Connectedness of families of sphere covers of a given type

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Abstract. There are now many applications of the following basic problem: Do all covers of the sphere by a compact Riemann surface of a "given type" compose one connected family? Or failing that, do they fall into easily discernible components? The meaning of "given type" usually uses the idea of a *Nielsen class* — a concept for covers that generalizes the genus of a compact Riemann surface. The answer has often been yes, and that answer has figured in many problems from the connectedness of the moduli space of curves of genus g (geometry) to Davenport's problem (arithmetic) and the genus 0 problem (group theory). This survey arose in response to the following special case asked by Brian Osserman. Do all genus zero covers of the sphere with r specific pure-cycles as branch cycles form one connected family?

Contents

2000 *Mathematics Subject Classification.* Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34.

1. Formulation of the problem in Nielsen Classes

1.1. Source of connectedness problems. The genus of a curve (compact, connected, Riemann surface) discretely separates decidely different algebraic relations in two variables to focus us on the connected moduli space $\mathcal{M}_g$. Yet, direct modern applications feature a data variable (function) on the curve. This data variable produces a monodromy group G embedded as a transitive subgroup of a symmetric group S_n, with n the degree of the data variable.

1.1.1. **Stage [I]**: *Using Conjugacy classes to detect variable relations.* A data variable also produces $r \geq 2$ (the case $r = 2$ is trivial compared to the others) conjugacy classes, denoted below $\mathbf{C} = \mathrm{C}_1, \dots, \mathrm{C}_r$ in G. You can pass a conjugacy class C in G to S_n, interpreting the result C^{S_n} as a disjoint cycle type. This map can be many-to-one. For example, as in §B.2, the projective linear group over a finite field has several conjugacy classes of cycles acting on the points of projective space. Still easier, any product of disjoint cycles of distinct odd lengths in S_n (for example $g = (1)(2\,3, 4)(5\,6\,7\,8\,9)$ with lengths 1, 3, 5 in S_9) has two representing conjugacy classes in A_n. The Riemann-Hurwitz formula recovers the genus of the curve from $\mathbf{C}^{S_n}$ (as in (2.1)).

Now we discuss why a data variable z on a Riemann surface X induces a relation. One consequence of Riemann's Existence Theorem (RET) is that there is another function w on X so that z and w generate all meromorphic functions on X. The points of X, excluding a finite number then identify with the pairs (z, w) modulo a nontrivial relation $f(z, w) = 0$. The simplest case is when X itself has genus 0, and (again from Riemann), there is a w giving z as a rational function in w (at least over $\mathbb{C}$). This isn't a trivial case: As §B.1 reminds, many renown theorems in number theory are instances asking for concise listing of rational functions $f(w) \in \mathbb{Q}(w)$ with the following property. Their reductions modulo infinitely many primes are one-one maps on the projective line over the residue field having the now ubiquitous use of encoding data into finite fields for protecting it.

Applications generalizing —Davenport covers §B.2 —give universal *relations* among among Poincaré series (counting functions). These problems epitimize asking of all two variables relations between data, which have a particular property. As our examples hint, some skill is involved in translating the original problem into data about groups and conjugacy classes, basically a version of Galois theory. So, in attacking some application, **Stage [I]** is finding which pairs $(G, \mathbf{C})$ could give the desired algebraic relations. My examples have a version over any number field —the finite fields appear by reduction modulo primes of the number field. A very successful conclusion to **Stage [I]** would consists of listing only those pairs $(G, \mathbf{C})$

for which a solution to the problem over *some* number field would be realized by the group-conjugacy class pair.

1.1.2. **Stage [II]**: *Putting structure in relations.* Solving a *data* problem P posed by these examples requires *collation*: Cataloging usefully algebraic relations that solve the problem. An effective technique starts with finding for each pair $(G, \mathbf{C})$ a list of parameter spaces $\mathcal{S}_P$ for the desired relations. That is, points of these spaces are *pointers* to solutions of the original data problem. There is a recognition problem. If someone gives you a set of algebraic relations with a data variable, can you detect if one in collection $\mathcal{S}_P$ contains it?

The spaces that have worked are versions, depending on our needs for this data variable, of *Hurwitz spaces*. The data above, $r \geq 3$ conjugacy classes $\mathbf{C}$ in the data variable monodromy G defines a *Nielsen class* (§1.2), a generalization of conjugacy class. The spaces to which this gives rise depend on any equivalence relation we put on the data problem solutions. For example, in looking for unique solutions we might equivalence two data variables if they differ by composition with a linear fractional transformation: *reduced* equivalence. This gives a space of dimension $r-3$. When $r = 4$, for example, this space is a curve and often we can be precise things about it. No matter what is r, this is what you would first want to know:

(1.1a) What are the geometric components of the space associated to $(G, \mathbf{C})$, its connected pieces; and
(1.1b) what are the definition fields of these components.

When $r = 4$, the following information has often been sufficient to nail the nature of the solutions.

(1.2a) What are the genuses of the components?; and
(1.2b) what are the cusps of each component?

