Connected Components of Sphere Cover Families of A_n-type

Michael D. Fried

Abstract. Our basic question: Restricting to covers of the sphere by a compact Riemann surface of a *given type*, do all such compose one connected family? Or failing that, do they fall into easily discerned components? The answer has often been "Yes!," figuring in the connectedness of the moduli space of curves of genus g (geometry), Davenport's problem (arithmetic) and the genus 0 problem (group theory). One consequence is that we then know the definition field of the family components. Our main goal is to explicitly describe specific projective sequences of such families, called **M(odular) T(ower)**s. This shows precisely why the Main **MT** Conjecture holds: high tower levels have general type and, for K a fixed number field, no K points.

We start with connectedness results for certain *absolute* Hurwitz spaces – examples of Liu-Osserman – of alterating group covers. The *inner* versions of these spaces are level 0 of our **MT**s. Connectedness results ensure certain cusp types – especially those defined by the *shift of a H(arbater)-M(umford)* representative – lie on a tower level boundary. Another type, a p-cusp, directly contributes to showing the Main **MT** Conjecture.

Modular curve towers have both p- and H-M related cusps, and no others. General **MT**s, can have another cusp type. This is like our examples, where $p = 2$, which have no p-cusps at level 0. Still, this 3rd type often disappears at higher levels, to be replaced by p-cusps. Our cusp description uses modular representations, rather than semi-simple representations. The *sh-incidence matrix*, from a natural *pairing* on cusps, simplifies displaying results. A *lift invariant* explains the nature of both components and cusps.

Contents

2000 *Mathematics Subject Classification.* Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34.

http://www.math.uci.edu/˜mfried has been revamped for adding/updating definitions/aids in understanding **MT**s. The end of §1.3.2 has the paths to two definition helpfiles.

1. Framework for the problem

The production of **MT**s is driven by two applications (see §6.1.1).

- Forming families of ℓ-adic representations over which we have control.
- Analyzing definition fields for regular Inverse Galois realizations.

The point of **MT**s was that these two goals were intimately related. To explain that, papers like [**D06**], [**DE06**], [**Fr95b**], [**Fr06a**] and [**FK97**] used the one well-known special case: modular curve towers.

The approach is not the usual take on modular curves (see Thm. 1.2). So, there's always something knew when viewing **MT**s as their generalization. That happens here (starting in §1.1.2) when we show how our approach to cusps gives meaning the phrase "over which we have control." We prove here that **MT** conjectures – in particular the Main Conjecture 1.4 – generalizing modular curve properties hold in explicit ways. This makes the towers of this paper available for [**Fr11b**]: finding new contexts where Serre's Open Image Theorem works.

§1.1 describes the main results as a by-product of understanding cusps. We use cusps types — introduced in [**Fr06a**, §3.2] — to derive connectedness results. They do so by giving a group/geometric character to components of the spaces comprising the **MT** levels.

1.1. A brief overview. Denote the Riemann sphere, inhomogenously uniformized by z, by $\mathbb{P}^1_z$. Let G be p-perfect group (§1.3.2) and let $\mathbf{C}$ be p' conjugacy classes of G (multiplicity of the appearance of conjugacy classes matters). Each such $(G, \mathbf{C}, p)$ produces a projective system of algebraic varieties $\{\mathcal{H}_k\}_{k=0}^{\infty}$.

1.1.1. *Equivalence notation.* Each $\boldsymbol{p} \in \mathcal{H}_k$ corresponds to an equivalence class of covers $\varphi : X \to \mathbb{P}^1_z$. Four different equivalences on covers correspond to four different, but related, spaces. The names for these equivalences are *absolute*, *inner*, and for each of those an additional equivalence called *reduced* (App. §A).

Our space notations for these might be $\mathcal{H}_k^{\rm abs,rd}$ for the absolute, reduced space, or $\mathcal{H}_k^{\rm in}$ for the inner (non-reduced) space. Some results hold for both absolute and inner spaces. In that case our notation will just refer to $\mathcal{H}^*$ or $\mathcal{H}^{*,rd}$.

In general those spaces that relate to a classical space are reduced spaces. Yet, for connected results there is a one-one relation between the components of the non-reduced and their reduced versions. Significantly, often that does not hold for the relation between absolute and inner spaces.

Definition 1.1. A **MT** is a projective system of absolutely irreducible components on $\{\mathcal{H}_k\}_{k=0}^{\infty}$.

We can always form **MT**s of inner spaces. Usually, though, we need an extra condition to form them for absolute spaces. The absolute case arises when G comes with a permutation representation T – defined by a subgroup we often denote as $G(T,1)$ – that extends to a representation on all the groups G_k. Each G_k is a p-Frattini cover of G (§2.1.2).

(1.1) The natural condition for extension is that $(|G(T,1)|, p) = 1$. Then (Schur-Zassenhaus), $G_k \to G$ restricted over $G(T,1)$ splits.

1.1.2. *Modular curve comparison.* Rational points on each $\mathcal{H}_k$ correspond to regular realizations of a Frattini covering group $G_k \to G$ with p-group kernel (of exponent p^k). One case G_k are dihedral groups $\{D_{p^{k+1}}\}_{k=0}^{\infty}$ (p odd) of respective orders $2 \cdot p^{k+1}$. The conjugacy classes are four repetitions of the involution (order 2) class. Here we get spaces $\mathcal{H}_k^{\rm abs,rd}$ and $\mathcal{H}_k^{\rm in,rd}$, because the natural representation is on a subgroup $G(1)$ generated by an involution, so $G(1)$ satisfies (1.1).

Theorem 1.2. [**Fr78**, §2] *and §2.5.2: A natural map*

$$\mathcal{H}_k^{\rm abs,rd} \to Y_0(p^{k+1}) \ (resp.\ \mathcal{H}_k^{\rm in,rd} \to Y_1(p^{k+1}))$$

to modular curves – minus their cusps – is an isomorphism of algebraic curves compatible with the maps of both spaces to the j-line.

General Hurwitz space components at level k correspond to (Artin or Hurwitz) braid orbits on the (combinatorial) *Nielsen class* attached to $(G_k, \mathbf{C}, p)$ (§1.3). To understand, however, level $k > 0$, it is necessary to nail level 0 ($G_0 = G$).

Some **MT** levels have nothing over them at the next level (an *obstructed* level; some tower levels may even be empty). Some **MT** levels have several components (even at level 0). Ex. 2.29 and Ex. 6.12 give, respectively, infinitely many examples of each situation, including distinctions between *absolute* and *inner* spaces.

Being obstructed is not modular curve-like. Yet, homological results show precisely when there is obstruction (Prop. 2.26). Invariance Prop. 2.23, generalizing spin structure on a Riemann surface, makes it especially effective when $p = 2$. §2.1.3 reformulates describing **MT**s, as a problem in classifying p group extensions of any finite p-perfect group. The condition that G is p-perfect (resp. that $\mathbf{C}$ are p' classes) is necessary for any level (resp. any level past the 0th) to be nonempty.

From here on, assume a **MT** refers to a projective system of components of the spaces in $\{\mathcal{H}_k\}_{k=0}^{\infty}$ with all levels nonempty.

Definition 1.3. We say a **MT** is over a field K when all levels (and maps between them) are defined over K.

