of $\omega_{RG}$ with obvious $R$-module splitting. Suppose we have covers $f_1 : H_1 \twoheadrightarrow G$ and $f_2 : H_2 \twoheadrightarrow G$ and a morphism $\varphi : H_1 \to H_2$ between them; let $t_1$ and $t_2$ be the respective transversals. The morphism $\Phi(\varphi)$ is defined as follows:

$$\Phi(\varphi)\left(k, \sum_{g\in G} a_g g\right) := \left(\varphi(k) + \sum_{g\in G} a_g \left(\varphi(t_1(g))t_2(g)^{-1}\right), \sum_{g\in G} a_g g\right).$$

Now define a functor $\Psi : \mathcal{C}_{RG}(\omega_{RG}) \to \mathcal{C}_{RG}(G)$. Let $c : N \twoheadrightarrow \omega_{RG}$ be a cover with $R$-module splitting $s$. Let $K$ be the kernel of $c$. The group structure on $K \times G$ is defined by:

$$(m, g)(n, h) := (m + gn + (gs(h-1) - s(g(h-1))), gh).$$

Projection onto the second coordinate is a cover of $G$ with obvious transversal. Suppose we have covers $c_1 : N_1 \twoheadrightarrow \omega_{RG}$ and $c_2 : N_2 \twoheadrightarrow \omega_{RG}$ and a morphism $\psi : N_1 \to N_2$ between them; let $s_1$ and $s_2$ be the respective $R$-module splittings. The morphism $\Psi(\psi)$ is defined as follows:

$$\Psi(\psi)(m, g) := (\psi(m) + (\psi(s_1(g-1)) - s_2(g-1)), g).$$

The functors $\Phi$ and $\Psi$ form Gruenberg-Roggenkamp's categorical equivalence.

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the second coordinate a transversal of the first coordinate; let the morphisms between two objects be the morphisms between the first coordinates of the two objects, i.e. the choice of transversal doesn't change the isomorphism class of the object.

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