Near completion of **Stage [I]** we might know solutions to the problem exist over some number field. We may know we have limited all pairs $(G, \mathbf{C})$ to those whose Nielsen classes contain solutions. Hurwitz spaces, however, need not be connected and while solutions may exist with varying number fields, many problems require algebraic relations over $\mathbb{Q}$, the regular version of the Inverse Galois Problem foremost among them (§B.3). As nonreduced Hurwitz spaces are nonsingular, they won't have $\mathbb{Q}$ points unless they have $\mathbb{Q}$ components. So minimally we need to detect the $\mathbb{Q}$ components. That lands us at **Stage [III]**: Decide if it is possible to list the points on these parameter spaces that define the solutions over a given field to the original problem.

We concentrate on **Stage [II]**, and especially questions (1.1). Question (1.1a) on connectedness has a combinatorial group formulation, and from its answer we often divine how to answer to (1.1b). Further, a description of cusps as in (1.2b) is an even easier form of group theory. This works for all values of $r \geq 4$ (as in §A.2). Yet, when $r = 4$ the spaces of $\mathcal{S}_P$ will be upper half-plane quotients covering the j-line, whose compactifications will have geometric cusps over $j = \infty$. This case allows rich comparison with the modular curves they generalize.

1.1.3. *Liu-Osserman pure-cycle Nielsen class problems.* Problem 2.2 (posed by Fu Liu and Brian Osserman [**LOs06**]) restricts the Nielsen classes to the case G is A_n or S_n, the genus of representing covers is 0, and the conjugacy classes are pure-cycles —each the class of one disjoint cycle. The context for the topic comes alive with related generalizations of their question.

§3 reminds of the whole genus 0 problem. This asks which groups G can appear as monodromy groups of a genus 0 data variable. The case $r = 4$ has always stood out. It requires much more than the topic of dessin d'enfants (three branch point covers) because now there is significant variation of the algebraic relations. For example, a formulation of modular curves has often entered in solutions of problems like those in the appendices (§5). Further, the Liu-Osserman special cases (2.2) have a significant modular curve-like property: Their reduced Hurwitz spaces embed naturally in $\mathbb{P}^1_j \times \mathbb{P}^1_j$ (§5.2). Here $\mathbb{P}^1_z \stackrel{\text{def}}{=} \mathbb{C}_z \cup \{\infty\}$ refers to the Riemann sphere uniformized, with by a variable z: The same variable used in a first course in complex variables. So, $\mathbb{P}^1_j$ refers to the copy of the Riemann sphere called the j-line from (usually the end of) 1st year graduate complex variables (§5.1).

Our main examples of (2) go with the most modern applications, where the disjoint cycles lengths are all odd, so $G = A_n$ for some n. Let $d_1, \dots, d_r$ be the lengths of the disjoint cycles. For this we often use the symbol $d_1 \cdots d_r$, with repetitions repeated as exponents. Detailed understanding of the special case 3^r, supports many results going beyond the restriction the genus is 0 (§4).

For the first time we also see the role of *Schur multipliers*, in the case of A_n appearing in the form of a half-canonical class. In turn, this alternating group example epitimizes a strengthening of the Conway-Fried-Parker-Völklein result (§6) whose gist is that if $\mathbf{C}$ repeats each class in its support sufficiently many times, then we know precisely the Hurwitz space components and their definition fields. This points to a conceptual affirmative answer to many problems generalizing those in [**LOs06**].

1.1.4. *The Main Conjecture on Modular Towers and the Strong Torsion.* We think the most compelling application is to **MT**s. Suppose a prime p divides $|G|$ but not $d_1 \cdots d_r$, and each d_i is repeated an even number of times. A special case of a general result then says the Nielsen class defines a projective system of nonempty reduced Hurwitz spaces for which we temporarily use the notation $\{\mathcal{H}_{d_1 \cdots d_r, p, k}\}_{k=0}^{\infty}$.

Any projective system of components on these spaces is called a **MT**, and we say it is defined over a number field K if all spaces with their system of maps has definition field K. This construction works much more generally, and it leads to a statement with this rough paraphrase: Projective systems of modular curves for the prime p are to the dihedral group D_p as **MT**s are to all p-perfect finite groups (more precisely stated in §B.4). The following statement is only serious if a **MT** has definition field K for some number field K.

Conjecture 1.1 (MCMT). Let $\{\mathcal{H}'_k\}_{k=0}^{\infty}$ be a **MT** over a number field K. Assume L is a number field containing K. Then, for k large, $\mathcal{H}'_k(L)$ is empty.

Cadoret has shown the Strong Torsion Conjecture (STC) on abelian varieties implies Conj. 1.1. There has not been much progress on the STC, beyond the famous case of elliptic curves called the Mazur-Merel Result. So, we are glad to have that tools to check the MCMT from group theory and geometry in special cases. When $r = 4$ there has been considerable recent progress on the MCMT, and the results of Liu-Osserman allow us to test (and sometimes prove) it in myriad cases (§5.2.1). Each case reflects on the STC and the RIGP. Our greatest motivation for extending results from [**LOs06**] come from this topic.