Main Conjecture 1.4 is a variant on [**FK97**, Main Conj. 0.1]. There is an exposition on it in [**D06**]. Most significant: Even if you didn't know what is a Hurwitz space, they and **MT** s are forced into existence by the formulation, for any finite group, of Conj. 1.6 on D_p [**Fr06a**, Prop. 1.1].

Conjecture 1.4. [**Fr06a**, §1.1.3]: The following hold for abelianed **MT** s:

(1.2a) High towers levels have *general type*.

(1.2b) For K a number field, high tower levels have no K points.

Property (1.2a) applies to the space $\bar{\mathcal{H}}_k^{\mathrm{in,rd}}$, the natural projective normalization of $\mathcal{H}_k^{\mathrm{in,rd}}$ as a cover of $\mathbb{P}^r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\mathrm{def}}{=} J_r$. It means that its sheaf of holomorphic 1-forms has a tensor product that embeds the space in some projective space.

Property (1.2b) – generalizing a modular curve property from Thm. 1.2 – is trivial unless the **MT** is defined over a number field. When $r = 4$, (1.2a) and (1.2b) are equivalent; both equivalent having the genus of $\mathcal{H}_k^{*,\mathrm{rd}}$ rise with k.

1.1.3. *The role of cusps.* The cusp types from §2.3.3 that naturally generalize those on modular curves are the g-p' cusps and the p-cusps. Example: The two cusps on $X_0(p)$ have width p and width 1. In our identifications, the former is a H(arbater)-M(umford) cusp (that happens to be a p-cusp) and the latter, the *shift* of the former, is a very special g-p' cusp.

The possibility of no p-cusps at any level of a **MT** – compare with Prop. 2.13 – is what makes the Main Conjecture hard. Connectedness results allow *recognizing* components of **MT** levels by *distinguishing cusps* (on the boundaries of their compactifications). To show how this works, we establish the Main Conjecture for infinitely many cases where G is an alternating group and $p = 2$.

§1.2 describes our connected results, and how they prove the Conj. 1.4. §1.3 reviews the framework. Proving Conj. 1.4 for *abelianized* **MT**s (§2.1) is a stronger result. Also, abelianized towers are more akin to modular curve towers. Tests for nonempty abelianized towers are simpler than for general towers (Prop. 2.26).

§2.2.3 has a more detailed list of results, summarizing what comes from a list of sh-incidence matrix Tables. These apply the *cusp pairing* on reduced Hurwitz spaces introduced in [**BF02**, §2.10]. Tables 2–7 display our main theorem (especially Table 7 in §5.2.3). These make all components, cusp-types and elliptic ramification contributions transparent. The remaining sh-incidence tables show the difference between assuming spaces of genus 0 covers (say, in the Liu-Osserman examples) and the case of higher genus covers.

1.2. Spaces whose components appear here. Liu and Osserman consider all connected covers, $\varphi : X \to \mathbb{P}^1_z$ with the following properties.

(1.3) The degree of $\deg(\varphi)$ is n, the genus, $\mathbf{g}_X$, of X is 0, and the cover has r specific *pure-cycles* as branch cycles (Def. 1.7).

We denote one of their examples by their pure-cycle lengths (with no loss) as $\boldsymbol{d} = (d_1, \dots, d_r)$. From *Riemann-Hurwitz*,

$$\frac{\sum_{i=1}^{r} d_i - 1}{2} = (n-1). \tag{1.4}$$

THEOREM 1.5. [**LOs06**, Cor. 4.11]*: The absolute space, $\mathcal{H}^{\rm abs}_{\boldsymbol{d}}$, of such covers form one connected family.*

1.2.1. *Context for Liu-Osserman.* Compare the Liu-Osserman genus 0 result with [**Fr11**, Thm. A and B]. There the r pure-cycles are all 3-cycles, but $\mathbf{g}_X$ (as in (1.3)) is any fixed non-negative integer. Here, if $\mathbf{g}_X > 0$ ($r \ge n$), then the absolute (and inner) spaces have exactly two components, distinguishable using our main tool, the *spin invariant* (related in this case to Riemann's half-canonical classes).

The spin invariant has many uses. Two used here:

(1.5a) Deciding, when $p = 2$, that a non p-cusp has above it only p-cusps.

(1.5b) Formulating a natural umbrella result containing both [**LOs06**] and [**Fr11**] (§6.3.3).

Our main results apply when all the pure-cycles have odd order and $r = 4$. Then, $G = A_n$ and, with no loss, the pure-cycle lengths are $d_1 \le d_2 \le d_3 \le d_4$ with $\frac{\sum_{i=1}^{4} d_i - 1}{2} = n-1$. We redo, while generalizing, part of their results for two reasons.

(1.6a) [**LOs06**, Cor. 4.11] is on *absolute* equivalence, but Inverse-Galois and **MT** s are on *inner* equivalence of Galois covers.

(1.6b) Redoing their hardest case, $r = 4$, using our combinatorial description of cusps shows quickly its advantage (Table 1 of Lem. 4.3).

Example of (1.6a): Several inner space components may map to the same absolute space component, as happens when $n \equiv 1 \mod 8$ in Prop. 4.1. This contrasts with our main result with $n \equiv 5 \mod 8$, when there is just one inner component.

When $p = 2$, Ex. 3.13 applies Invariance Prop. 2.23 to describe exactly which of the Liu-Osserman examples are the bottom level of at least one abelianized **MT**. When $p \ne 2$ is a prime dividing $n!/2$ (but none of the d_i s), then each Liu-Osserman example is the bottom level of at least one **MT**.

§E.3 examples show that if you don't assume spaces of genus 0 covers, the story is richer. Yet, the lifting invariant still tells much of the story.

1.2.2. *Connecting to the R(egular) I(nverse) G(alois) P(roblem).* Each space in §1.2.1 occurs in the RIGP. Here is why this is also a modular curve-like property.

Spaces $\mathcal{H} = \mathcal{H}^{\rm in}$ attached to inner equivalence and a centerless group G come with a uniquely defined Galois cover $\Psi : \mathcal{Y} \to \mathcal{H}^{\rm in} \times \mathbb{P}^1_z$, with group G. Attached to a K point $\boldsymbol{p} \in \mathcal{H}$ is the fiber $\Psi_{\boldsymbol{p}} : \mathcal{Y}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$. This is a geometric cover attached to a K regular realization of G.

Assume p is an odd prime. For modular curves, an old story gives an RIGP way to look at the K rational points of $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$. Any $\boldsymbol{p} \in Y_1(p^{k+1})(K)$

corresponds to a regular realization of the dihedral group $D_{p^{k+1}}$ of order $2p^{k+1}$ with four *involution* (order 2) branch cycles ([**Fr78**, §2], for notation §1.3).

A $\mu(m)$ point on an abelian variety, A, is an m-division point $a \in A$ for which the group $G_{\mathbb{Q}}$ acts the same on the group generated by a as it does on the multiplicative group generated by $e^{2\pi i/m}$. [**DFr94**, Thm. 5.1] says that finding a $\mu(m)$ point on a hyperelliptic Jacobian of dimension $\frac{r}{2}-(m-1)$ is equivalent to finding a $\mathbb{Q}$ regular realization of D_m with where $\mathbf{C}$ consists of r involution conjugacy classes, if m is odd. As a consequence of Mazur's Theorem, you need at least six branch points to find a $\mathbb{Q}$ regular realization of D_m if odd m exceeds 7. The following is an extremely special case of Conj. (1.2b).