1.2. Braid group actions on Nielsen classes. §1.2.1 defines Nielsen classes from the data of r conjugacy classes in group G, the basic objects on which the braid group B_r acts. §1.2.2 gives notation for subgroups of B_4 quotients that make precise the case $r = 4$. Here reduced Hurwitz spaces are rich generalizations of modular curves. The comparison has been illuminating in *both* directions. Finally, §1.2.3 gives an overview of how our examples work.

1.2.1. *Nielsen classes and braid groups.* For $\boldsymbol{g} \stackrel{\text{def}}{=} (g_1, \dots, g_r) \in G^r$ we use the following conditions, collectively phrased as $\boldsymbol{g}$ *generates (G) with product-one*.

(1.3a) *Generation* —$\langle g_1, \dots, g_r\rangle = G$; and

(1.3b) *product-one* —$\prod g_1 \cdots g_r \stackrel{\text{def}}{=} \Pi(\boldsymbol{g}) = 1$.

Also, $\boldsymbol{g}$ defines a set $\mathbf{C}$ (with multiplicity) of conjugacy classes in G. Given r conjugacy classes $\mathbf{C}$, $\boldsymbol{g} \in \mathbf{C}$ means $\boldsymbol{g}$ defines $\mathbf{C}$. Example:

$$\boldsymbol{g}_{\text{HM}} = ((1\,2\,3), (1\,3\,2), (1\,4\,5), (1\,5\,4)) \tag{1.4}$$

generates A_5 with product-one, and defines $\mathbf{C}_{3^4}$, the repetition of the 3-cycle conjugacy class with multiplicity 4.

Definition 1.2. For $g \in G$, define the Nielsen class (of $(G, \mathbf{C})$):

$$\{\boldsymbol{g} \in \mathbf{C} \mid \boldsymbol{g} \text{ generates with product-one}\} \stackrel{\text{def}}{=} \text{Ni}(G, \mathbf{C}).$$

A combinatorial *braid group* $B_r = \langle Q_1, \dots, Q_{r-1}\rangle$ naturally acts on $\text{Ni}(G, \mathbf{C})$ with the *twisting* action of the generators illustrated as here:

$$Q_2 : \boldsymbol{g} \mapsto (g_1, g_2 g_3 g_2^{-1}, g_2, g_4, \dots, g_r).$$

Check: The action of $Q^{(r-1)} \stackrel{\text{def}}{=} Q_1 \cdots Q_{r-1} Q_{r-1} \cdots Q_1$, conjugates $\boldsymbol{g}$ by g_1. In general, applying all conjugates of $Q^{(r-1)}$ in B_r to $\boldsymbol{g}$ gives the collection of conjugates

$$\{g\boldsymbol{g}g^{-1} \stackrel{\text{def}}{=} (g g_1 g^{-1}, \dots, g g_r g^{-1})\}_{g \in G}$$

of $\boldsymbol{g}$. The group formulation of our main problem is to decide what are the orbits of B_r on Nielsen classes. It simplifies our problem (especially the cases $r = 3$ and 4) to consider inner Nielsen *classes* $\text{Ni}(G, \mathbf{C})^{\text{in}}$, the quotient $\text{Ni}(G, \mathbf{C}) \mod G$. On this set B_r acts through H_r, the *Hurwitz monodromy* group: the combinatorial group quotient of B_r by the relation $Q^{(r-1)}$ —on inner Nielsen classes.

Typically we denote the image in H_r of $Q \in B_r$ in H_r by q whenever this helps explain in which group we are operating. Significantly, H_r is the fundamental group of a space many mathematicians use: The set U_r of monic polynomials of degree r with no repeated roots. All the Hurwitz spaces that appear in this paper are thereby naturally presented as covers either of U_r, or of its quotient by an action of $\text{PGL}_2(\mathbb{C})$, the group of linear fractional transformations. The corresponding covers of $U_r/\text{PGL}_2(\mathbb{C})$ are *reduced* Hurwitz spaces. We as well are asking about their connected components. Comments on forming these spaces are in App. A.1.

1.2.2. *Cusps when* $r = 4$. When $r = 4$, reduced Hurwitz spaces are upper half-plane quotients covering the j-line, ramified (in our normalization) only at 0, 1. So, they have natural compactifications over $\mathbb{P}^1_j$ with cusps over $j = \infty$. The exposition accepts these facts without giving all details. Still, we will compare examples with modular curve cusps using the group $\bar{M}_4$ below.

A graphic aspect appears in our examples from the **sh**(ift)-incidence matrix, a pairing on reduced Hurwitz space cusps. We start with how group theory interprets cusps using observations in H_4 coming from the braid group relations from B_4.