Conjecture 1.6. [**DFr94**, §5.2]: For any odd prime p, there is no integer r_0, for which there are $\mathbb{Q}$ regular realizations of $D_{p^{k+1}}$ with at most r_0 branch points.

1.2.3. *The values of* $\boldsymbol{d}$. We restrict to the $\boldsymbol{d}$ in Liu-Osserman which are 2-perfect §1.3.2. When even one of the d_i is even, the monodromy group of a cover in the Nielsen class is S_n, which is not 2-perfect . So, we must have all the d_i s odd.

Standard definitions of modular curves use congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, making it a numerical matter to find their genuses. When $r = 4$, finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ define (reduced) **MT** levels [**BF02**, Prop. 4.4]. Yet, they are rarely congruence subgroups; certainly not in the Liu-Osserman cases. Still, the (compactified) levels have a genus (usually different from that of the curves their points parametrize). Conj. 1.4 holds if the and only if some tower level genus exceeds 1 ([**Fr06a**, Prop. 5.1] or Prop. 2.13).

When all the d_i s are odd, then $G = A_n$ for some n. For the most refined results, we assume all d_i are equal. We denote the conjugacy classes then by $\mathbf{C} = \mathbf{C}_{(\frac{n+1}{2})^4}$, four repetitions of an $\frac{n+1}{2}$-cycle (odd only if $n \equiv 1 \mod 4$). §5.2.3 (for the very different cases $n \equiv 5 \mod 8$ and $n \equiv 1 \mod 8$) prove the Main Conjecture as a corollary of computing the genuses (Prop. 5.15) of the reduced absolute and inner level 0 Hurwitz spaces. A graphic understanding of cases — the first showing the Main Conjecture holds for infinitely many distinct, non-modular curve, examples — comes from explicit sh-incidence matrices.

The case $n \equiv 1 \mod 8$ has special interest because there are two braid orbits. That is $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ is not connected. The explanation is not from the lift invariant. Rather, the outer automorphism of A_n is unbraidable. It permutes the two components. This produces infinitely many reduced Hurwitz spaces where $G = A_n$ and with two components whose definition field is not immediate. Yet, §6.3.4 shows they are conjugate over a natural quadratic extension of $\mathbb{Q}$.

When $n \equiv 5 \mod 8$, $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ has just one component, generalizing level 0 of the case ($n = 5$) guiding [**BF02**]. The difference: [**BF02**, §9] went deeply into level 1 — including the sh-incidence matrix for it. While here we extract less information about level 1, we still manage modular curve parallels, as in §1.2.4).

1.2.4. *2-cusps and strengthening the modular curve analogy.* None of the spaces $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ has any 2-cusps. Yet, each **MT** (for $p = 2$) over them does have a 2 cusp by level 1 (Cor. 6.7). We expand on the §1.1.3 discussion on modular curve cusps. Cusps on a **MT** form a projective tree. To tackle the nature of the tree, we compare it with the cusp tree of modular curves. For that purpose we call the type of subtree that arises over the long cusp of $X_0(p)$ a *p-Spire*.

For $n \equiv 5 \mod 8$, the H-M cusps at level 1 are the base of a 2-Spire (Cor. 6.8), a property stronger than the Main Conjectures when $r = 4$. Considering when there is a p-Spire is meaningful for any $r \geq 4$.

§6.2.1 gives an approach to proving the Main Conjectures for Liu-Osserman examples for primes different from 2. Right now it lacks a piece of modular representation theory. Still, §6.2.2 shows it working for $(A_5, \mathbf{C}_{3^4})$ and the prime 5.

We didn't do all the Liu-Osserman A_n odd pure-cycle genus 0 classes. §6.3 describes other phenomena for analogous results about the cusps for the remainder. Then, §E.3 continues on what the lift invariant has to say about dropping the genus 0 condition. This big topic should be helpful on understanding the major issues in the Conway-Fried-Parker-Völklein Thm. (§E.2) that still stands out as the most definitive result on connectedness of Hurwitz spaces. The [**FV91**, App.] result is roughly: If you repeat *all* conjugacy classes *sufficiently* many times, then there is one connected component of the Hurwitz space (absolute or inner) of covers of the sphere in a given Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. We engage expectations for what *sufficiently* means. Also, using all classes doesn't even include the easy classical results of the connectedness of the moduli of genus g curves, and it defies modern applications. So, we also drop that.

For each fixed $n \equiv 5 \mod 8$, and conjugacy classes $\mathbf{C}_{(\frac{n+1}{2})^4}$, the only primes that seem to play a role in our **MT** are those dividing $|A_n|$, not dividing $\frac{n+1}{2}$. Building on the examples of [**Fr06a**, §6], §6.1.4 reminds that this is wrong. Our example here, generalizes that, but these are still just examples. Take $V = (\mathbb{Z})^{n-1}$ the $n-1$ dimensional irreducible representation of A_n: the standard representation modulo the trivial representation. Then, p not dividing $\frac{n+1}{2}$ consider the **MT** for p with base $\mathrm{Ni}(V/pV \times^s A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$. In this modular curve analogy:

$$\mathbb{Z}/2 \leftrightarrow A_n, \mathbb{Z} \leftrightarrow V \text{ and, the primes dividing } \frac{n+1}{2} \text{ are exceptional,}$$

as $p = 2$ is exceptional for modular curves. Considering when there is a p-Spire is meaningful for any $r \geq 4$, and for any allowable primes p.

1.3. Classical $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$ Generators. Let $\varphi : X \to \mathbb{P}^1_z$ be a (nonconstant) function on a compact Riemann surface X. Then, φ defines a number of quantities:

(1.7a) A group G for a minimal Galois closure cover $\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z$: Automorphisms $(\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z))$ of $\hat{X}$ commuting with $\hat{\varphi}$;
(1.7b) Unordered branch points $\boldsymbol{z} = \{z_1, \dots, z_r\} \in U_r$;
(1.7c) Conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G; and
(1.7d) A *Poincaré extension* of groups:

$$\psi_{\hat{\varphi}} : M_{\hat{\varphi}} \to G \text{ with } \ker_\psi \stackrel{\text{def}}{=} \ker(M_{\hat{\varphi}} \to G) = \pi_1(\hat{X}).$$

Further, (1.7a) produces a permutation representation of G, by its action on the cosets of $\mathrm{Aut}(\hat{X}/X)$ in $\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$. The coset of the identity is canonical, but other cosets may have no natural labeling.

1.3.1. *Homomorphisms and Nielsen classes.* Order the points in $\boldsymbol{z}$ to consider (App. A) a set of classical generators, $\mathcal{P}$, of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$. We don't order the conjugacy classes: $\mathrm{C}_i \leftrightarrow z_{\pi(i)}$ for some $\pi \in S_r$. The isotopy class of $\mathcal{P}$ consists of r (ordered) elements generatoring $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$. Denote these by $\bar{\boldsymbol{g}} = (\bar{g}_1, \dots, \bar{g}_r)$; their images in $M_{\hat{\varphi}}$ also as $\bar{\boldsymbol{g}}$; and their images in G by $(g_1, \dots, g_r) = \boldsymbol{g}$.

Then, $\boldsymbol{g}$ is in the *Nielsen class* of $(G,\mathbf{C})$:

$$\mathrm{Ni}(G,\mathbf{C}) \stackrel{\text{def}}{=} \{\boldsymbol{g}\in\mathbf{C} \mid \langle\boldsymbol{g}\rangle = G, \Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} g_1\cdots g_r = 1\}.$$

In English: Ordered r-tuple generators (satisfying *generation*) of G having product one, and falling (in some order, multiplicity counted) in the conjugacy classes $\mathbf{C}$.

Given classical generators $\bar{\boldsymbol{g}} = (\bar{g}_1,\dots,\bar{g}_r)$ of M_φ, $\boldsymbol{g}\in \mathrm{Ni}(G,\mathbf{C})$ is exactly what we need to form $M_{\hat\varphi}\to G$, by mapping $\bar{g}_i\mapsto g_i$, $i=1,\dots,r$. The notation $M_{\bar{\boldsymbol{g}}}$ applied to $M_{\hat\varphi}$ is useful, and then it is convenient to rename $\psi_{\hat\varphi}$ to $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$.

The classical generators of $\pi_1(\mathbb{P}^1_z\setminus\boldsymbol{z},z_0)$ form a homogeneous space for the action of the combinatorial *Hurwitz monodromy* group. We use the phrase *braid* equivalence of these homomorphisms by this action (§2.1.5). When there is just one such equivalence class, we call the space of such homomorphisms *connected*. This corresponds to an actual *Hurwitz* space $\mathcal{H}(G,\mathbf{C})$ being connected.

We use four Hurwitz space attached to any Nielsen class: $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$, $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs}}$ (from a permutation representation of G) and their reduced versions $\mathcal{H}(G,\mathbf{C})^{\mathrm{in,rd}}$ and $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs,rd}}$. Each corresponds to a further equivalence on $\mathrm{Ni}(G,\mathbf{C})$ (§2.1.5). Connectedness of $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ and $\mathcal{H}(G,\mathbf{C})^{\mathrm{in,rd}}$ (resp. $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs}}$ and $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs,rd}}$) are equivalent, though we emphasize cusps belonging to the reduced spaces. Connected absolute spaces, however, don't imply connected inner spaces.

App. A summarizes the literature on this correspondence and these spaces. Most of this paper is about the braid orbits. Applications depend on our figuring from this useful properties of the Hurwitz space, or their reduction by a $\mathrm{PSL}_2(\mathbb{C})$ action, so it has dimension $r-3$. Usually, initial data about some problem produces a collection of groups that could be the monodromy group of a cover solving the problem. So, applications are about collections of related Nielsen classes.

For G a p-perfect group, you can't get their Hurwitz spaces from Kummer theory; they come from nonabelian covers of $\mathbb{P}^1$. The essential data about Hurwitz spaces we use comes through connectedness results. Properties come from knowing about cusps – very nicely when $r=4$ – through the braid class of our homomorphisms $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$. When the spaces are connected, or their components separate by discrete invariants, we know their definition fields.

1.3.2. *Notation. Equations*: If V is quasi-projective and K is a field, then $V(K)$ is the points on V with coordinates in K.

Orbits: There are two general orbits on a Nielsen class $\mathrm{Ni}(G,\mathbf{C})$ (§1.3.1): Braid orbits (for the action of the braid group), and cusp orbits for the action of a cusp (sub-) group (§2.2) of the braid group. We denote the former by O, and the latter by either Cusp or ${}_{\mathbf{c}}O$ (the second more common). Both will often have distinguishing subscript and superscript decoration.

Permutations: Denote the cyclic group of order N by $\mathbb{Z}/N$. For any finite group G, with conjugacy classes $\mathbf{C}=\{\mathrm{C}_1,\dots,\mathrm{C}_r\}$, denote the least common multiple of orders of elements in $\mathbf{C}$ by $N_{\mathbf{C}}$. We say G is p-perfect if $p||G|$, but there is no surjective homomorphism $G\to\mathbb{Z}/p$.

DEFINITION 1.7. A *pure-cycle* conjugacy class in $G\le S_n$ is one in which each element in the class has exactly one nontrivial (length greater than 1) disjoint cycle.

Displays can simplify using $x_{i,j}$ for $(i\,i{+}1\,\cdots\,j)$ (assuming $1\le i<j\le n$). The inverse of this element is $x_{i,j}^{-1}=x_{j,i}$. When operations are confined to a segment of integers within a permutation we use $|\quad|$s around the segment, as in Ex. 3.10, $\boldsymbol{w}_2=|6\,7\,8|$. Often we will have $(\dots x_{i,j}\dots)$ where the $\dots$ on either side indicate

some integers adjoining the segment $|i \dots j|$. It is understood the parens around $x_{i,j}$ have been dropped, so the expression is still a cycle. Example: $(x_{1,3}\,4\,5) = x_{1,5}$.

LEMMA 1.8. *Consider $x_{a,b}$ with $a < b$ and $b - a \equiv 0 \mod 4$ (resp. $\equiv 2 \mod 4$). Then, $(b\,a)(b{-}1\,a{+}1)\cdots(b'\,a')$ with $(b'\,a') = (b{-}\frac{b-a-2}{2}\,a{+}\frac{b-a-2}{2})$, an even (resp. odd) permutation, conjugates $x_{a,b}$ to its inverse. It has parity $(-1)^{\frac{b-a}{2}}$.*

Also, if $b - a + 1 \equiv 0 \mod 4$ (resp. $\equiv 2 \mod 4$) then $(b\,a)(b{-}1\,a{+}1)\cdots(b'\,a')$ with $(b'\,a') = (b{-}\frac{b-a-1}{2}\,a{+}\frac{b-a-1}{2})$, an even (resp. odd) permutation, conjugates $x_{a,b}$ to its inverse. It has parity $(-1)^{\frac{b-a+1}{2}}$.

Previous papers (like [**Fr06a**]) used a right action of permutations on integers because several commuting group actions forced both left and right actions. groups. Throughout this paper we act on the left of integers.

One frequent computation (as in Prop. 5.1) has a cycle α of consecutive integers *conjugating* another cycle β containing a subsegment of those integers. As an example, take $\alpha = (1 \dots k)$ and $\beta = (b_1 \dots b_t\, i\, i{+}1 \dots j)$, $1 \le i < j \le k$, and the b_is disjoint from $\{1, \dots, k\}$, with it understood $i\, i{+}1 \dots j$ is a sequence of consecutive integers. Then, the act of conjugating β by α is to form $\alpha\beta\alpha^{-1}$. This produces $\beta' = (b_1 \dots b_t\, i{+}1\, i{+}2 \dots j{+}1)$, $1 \le i < j \le k$ with the proviso that if $j = k$, you replace $j{+}1$ by 1. This and similar types, and iterations (by α^u) of such conjugations will be regarded as instantaneously recognizable, with the operation referred to as *translation of a segment*.

The acronym **R-H** (App. A) is for the Riemann-Hurwitz formula. It gives the genus of a sphere cover from a branch cycle description for it. The genus of covers in a given Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ is constant, denoted $\mathbf{g}_{G,\mathbf{C}}$.