The *twist* action of $H_4 = \langle q_1, q_2, q_3\rangle$ generators on $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$ is above. Here is the respective effect of q_1 and q_2 on $\boldsymbol{g}_{\mathrm{HM}}$ in (1.4):

$$((1\,3\,2),(1\,2\,3),(1\,4\,5),(1\,5\,4)) \text{ and } ((1\,3\,2),(3\,4\,5),(1\,2\,3),(1\,5\,4)).$$

As with modular curves, when $r = 4$, much data about the space attached to an H_4 orbit comes from cusps. Two groups figure in the definition of cusps:

(1.5a) $\mathcal{Q}'' = \langle q_1q_3^{-1}, (q_1q_2q_3)^2\rangle$ (Klein Image), a normal subgroup of H_4; and

(1.5b) $\mathrm{Cu}_4 \stackrel{\mathrm{def}}{=} \langle q_1q_3^{-1}, (q_1q_2q_3)^2, q_2\rangle = \langle \mathcal{Q}'', q_2\rangle$ (Cusp group).

The group $\bar{M}_4 \stackrel{\mathrm{def}}{=} H_4/\mathcal{Q}''$ is actually $\mathrm{PSL}_2(\mathbb{Z})$. [**BF02**, §2.4.2] has normalizations that identify the monodromy generators of the j-line covers from a Nielsen class. These are images in $\bar{M}_4$ of the following three elements:

(1.6) $q_2 \mapsto \gamma_\infty$ (local cusp generator); $q_1q_2q_3$ (shift) $\mapsto \gamma_1$ (order 2, for ramification over $j = 1$); $q_1q_2 \mapsto \gamma_0$ (order 3, for ramification over $j = 0$).

We can see these orders from the braid relations. Example for γ_0: Use the braid relation $q_1q_2q_1 = q_2q_1q_2 \mod \mathrm{Cu}_4$, $q_1 = q_3 \mod \mathrm{Cu}_4$, and $1 = q_1q_2q_3q_3q_2q_1$ (image of $Q^{(2)}$ above). Then,

$$1 = q_1q_2q_1q_1q_2q_1 = q_1q_2q_1q_2q_1q_2 = (q_1q_2)^3 = \gamma_0^3 \mod \mathrm{Cu}_4.$$

The definition of an inner *reduced* Nielsen class is the set given by the quotient action $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}/\mathcal{Q}'' \stackrel{\mathrm{def}}{=} \mathrm{Ni}(G,\mathbf{C})^{\mathrm{in,rd}}$. The orbits of $\bar{M}_4$ on $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in,rd}}$ correspond one-one with the orbits of H_4 on $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$, and the lengths of the orbits are the degrees of the corresponding reduced space components over the j-line.

We give the the cusps for the *elementary* modular curves $X_0(p^{k+1})$ and $X_1(p^{k+1})$ (p odd), defined respectively as compactifications of quotients of the upper half-plane $\mathbb{H}$ by the respective congruence subgroups:

$$\Gamma_0(p^{k+1}) \stackrel{\mathrm{def}}{=} \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \mod p^{k+1}; \text{ and}$$

$$\Gamma_1(p^{k+1}) \stackrel{\mathrm{def}}{=} \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \mod p^{k+1}.$$

Classically you list cusps by selecting good coset representatives and then computing $\gamma_\infty \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ orbits on them. §A.2 shows how to list these cusps in the framework of Nielsen classes, and then §A.3 gives examples of nonmodular curves where you would be hard-pressed to find a classical approach.

1.2.3. *Example rubric.* We can label pure-cycle Nielsen classes as follows. Put in the distinct integers $d_1^* < ... < d_s^*$ that appear as disjoint cycle lengths. If, however, all these lengths are odd, and $d_s^* = n-1$ (n even) or n (n odd), then you might have to consider two conjugacy classes respectively referred to as $d_s^*(1)$ and $d_s^*(2)$. Now consider just Nielsen classes with support in $d*_1 \cdots d*_s$: The corresponding conjugacy classes appear with multiplicity, say respectively, $(m_1, ..., m_s)$. Denote this Nielsen class $\mathrm{Ni}_{\boldsymbol{d}^*,\boldsymbol{m}}$. The tacit assumption is this: The group G is A_n if all the d_i^* s are odd, and otherwise S_n. We state results for both cases, though concentrate on the former case as justified by its many applications and its educative value, for this is where interesting invariants and distinguishing the cases of two —versus one —Hurwitz space component(s) arise in clear abundance.

Fix odd $d*_1 \cdots d*_s$, and consider on the set

$$\mathcal{D}_{d_1^*\cdots d_s^*} \stackrel{\mathrm{def}}{=} \{\boldsymbol{m} \geq (1,1,\dots,1) \mid d^*{}_1^{m_1} \cdots d^*{}_s^{m_s}\}.$$