2. Tools and MT definitions

§2.1 introduces the braid group and certain of its quotients and subgroups, especially for a natural equivalence on group extensions. Definitions of **MT**s and their abelianization appear here. §2.2 has the combinatorial definition of cusps used in the paper's precise results. Their relation to the Main Conjecture is in §2.3.

§2.4 has the main homological tool, the *spin lift invariant* and how it applies to deciding braid orbits and existence of **MT**s. Finally, §2.5 introduces the precise Nielsen classes for our main result. Here there are examples of how to apply the lift invariant for information on cusps at the next level.

2.1. Braid actions and MTs.

We start with braid actions on sphere covers.

2.1.1. *Deformation equivalence of extensions.* If $\varphi : X \to \mathbb{P}^1$ is Galois with group G with G abelian, we could write equations for it by hand. From, however, G being p-perfect, it isn't. Further, why deal one cover at-a-time? Consider all covers with $(G, \mathbf{C})$ as their data: In the Nielsen class.

Our topological need: To devine connected components of all covers in a given Nielsen class. Each component has a cover with any a priori fixed (collection of r distinct) branch points $\boldsymbol{z}^0$. That is, any cover (with branch points $\boldsymbol{z}$) deforms through covers with r branch points to a cover with branch points $\boldsymbol{z}^0$. Further, if $(\boldsymbol{g}, \mathbf{C})$ is associated to it, consider a §1.3.1 homomorphism: $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$. Then, $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ and any of its extensions deform with it. This, the identification of *Hurwitz Monodromy group* H_r with $\pi_1(U_r, \boldsymbol{z}^0)$, and the explicit action (with representing paths) on $\bar{\boldsymbol{g}}$ in (2.1) is in [**Fr77**, §4].

For further help, H_r — related to classical braid group discussions — and their consequences are reviewed in [**BF02**, §2.2]. (Proofs are compatible with our use in [**Fr08a**, Chap. 4, 5]; exposition in html definition files as in §E.1.) We especially use H_4 though, generally: H_r is the group of automorphisms of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$ that preserves an (transitive) action on classical generators. Given classical generators, it identifies with $\pi_1(U_r, \boldsymbol{z}^0)$.

We give the generators of H_r by their actions on $\bar{\boldsymbol{g}}$:

(2.1a) Shift: $\mathbf{sh} : \bar{\boldsymbol{g}} \mapsto (\bar{g}_2, \dots, \bar{g}_r, \bar{g}_1)$; and

(2.1b) 2nd Twist: $q_2 : \bar{\boldsymbol{g}} \mapsto (\bar{g}_1, \bar{g}_2\bar{g}_3\bar{g}_2^{-1}, \bar{g}_2, \bar{g}_4, \dots)$.

For each $i = 1, \dots, r-1$ there is an i-twist $q_i \stackrel{\text{def}}{=} \mathbf{sh}^{i-2} q_2 \mathbf{sh}^{-i+2}$ ($i \mod r-1$). Our formulas are best seen using $i = 2$ when $r = 4$.

2.1.2. **MT** *definitions.* Denote the maximal p-Frattini cover of G with elementary p group kernel by $G_1 \to G = G_0$. Let $G_{k+1} = G_1(G_k)$. Note: We drop most p notation. Still, if you change p, the new G_k for $k > 0$ is a different group.

DEFINITION 2.1 (**MT**). A projective system of H_r orbits on $\{\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$ is a M(odular) T(ower). Let $\ker_{k,0} = \ker(G_k \to G_0 = G)$. An *abelianized* **MT** is similarly a projective system, except the braid orbits are in the Nielsen classes from replacing G_k by $G_k/(\ker_{k,0}, \ker_{k,0}) = G_{k,\mathrm{ab}}$ (as in [**BF02**, Prop. 4.16]).

Denote the projective limit of all $G_{k,\mathrm{ab}}$s by ${}_p\tilde{G}/(\ker_0, \ker_0) = {}_p\tilde{G}_{\mathrm{ab}}$. Though $G_{1,\mathrm{ab}} = G_1$, for $k > 1$, the natural map $G_k \to G_{k,\mathrm{ab}}$ has (known) degree exceeding 1 if and only if $\dim_{\mathbb{Z}/p} \ker(G_1 \to G) > 1 \Leftrightarrow G_0$ is not p super-solvable [**BF02**, §5.7].

Let $M_{\bar{\boldsymbol{g}},\mathrm{ab}}$ be the natural quotient of $M_{\bar{\boldsymbol{g}}}$ with $\ker(M_{\bar{\boldsymbol{g}},\mathrm{ab}} \to G)$ the homology of the Riemann surface for which $\ker(M_{\bar{\boldsymbol{g}}} \to G)$ is its fundamental group. Finding extensions of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$ to ${}_p\tilde{G}_{\mathrm{ab}}$ is equivalent to its extension to $M_{\bar{\boldsymbol{g}},\mathrm{ab}} \to G$.

Any p-perfect group G has a universal central p extension $\psi^* : R^*_{G,p} \to G$. Universal here means that if $\mu_{H,G} : H \to G \to 1$ is a central p extension, than a unique map $\psi : R^*_{G,p} \to H$ commutes between ψ^* and $\mu_{H,G}$. Let $\mu_k : R_k \to G_k$ be the universal exponent p central extension of G_k:

(2.2) $G_{k+1} \to G_k$ factors through μ_k, and $\ker(R_k \to G_k)$ is the max. elementary p-quotient of the Schur multiplier of G_k.

2.1.3. *Group form of* **MT***s.* For a prime p dividing $|G|$, we ask the following.

(2.3a) When does $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ extend to all covers $H \to G$ with p-group kernel?

(2.3b) How does this depend on $\boldsymbol{g}$?

(2.3c) What equivalence reasonably describes all extensions of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$?

2.1.4. *Basic Reductions.* The following reductions apply to considering when there is an affirmative answer to (2.3a).

(2.4a) Complete $M_{\bar{\boldsymbol{g}}}$ so $\ker_{\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}}$ is the pro-p completion of $\pi_1(X)$.

(2.4b) Restrict to p-Frattini covers $H \to G$ (no H proper in G maps onto).

(2.4c) Any $g \in \mathbf{C}$ must have order prime to p.

(2.4d) G must be p-perfect (it has no $\mathbb{Z}/p$ quotient).

Equivalent: When are all p-Frattini covers $H \to G$ achieved by unramified extensions $Y_H \to X$ extending $X \to \mathbb{P}^1$?

Here is the source of the (2.4) reductions. For (2.4b), consider a p extension

$$\mu : H \to G \to 1.$$

Take any subgroup $H^* \le H$ for which μ_{H^*} is still surjective. A minimal such is a p-Frattini cover of G. If you can extend $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ to that, you can extend it through μ.

Any element of order p in $G = G_0$ has all its lifts to G_1 of order p^2 [**FK97**, Lifting Lem. 4.1]. That explains (2.4c).

Finally, for (2.4d), we now know all elements of $\mathbf{C}$ are p'. So, entries of a Nielsen class element cannot generate if G has $\mathbb{Z}/p$ as a quotient (as in [**Fr06a**, Lem. 2.1]).

Since the G_k s (of §2.1.2) are co-final in all p-Frattini covers of G, goal (2.3a) needs only for H to run over the G_k s.

2.1.5. *Braid Comments.* Through (2.1), the H_r action on $\bar{\boldsymbol{g}}$ extends to the image of $\bar{\boldsymbol{g}}$ in any quotient group G of $M_{\bar{\boldsymbol{g}}}$. Then, it acts compatibly on these sets:

(2.5a) Inner Nielsen Classes: $\mathrm{Ni}(G,\mathbf{C})/G \stackrel{\mathrm{def}}{=} \mathrm{Ni}^{\mathrm{in}}$.

(2.5b) Absolute Nielsen classes: $\mathrm{Ni}(G,\mathbf{C})/N_{S_n}(G) \stackrel{\mathrm{def}}{=} \mathrm{Ni}^{\mathrm{abs}}$. with $G \le S_n$ giving a permutation representation.

(2.5c) Poincaré extensions: $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$.

Since we want extensions of homomorphism $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$, the action (starting from $q \in H_r$ acting on $\bar{\boldsymbol{g}}$, from the left) is given by $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} \mapsto \psi_{\bar{\boldsymbol{g}},(\boldsymbol{g})q^{-1}}$, an action on the right. Any extension properties of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ are preserved by a braid orbit.

Problem 2.2 (H^1 action). Given $(G,\mathbf{C},p)$, understand projective systems of H_r orbits on $\{\mathrm{Ni}(G_{k,\mathrm{ab}},\mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$.

2.2. Cusp types and sh-incidence. We understand H_r orbits (and reduced Hurwitz spaces) through the *cusps* that lie on an orbit. For each M(odular) T(ower) (Def. 2.1), there is a prime p (dividing $|G|$), and a notion of p-cusp that starts the story of distinguishing cusp types.

2.2.1. *Hurwitz space Cusps.* A combinatorial definition of cusps gives them as an H_r suborbit of a *cusp group* $\mathrm{Cu}_r < H_r$.

(2.6a) For $r \ge 5$: $\mathrm{Cu}_r = \langle q_2 \rangle$.

(2.6b) For $r = 4$: with $\mathcal{Q}'' = \langle \mathbf{sh}^2, q_1 q_3^{-1} \rangle$, $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}'' \rangle$.

Much data is from the conjugacy class of Cu_r. So —except for normalizations related to identifications with upper half-plane objects —if done consistently, we could substitute q_i for the appearance of q_2 in Cu_r.

Definition 2.3. A *p-cusp* is the Cu_r orbit of $\boldsymbol{g} \in {}_{\mathbf{c}}O$ for which $p^{\mu_p(\boldsymbol{g})} || \mathrm{ord}(g_2 g_3)$, $\mu_p(\boldsymbol{g}) > 0$ (p-multiplicity of $\boldsymbol{g}$).

The definition doesn't depend on the representive of the p-cusp, as changing the representative changes (g_2,g_3) to (hg_2h^{-1}, hg_3h^{-1}) with h a power of g_2g_3. For $r = 4$, to see that being a p-cusp is independent of the representative, you would substitute (g_4,g_1) (resp. (g_1,g_4)) for (g_2,g_3) to see the condition for a p-cusp is unchanged by applying $\mathbf{sh}^2$ (resp. $q_1q_3^{-1}$) to $\boldsymbol{g}$. When $r = 4$, we call $\mathrm{ord}(g_2g_3)$ the *middle product* of $\boldsymbol{g}$, denoted $(\boldsymbol{g})\mathbf{mp}$.

2.2.2. *Other cusp types for* $r = 4$. See App. B for $r > 4$.

(2.7a) g(roup)-p': $U_{1,4}(\boldsymbol{g}) = \langle g_1, g_4 \rangle$ and $U_{2,3}(\boldsymbol{g}) = \langle g_2, g_3 \rangle$ are p' groups

(2.7b) o(nly)-p': not a p cusp, but $U_{1,4}(\boldsymbol{g})$ or $U_{2,3}(\boldsymbol{g})$ are not p'.

For even $r = 2s$, *H(arbater)-M(umford)* cusps have a cusp orbit representative of form $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$. When $r = 4$, its shift $(g_1^{-1}, g_2, g_2^{-1}, g_1)$ is a representative for a g-p' cusp, no matter what is p since the middle product is 1.

Having an H-M rep. requires classes that are pairable: $\mathrm{C}_1, \mathrm{C}_1^{-1}, \dots, \mathrm{C}_s, \mathrm{C}_s^{-1}$ where C^{-1} denotes the class of the inverse of an element in C. Consider a Liu-Osserman pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{abs}}$ (§1.2). So, with $d_1 < d_2 < \cdots < d_u$, necessarily the conjugacy classes defining the Nielsen class have form $\mathbf{C}_{\boldsymbol{d}}$, with $\boldsymbol{d} = d_1^{e_1} \cdots d_u^{e_u}$ and each e_i even.

The only time there are two distinct conjugacy classes of length d_i is when $G = A_n$, and $d_u = n$ (if n is odd) or $d_u = n{-}1$ (when n is even). In these cases denote the conjugacy class pairs by $\mathrm{C}(n)', \mathrm{C}(n)''$ (resp. $\mathrm{C}(n{-}1)', \mathrm{C}(n{-}1)''$).

Proposition 2.4 (g-p' **MT**). *If a braid orbit O_0 has a g-p' cusp, then a* **MT**, $\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, *lies over it.*

Consider a pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{abs}}$ *(§1.2), with all the d_i s odd, and G a transitive, but not cyclic, subgroup of A_n. Then, Prop. 3.11 says $G = A_n$. This Nielsen class contains an* H-M *rep. if and only if one of the following:*

- (2.8a) $n \equiv 1 \mod 4$ *(resp.* $n{-}1 \equiv 1 \mod 4$*),* $d_u = n$ *(resp.* $d_u = n{-}1$*) and exactly half the e_u conjugacy classes of length d_u are equal* $\mathrm{C}(n)'$ *(resp.* $\mathrm{C}(n{-}1)'$*); or*
- (2.8b) $n \equiv 3 \mod$ *(resp.* $n{-}1 \equiv 3 \mod 4$*),* $d_u = n$ *(resp.* $d_u = n{-}1$*) and each of the conjugacy classes* $\mathrm{C}(n)'$ *and* $\mathrm{C}(n)''$ *(resp.* $\mathrm{C}(n{-}1)'$ *and* $\mathrm{C}(n{-}1)''$*) appear with even multiplicity.*

Proof. The first result applies to the general definition of g-p' cusp as in App. B [**Fr06a**, Fratt. Princ. 3.6]. The necessity of (2.8) is a consequence of the definition of H-M rep., and the congruence condition for declaring when $\mathrm{C}(n)', \mathrm{C}(n)''$, etc. each contain the inverse of any element in them (say, Lem. 1.8).