It makes sense to consider an alternative to $\mathrm{Ni}_{\boldsymbol{d}^*,\boldsymbol{m}}$, where we replace A_n by the nonsplit degree two to extension $\mathrm{Spin}_n \to A_n$. For $n = 5$, for example, the natural map $\mathrm{SL}_2(\mathbb{Z}/5) \to \mathrm{PSL}_2(\mathbb{Z}/5)$ represents this cover is represented by identifying A_5 with $\mathrm{PSL}_2(\mathbb{Z}/5)$. As in §**??**, consider the odd order conjugacy classes that lift $d*_1 \cdots d*_s$ to Spin_n by the same symbol. Then denote the Nielsen class by substituting Spin_n for A_n by $\mathrm{Ni}^{\mathrm{sp}}_{\boldsymbol{d}^*,\boldsymbol{m}}$. This gives a natural one-one (but not necessarily onto) map $\mathrm{Ni}^{\mathrm{sp}}_{\boldsymbol{d}^*,\boldsymbol{m}} \to \mathrm{Ni}_{\boldsymbol{d}^*,\boldsymbol{m}}$. To this map we associate three possible symbols: $\oplus$ if it is onto, $\ominus$ if $\mathrm{Ni}^{\mathrm{sp}}_{\boldsymbol{d}^*,\boldsymbol{m}}$ is empty, and $\oplus\ominus$ if neither of the first two happen. If the symbol attached to $\boldsymbol{m}$ is $\oplus\ominus$, then there must be at least two braid orbits on $\mathrm{Ni}_{\boldsymbol{d}^*,\boldsymbol{m}}$ (two Hurwitz space components). Applying Conway-Fried-Parker-Völklein (C-F-P-V, §B.3) to this particular case says that if all the m_i s are suitably large, there are *exactly* two braid orbits on $\mathrm{Ni}_{\boldsymbol{d}^*,\boldsymbol{m}}$(two Hurwitz space components; one on $\mathrm{Ni}^{\mathrm{sp}}_{\boldsymbol{d}^*,\boldsymbol{m}}$) and these two components are represented by the symbol $\oplus\ominus$. There are two improvements here on this general result which is blind to what is G or $\mathbf{C}$. For fixed $d*_1 \cdots d*_s$:

(1.7a) There is an algorithm giving the precise $\boldsymbol{m}$ s for each of the symbols $\oplus$, $\ominus$ and $\oplus\ominus$ (Thm. 4.1).

(1.7b) There is evidence the symbol tells precisely which component possibilities occur.

In §4 we also do this for the case where the d_i^* s include some even integers, but then you replace Spin_n by the representation cover of S_n. If we knew that this analog determined the components (braid orbits) exactly, that would be the exact analog of [**Fri06b**, Thms 1.2 and 1.3] for the case $d*_1 \cdots d*_s$ is 3 —Nielsen classes of 3-cycles which is the most precise version of (1.7b) in this case.

Notice, however, in any case, the analog of Figure 1 depends on knowing which lifting invariant values occur for which m_i s. I can give a serious result when all the d_i^*s are odd precisely because I can use a trick to apply [**Fri06b**, Thm. 1.3]. That is the one original result I'm putting in this survey. This is what I meant when I said I was using the Fried-Serre formula. This is one kind of generalization of the Fried-Serre formula (something I flirted with in the Bailey-Fried paper).

[§5 strengthens [**LOs06**,] (the genus now is 0) in the case $r = 4$ to give test cases for the Main Conjecture on Modular Towers. The result here is that the cusps are all $2'$ cusps and then it inspects which of those cusps have $2'$ cusps above them at level 1.]

Example 1.3 (Dihedral and Alternating cases). If $G = D_{p^{k+1}}$ with p odd, and $\mathbf{C}^* = \{\mathrm{C}_2\}$ (conjugacy class of an involution), then $i \mapsto \mathbf{C}_{2^{r_i}}$ is one-one and onto, with the r_i s running over all even integers ≥ 4. Also, H_i^{rd} identifies with the space of cyclic p^{k+1} covers of hyperelliptic jacobians of genus $\frac{r_i-2}{2}$ [**?**, §5].

If $G = A_n$ with $\mathbf{C}^* = \{\mathrm{C}_3\}$, class of a 3-cycle, then $i \mapsto \mathbf{C}_{3^{r_i}}$ with $r_i \geq n$ is two-one. Denote indices mapping to r by $i_r^{\pm}$. Those covers in $\mathcal{H}_{i_r^{\pm}}$ are Galois closures of degree n covers $\varphi : X \to \mathbb{P}^1_z$ with 3-cycles for local monodromy. Also, write the divisor $(d\varphi)$ of the differential of φ as $2D_\varphi$. Then, $\varphi \in \mathcal{H}_{i_r^+}$ (resp. $\mathcal{H}_{i_r^-}$) if the linear system of D_φ has even (resp. odd) dimension; it is an even (resp. odd) θ characteristic. For $r_i = n-1$ the map $i \mapsto \mathbf{C}_{3^{r_i}}$ is one-one.

Together with the 3-cycle case what emerges is that all examples are as connected as they can be. I always wondered why, and the alternating group and 3-cycles paper gives must to ponder going beyond the genus case.

At the minimum you will get from it more evidence that your problem has much application and is true in greater generality than you have first conjectured – though we must still see on that. There is a statement in the alternating groups paper I sent previously that will play a role in my coming essay. It's forerunner was used in a paper with Helmut Voelklein that appeared in the Annals in 1992 which guides a lot of proven cases of connectedness of these spaces.