Finally, to fulfill an H-M rep. under these conditions requires only producing transitive pure-cycles whose lengths in order are given by the symbol $d_1^{e_1'} \cdots d_u^{e_u'}$ with e' denoting $\frac{e}{2}$. Since, $\sum_{i=1}^{u} \frac{e'}{2}(d_i{-}1) \ge n{-}1$, this is easy. Begin with $g_1 = (1 \dots d_1)$. Then continue inductively, starting the next pure-cycle — and its increasing integer sequence — with the last integer occuring in the previous pure-cycle. When you get to n, cycle around to 1. Example: For $n = 7$, $r = 6$ and $\boldsymbol{d} = 3^4 \cdot 5^2$, so $e_1' = 2, e_2' = 1$, take $g_1 = (1\,2\,3), g_2 = (3\,4\,5)$, $g_3 = (5\ 6\,7\,1\,2)$. □

Remark 2.5 (When $G = S_n$, or $\mathbf{g}_{G,\mathbf{C}_{\boldsymbol{d}}} > 0$). If in Prop. 2.4 – the pure-cycle case – one of the d_i s is even then G is no longer in A_n. There are two changes.

- (2.9a) A complication: You expect $G = S_n$, but there are some exceptions.
- (2.9b) A simplication: In S_n, their shape determines conjugacy classes, so there is no need for a Lem. 1.8.

I comment on (2.9a), using Rem. 3.14 which I distilled from [**LOs06**, Thm. 5.3]. Cyclic groups are one exception. Liu-Osserman have $\mathbf{g}_{G,\mathbf{C}} = 0$. But, **R-H** shows for $r \ge 3$, pure-cycles can't generate a cyclic (transitive) subgroup of S_n. Also, if $r = 3$, we could have $n = 5$, and $\boldsymbol{d} = 4^2 \cdot 5$ where G is the non-standard representation of S_5 in S_6: S_5 acting by conjugation on the normalizer of a 5-Sylow.

If we expand beyond the genus 0 case, we can also have $A_5 = G$ in this degree six representation (with $\mathbf{C}$ among the two 5-cycle conjugacy classes). So, for $\mathbf{g}_{G,\mathbf{C}} > 0$, these sporadic pure-cycle cases require careful accounting.

2.2.3. *sh-incidence on our main example.* Thm. 2.9 summarizes the main data we get from the explicit production of the *sh-incidence* pairing on cusps for the Nielsen classes denoted $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $n \equiv 1 \mod 4$.

Definition 2.6. For $r = 4$, and $* =$ abs or in, the *reduced $*$ Nielsen classes* are elements of the quotient $\mathrm{Ni}(G, \mathbf{C})^*/\mathcal{Q}''$. Denote these by $\mathrm{Ni}(G, \mathbf{C})^{*,\mathrm{rd}}$; (reduced) cusps correspond to Cu_4 orbits on $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$. The *length* (or *width*) of a cusp is the length of that Cu_4 orbit.

This combinatorial cusp definition produces a pairing on cusps.

DEFINITION 2.7 (**sh**-incidence). Given two cusp orbits ${}_{\mathbf{c}}O_1, {}_{\mathbf{c}}O_2$ (elements are $*,\mathrm{rd}$ representatives) the pairing maps $({}_{\mathbf{c}}O_1, {}_{\mathbf{c}}O_2) \mapsto |{}_{\mathbf{c}}O_1 \cap {}_{\mathbf{c}}O_2|$. [**BF02**, §2.10]: It makes sense for all $r \ge 4$; each matrix block corresponds to a component of the corresponding reduced Hurwitz space.

LEMMA 2.8. *Assume* $r=4$. *If* $\boldsymbol{g}_1^{\mathrm{in}} = ((\boldsymbol{g}_2)\mathbf{sh})^{\mathrm{in}}$ *implies* $((\boldsymbol{g}_1)\mathbf{sh})^{\mathrm{in}} = ((\boldsymbol{g}_2\mathbf{sh}^2)^{\mathrm{in}}$, *which is reduced equivalent to* $\boldsymbol{g}_2^{\mathrm{in}}$. *So, the* **sh**-*incidence matrix is symmetric.*

The *mapping class* group $\bar M_4 \stackrel{\mathrm{def}}{=} H_4/\mathcal{Q}''$ (from (2.6b)) has orbits on $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$ with * any appropriate equivalence; here either inner or absolute.

PROPOSITION 2.9. [**BF02**, Prop. 2.3]*: There is a one-one correspondence between* H_r *orbits* O *on a reduced Nielsen class* $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$ *and components* $\mathcal{H}_O$ *of the reduced Hurwitz space* $\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$. *Each* $\mathcal{H}_O$ *is a natural* J_r *cover.*

When $r=4$, $\mathcal{H}(G,\mathbf{C})_O$ *is an upper-half plane quotient, j-line cover, whose only possible ramified points are over 0 (resp. 1) of index dividing 3 (resp. 2).*

Its projective normalization $\bar{\mathcal{H}}_O$ *is a cover of* $\mathbb{P}^1_j$, *whose points over* $j=\infty$ *(cusps) correspond one-one with the reduced cusps in Def. 2.6 that lie on* O. *They have ramification indices equal to the corresponding cusp lengths.*

Assume $r=4$. Then, $\mathcal{Q}''$ acts through a Klein 4-group $K_4 = \langle q_1q_3^{-1}, \mathbf{sh}^2\rangle/\langle \mathbf{sh}^4\rangle$ [**BF02**, §2.10]. The expected length of the Cu_4 orbit on $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})^{\mathrm{in},\mathrm{rd}}$ is the length of the q_2 orbit of $\boldsymbol{g}$ as an element of $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$ – denote that $(\boldsymbol{g})\langle q_2\rangle$ – and that is usually $2\cdot(\boldsymbol{g})\mathbf{mp}$. That means the cusp widths are often easy to compute, but there are two modifications.

(2.10a) The length of $(\boldsymbol{g})\langle q_2\rangle$ is $(\boldsymbol{g})\mathbf{mp}$ if and only if $(\boldsymbol{g})\mathbf{mp}$ is odd and the condition of Princ. 3.5 holds.

(2.10b) The K_4 action could equivalence elements in $(\boldsymbol{g})\langle q_2\rangle$, pairs (respectively) 4-tuples, giving an orbit length of $\frac{(\boldsymbol{g})\mathbf{mp}}{2}$ (resp. $\frac{(\boldsymbol{g})\mathbf{mp}}{4}$).

Proving Thm. 2.10 – a summary of level 0 results – starts by listing absolute (§4.1.2) and inner (§4.1.3) cusps. The **sh**-incidence pairing immensely simplifies detecting connected components. Example: In the absolute cases, one long cusp intersects all other cusps, guaranteeing one matrix block, so one component. For all level 0 absolute results, proviso (2.10a) applies (Lem. 4.3), but significantly it is more complicated for inner results (Lem. 4.4). On the other hand, (2.10a) does not hold, because another rare event occurs: K_4 acts trivially (Lem. D.3; the reduced spaces lack fine moduli properties).