2. The Liu-Osserman problem

Call a product of disjoint cycles in S_n is *pure-cycle* if it has exactly one disjoint cycle of length exceeding one.

We say of a Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$ it is *pure-cycle* if all conjugacy classes are pure-cycle. Should we want to indicate a pure-cycle has length d, we refer to it as a d-cycle. Often we assume $G \le S_n$ is a transitive subgroup. Then, we say the Nielsen class pure-cycle and transitive, and apply these words to the covers they produce from RET. For such it is often convenient to indicate the Nielsen class by $\mathrm{Ni}(G,\mathbf{C}_{d_1\cdots d_r})^{\mathrm{abs}}$ if $d_1,\dots,d_r$ are the lengths of the pure-cycles.

2.1. Genus formulas.

If G is transitive, there is a necessary condition that $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$ is nonempty:

(2.1) The genus $\mathbf{g} = \mathbf{g}_{d_1\cdots d_r} \stackrel{\mathrm{def}}{=} \frac{\sum_{i=1}^r d_i}{2} - (n-1)$ is a non-negative integer.

From Riemann-Hurwitz $\mathbf{g}_{d_1\cdots d_r}$ is the genus of any cover in the Nielsen class. Suppose $G \le S_n$ and $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})$, a pure cycle Nielsen class $\mathrm{Ni}(G,\mathbf{C})$, with the image of $\mathbf{C}$ in S_n equal to $\mathbf{C}^{S_n} \stackrel{\mathrm{def}}{=} \mathbf{C}_{d_1\cdots d_r}$. Suppose $\varphi: X \to \mathbb{P}^1_z$ corresponds to $\boldsymbol{g}$ in this Nielsen class.

Suppose G is a transitive, but not a primitive, subgroup of S_n. Then φ decomposes as $X \to X' \xrightarrow{\varphi'} \mathbb{P}^1_z$, with the degree of $X \to X'$ equal to $1 < m < n$ dividing n. Since the Nielsen class is pure-cycle, by reordering the d_i s we have the following setup. Above each of the branch points $z_i \in \boldsymbol{z}$, there is exactly one ramified point $x_i \in X$ having image $x_i' \in X'$, and for these the following hold:

(2.2a) for $1 \le i \le r'$, x_i/x_i' has ramification index m (totally ramified) and x_i'/z_i has ramification index d_i/m; and

(2.2b) for $r'+1 \le i \le r$, x_i/x_i' has ramification index d_i (x_i'/z_i doesn't ramify).

So, $X' \to \mathbb{P}^1_z$ is a cover in the Nielsen $\mathrm{Ni}(G',\mathbf{C}')$ with G' a transitive subgroup of $S_{\frac{n}{m}}$ and $(\mathbf{C}')^{S_{\frac{n}{m}}} = \mathbf{C}_{\frac{d_1}{m}\cdots\frac{d_{r'}}{m}}$.

THEOREM 2.1. *Continue the previous notation with φ,φ',m,r'. Apply R-H to φ' to compute the genus $\mathbf{g}'$ of X' as*

$$\begin{aligned}\mathbf{g}_{\frac{d_1}{m}\cdots\frac{d_{r'}}{m}} &= \frac{1}{2m}\Big(\sum_{i=1}^{r'} d_i - m - 2(n-m)\Big)\\ &= \frac{1}{2m}(2\mathbf{g}_{d_1\cdots d_r} - \sum_{i=r'+1}^{r} d_i - 1 - (r'-2)(m-1)).\end{aligned}$$

Now suppose that no such m exists (G is primitive) and G contains a length d pure-cycle with $d \le (n-d)!$. Then, $G = A_n$ if all the d_i s are odd and S_n otherwise.

PROOF. Formula (2.1) is just manipulation with R-H. The last paragraph follows from [**Wm73**]. □

[**LOs06**, Thm. 5.3] is the case $\mathbf{g}_{d_1\cdots d_r} = 0$ in (2.1), from which one deduces G is primitive: there is no such m since the right side would then be negative. We will use the formula to conclude primitivity of G in many odd order pure-cycle Nielsen classes, thus forcing $G = A_n$.

2.2. The Liu-Osserman Theorem. The following is [**LOs06**, Thm. 1.2].

THEOREM 2.2. *Suppose an absolute Nielsen class is transitive, pure-cycle and genus 0. Then it consists of one braid orbit.*

We are going to use and the Main Theorem of [**Fri06b**] to

2.3. Nonempty Nielsen classes. Consider the transitive pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})$. Here we want to inspect when condition (2.1) is sufficient to guarantee the Nielsen class is nonempty. Notice that by braiding, we may assume with no loss a normalizing condition:

$$d_1 \le d_2 \le \cdots \le d_r. \tag{2.3}$$

Notice the next result does not assume $\mathbf{g} = 0$.

PROPOSITION 2.3. *If $r = 3$ and $\mathbf{g}_{d_1 \cdots d_r} = 0$, then there is a unique element in $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_3})$ satisfying* (2.3).

PROOF. We assume $d_1 \le d_2 \le d_3$. Assume the genus is $\mathbf{g}$. Let $g_1 = (1 \dots d_1{-}u \dots d_1)$ for some integer $1 \le u \le d_1{-}1$. Now consider the following two elements based on another integer t:

$$\begin{aligned} g_2 &= (d_1\, d_1{-}1 \dots d_1{-}u\, n \dots n{-}t), \text{ and} \\ g_3 &= (1 \dots d_1{-}u{-}1\, n \dots n{-}t\, d_1)^{-1}. \end{aligned} \tag{2.4}$$

Note these properties:

(2.5a) (g_1, g_2, g_3) has product-one.
(2.5b) The genus is ??.
(2.5c) This represents the unique element in $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_3})$.

□

When $r = 4$, the reduced Hurwitz space of a pure-cycle Nielsen class has a birational embedding in $\mathbb{P}^1_j \times \mathbb{P}^1_j$. To see that consider such a cover $\varphi : X \to \mathbb{P}^1_z$. Then, map the four branch points $\varphi_{\boldsymbol{z}}$ to their j invariant $j_{\varphi_{\boldsymbol{z}}}$. Above each branch point z_i is a unique ramified point x_i. So, that gives the j invariant of $\boldsymbol{x}$, which we denote $j_{\varphi_{\boldsymbol{x}}}$. The birational embedding is $\varphi \mapsto (j_{\varphi_{\boldsymbol{z}}}, j_{\varphi_{\boldsymbol{x}}})$.

3. The genus 0 problem

3.1. Polynomial case. Many applications that arose in early years considered covers by genus 0 curves. The serious applications included cases where the covers were polynomials covers. The enclosed "Two genus 0 problems of John Thompson" is based on 4 branch point polynomial covers that arose in considering many problems when I was starting out (especially Davenport's problem, and separately Schinzel's problem). Polynomial covers (of course) have one cycle (at infinity) in their branch cycles. Yet, in these problems the other branch cycles were not cycles. As in the Davenport problem case, they often had several unconnected (braid group not transitive) families, though a marvelous thing happened. They were accounted for as a collection by one of my early theoretical results called "The Branch Cycle Lemma." This showed the families were conjugate by action of the absolute Galois group of Q. In the enclosed paper, you also see the parametrization spaces of all the families are the same in a provocative way: When you go from the covers to the associated vector bundles, the distinctions between the families disappear.

3.2. Use of the Branch Cycle Lemma.

4. The Alternating Group Case

I've thought about your problem (all branch cycles are cycles and the cover is genus 0), and so far I have no counterexamples. The case of 4 branch point covers has always been the concentration point – except for this paper on alternating groups where the number r of branch points is arbitrary – that feeds into a large group of applications.

4.1. Possible groups G for pure-cycle Nielsen classes. For $g_{d_1\cdots d_r} > 0$ it is possible that many more values of r will produce transitive subgroups of S_n that don't contain A_n. What we have to show: 1. Primitivity. 2. If $d_1 \le (n-d_1)!$ we get the desired conclusion. For the latter apply [**Wm73**] stating that if a primitive subgroup of S_n contains a cycle of order d, with $d \le (n-d)!$, then it must be A_n or S_n.

THEOREM 4.1. *Algorithm to figure on the symbol of* $\boldsymbol{m}$ *for a given* $d_1^* \cdots d_s^*$.

5. Application case: 4 branch points

5.1. Modular Curves. Assume p is an odd prime, and consider the dihedral group $D_{p^{k+1}}$ of order $2\cdot p^{k+1}$. You can identify it with There is a natural birational embedding of the modular curves $X_0(p^{k+1})$ (p odd) into $\mathbb{P}^1_j \times \mathbb{P}^1_j$. Here is how we compare this with our other examples. Consider the Nielsen class $\mathrm{Ni}(\mathbb{Z}/p^{k+1} \times^s \{\pm 1\}, \mathbf{C}_{2^4})^{\mathrm{abs}}$ of p^{k+1} degree covers with 4 branch points whose conjugacy classes are all repetitions of the class of -1 in

5.2. Pure-cycle cases of the MCMTs. We now have reason to try out Liu-Osserman on their special case $r = 4$ and $\mathbf{g} = 0$, which by itself is a model for explicitly describing reduced Hurwitz spaces. Then, §5.2.2 considers what generalizing their results does for the MCMTs. Finally, §5.2.3 considers what are the implications for the STC.

5.2.1. *The MCMTs for the genus 0 pure-cycle case.* Theorem4.2. In Situation 4.1, the cardinality of a Nielsen class is $\min_{1\le i\le 4}(d_i(n+1-e_i)$. Notice we have g_2g_3 in place of there use of g_3g_4. Moreover, the possible $\boldsymbol{g}$ are classified as follows with $g \stackrel{\text{def}}{=} g_2g_3 = (\boldsymbol{g})\mid$:

(5.1a) if g is trivial or a single cycle $(k\, k+1\, \dots\, d_2+d_3-k)$, then

$$g_4 = (n\, n{-}1\, \dots d_2{+}d_3{+}1{-}k\, g^{-(n+2-k-d_4)}(\ell)\, g^{-(n+3-k-d_4)}(\ell)\, \dots\, g^{-(d_2+d_3+1-2k)}(\ell) = \ell),$$

$$g_1 = (d_2+d_3+1-k\, d_2+d_3+2-k\, \dots\, n-1\, n\, \ell\, g^{-1}(\ell)\, \dots\, g^{-(n+1-k-d_4)}(\ell)),$$

$$g_2 = (k\, k-1\, \dots\, 2\, 1\, d_3+1\, d_3+2\, \dots\, d_2+d_3-k),$$

$$g_3 = (1\, \dots\, d_3),$$

where we allow any k with $d_2+d_3-n \le k \le d_2$ and $k \le n+1-d_1$, we allow ℓ to vary in the range $k \le \ell \le d_2+d_3-k$.

(5.1b) if g is a product of two disjoint cycles, then

$$g_4 = (m + d_4 - 1\, m + d_4 - 2\, \dots\, m + 1\, m),$$

$$g_1 = (n\, n - 1\, \dots\, m + d_4\, m + n + k - d_2 - d_3\, m + n - 1 + k - d_2 - d_3\, \dots\, k),$$

$$g_2 = (k\, k{-}1\, \dots\, 1\, d_3{+}1\, d_3{+}2\, \dots\, m{+}d_4{-}1\, m\, m{-}1\, \dots\, m{+}n{+}1{+}k{-}d_2{-}d_3\, m{+}d_4\, m{+}d_4{+}1\, \dots\, n),$$

$$g_3 = (1\, \dots\, d_3),$$

where we allow any k with $1 \le k \le d_2 + d_3 - n - 1$, and any m with $d_3 - d_4 + 1 \le m \le n + 1 - d_4$ and $m \le d_3$.

5.2.2. *The MCMTs for* $\mathbf{g} > 0$ *in the pure-cycle case.*

5.2.3. *What the MCMTs says about the STC.*

6. Guided by the Conway-Fried-Parker-Völklein result

6.1. Limit components. An addition to [**FV91**] says this (see App. **??**).

Theorem 6.1 (Branch-Generation Thm.). *Assume G centerless and $\mathbf{C}^*$ a distinct rational union of (nontrivial) classes in G. An infinite set $I_{G,\mathbf{C}^*}$ indexes distinct absolutely irreducible $\mathbb{Q}$ varieties $\mathcal{R}_{G,\mathbf{C}^*} \stackrel{\text{def}}{=} \mathcal{R}_{G,\mathbf{C}^*,\mathbb{Q}} = \{\mathcal{H}_i\}_{i \in I_{G,\mathbf{C}^*}}$ with:*

(6.1a) *a finite-one map $i \in I_{G,\mathbf{C}^*} \mapsto {}_i\mathbf{C}$, r_i conjugacy classes of G supported in $\mathbf{C}^*$; and*

(6.1b) *the RIGP holds for G with conjugacy classes $\mathbf{C}$ supported in $\mathbf{C}^*$ $\Leftrightarrow$ $i \in I_{G,\mathbf{C}^*}$ with $\mathbf{C} = {}_i\mathbf{C}$ and $\mathcal{H}_i$ has a $\mathbb{Q}$ point.*

The emphasis is on $I_{G,\mathbf{C}^*}$ being infinite. Realizations come by augmenting existence of $\mathcal{R}_{G,\mathbf{C}^*}$ with info on the varieties $\mathcal{H}_i$, $i \in I_{G,\mathbf{C}^*}$. Given $\mathbf{C}$, the collection of $\boldsymbol{g} \in \mathbf{C}$ that generate with product-one is called the *Nielsen class* of $(G, \mathbf{C})$. Denote it $\mathrm{Ni}(G, \mathbf{C})$. Each $i \in I_{G,\mathbf{C}^*}$ corresponds to a unique Nielsen class $\mathrm{Ni}(G, {}_i\mathbf{C})$ with ${}_i\mathbf{C}$ having r_i elements (see §**??**). The reduced space $\mathcal{H}_i^{\mathrm{rd}}$ equivalences field extensions if they differ by a change $z \mapsto \alpha(z)$, $\alpha \in \mathrm{PGL}_2(\mathbb{C})$. Its dimension is $r_i - 3$.

Appendix A. Hurwitz spaces

A.1. Inner, absolute and reduced equivalence.

A.2. sh-incidence and modular curve cusps. In the next subsections, there are two different copies of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, γ_∞ in $\mathrm{SL}_2(\mathbb{Z})$ and also in $D_{p^{k+1}}$; and they shouldn't be confused. We refer, therefore, to the first as γ_∞.

A.3. Some nonmodular curve cusps.

Appendix B. Applications

B.1. Maps that are one-one.

B.2. Relations among zeta functions.

B.3. The Regular Inverse Galois Problem —RIGP.

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