THEOREM 2.10. *Absolute cusps in* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs},\mathrm{rd}}$ *— all fall in one braid orbit, with a unique cusp of each odd length between 1 and* n. *Each cusp "meets" any other (including itself) at most twice (precisely in §5.1.3). For* $n \equiv 1 \mod 8$, $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in},\mathrm{rd}}$ *has two braid orbits (two Hurwitz space components; conjugate over* $\mathbb{Q}$*), each having sh-incidence pairing exactly as in the absolute case.*

For $n \equiv 5 \mod 8$, $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in},\mathrm{rd}}$ *and each odd* k, $1 \le k \le n$, *there are either two length* k *cusps, or one length* $2k$ *cusp. The* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in},\mathrm{rd}}$ *sh-incidence derives – mainly – from that of* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs},\mathrm{rd}}$ *by substituting in* *sh-incidence entries of the latter one of symbols* $\begin{smallmatrix}1|1\\ \hline 1|1\end{smallmatrix}$, $\begin{smallmatrix}2|0\\ \hline 0|2\end{smallmatrix}$ *or* $\begin{smallmatrix}0|2\\ \hline 2|0\end{smallmatrix}$*. Most important are rules for level 0 cusps to have only 2-cusps above them at level 1 (§6.1.3).*

§2.2.4 has general tools helping with elliptic fixed points and, from that and Thm. 2.10, computing the genuses of reduced Hurwitz spaces. The actual computations in our examples are in §5.3 (Prop. 5.15).

Our understanding of higher tower levels is dominated by the need to locate p-cusps, the topic of §2.3. It is from this that we can see a sense in which general **MT**s have some resemblance to modular curve towers.

2.2.4. *Elliptic fixed points and computing the genus.* [**Fr06a**, §3.1.2] has a formula for the genus, $\boldsymbol{g}_O$, of the compact reduced Hurwitz space component $\bar{\mathcal{H}}^*_O$ for O. Induce permutation actions of q_1q_2, or **sh**, or q_2 on O. Call these, respectively, $\gamma'_0, \gamma'_1, \gamma'_\infty$. Rem. 2.12 shows why γ'_0 has orbit lengths 1 or 3. Then**R-H** says:

$$2(|O| + \mathbf{g}_O - 1) = \mathrm{ind}(\gamma'_0) + \mathrm{ind}(\gamma'_1) + \mathrm{ind}(\gamma'_\infty). \tag{2.11}$$

The next lemma shows fixed points of either γ'_0 or γ'_1 (as in (2.11)) contribute to the main diagonal of the **sh**-incidence matrix.

Lemma 2.11. *The* **sh** *incidence pairing applied to all cusps in a reduced Nielsen class has its irreducible blocks corresponding one-one to the braid orbits (components). Further, we can replace the shift (represented by $q_1q_2q_1$) by q_1q_2 (representing γ_0) to form the matrix. More precisely, the set $({}_{\mathbf{c}}O')q_1q_2$ is the same as $({}_{\mathbf{c}}O')\mathbf{sh}$, and therefore their intersections with ${}_{\mathbf{c}}O$ are the same.*

Thus, any γ'_0 or γ'_1 fixed point in the Nielsen class contributes to the diagonal of the matrix. If all braid orbits on $|\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}|$ have length exceeding 1, then reduced Nielsen classes fixed by γ'_0 are distinct from such classes fixed by γ'_1.

The set ${}_{\mathbf{c}}O \cap ({}_{\mathbf{c}}O)\mathbf{sh}$ is preserved by the shift. In particular, if $|{}_{\mathbf{c}}O \cap ({}_{\mathbf{c}}O)\mathbf{sh}|$ is odd, then ${}_{\mathbf{c}}O \cap ({}_{\mathbf{c}}O)\mathbf{sh}$ contains at least one γ'_1 fixed point.

Proof. The 1st sentence is [**BF02**, Lem. 2.26]. We show the second. If q_1q_2 mod K_4 fixes $\boldsymbol{g} \in {}_{\mathbf{c}}O'$, then $q_2q_1q_2$ maps: $\boldsymbol{g}' = (\boldsymbol{g})q_2^{-1} \in {}_{\mathbf{c}}O' \mapsto \boldsymbol{g} \in ({}_{\mathbf{c}}O')q_2q_1q_2$. One braid relation is $q_2q_1q_2 = q_1q_2q_1$, the shift mod K_4. So, $\boldsymbol{g} \in {}_{\mathbf{c}}O' \cap ({}_{\mathbf{c}}O')\mathbf{sh}$. Since γ'_0 and γ'_1 generate $\bar{M}_4$ acting on reduced Nielsen classes, any simultaneous fixed point of both is a length 1 braid orbit. We excluded this.

Since $\mathbf{sh}^2$ is trivial on reduced Nielsen classes, applying the shift to ${}_{\mathbf{c}}O' \cap ({}_{\mathbf{c}}O')\mathbf{sh}$ gives $({}_{\mathbf{c}}O')\mathbf{sh} \cap {}_{\mathbf{c}}O'$, the same set. This concludes the proof. □

Remark 2.12. Applying $(q_1q_2)^3$ to $\boldsymbol{g} \in \mathrm{Ni}(G,\mathbf{C})$ conjugates $\boldsymbol{g}$ by its 4th entry.

2.3. Higher tower levels and p-cusps. Compatible with Def. 1.1, A **MT** corresponds to a projective system of braid orbits $\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k,\mathbf{C})^*\}_{k=0}^{\infty}$, $* =$ in or abs. There are two sequences of spaces attached to this: a projective system (tower) of ordinary (resp. reduced) Hurwitz spaces $\{\mathcal{H}_k\}_{k=0}^{\infty}$ (resp. $\{\mathcal{H}_k^{\mathrm{rd}}\}_{k=0}^{\infty}$), for each k a nonsingular (resp. normal) absolutely irreducible affine variety of dimension r (resp. $r-3$) covering $U_r \subset \mathbb{P}^r$ (resp. J_r). Each $\mathcal{H}_k^{\mathrm{rd}}$ is the quotient of $\mathcal{H}_k/\mathrm{PGL}_2(\mathbb{C})$ of $\mathcal{H}_k$ by the connected group $\mathrm{PGL}_2(\mathbb{C})$.

Prop. 2.13 applied to the precise form of the last line of Thm. 2.10 gives the Main Conjecture 1.4: Getting p-cusps (2-cusps in our cases) is the main point. The Main Conjecture is trivial unless G is p-perfect, for otherwise $\mathrm{Ni}(G,\mathbf{C})$ is empty when $\mathbf{C}$ are p' conjugacy classes [**Fr06a**, Lem. 2.1]. It is trivial, too, unless the **MT** has some number field K as definition field. That is, all levels, simultaneously including the maps between them, have K as definition field. [**Fr06a**, Prop. 3.3] reduces the Main Conjecture in general to this case:

(2.12) The prime p does not divide the order of the center of G.

From here we assume these things:

(2.13) G is p-perfect and has p' center.

2.3.1. *Reductions on Conj. 1.4.* Prop. 2.13 shows the Main Conj. for $\{\mathcal{H}_k\}_{k=0}^{\infty}$ follows from it holding for $\{\mathcal{H}_k^{\rm rd}\}_{k=0}^{\infty}$. When $r = 4$ it shows the main ingredient in getting explicit lower bounds on the genuses from knowing about cusps.

PROPOSITION 2.13. *As in §1.1.2, let $\bar{\mathcal{H}}_k^{\rm rd}$ be the unique (projective) normalization of $\bar{J}_r$ in the function field of $\mathcal{H}_k^{\rm rd}$. If there is at least one p-cusp at level k_0, the relative degree $\deg(\bar{\mathcal{H}}_{k+1}^{\rm rd}/\bar{\mathcal{H}}_k^{\rm rd}) \stackrel{\rm def}{=} d_{k+1,k}$, is some integer multiple of p for $k \geq k_0$. It is always true that $\limsup_{\leftarrow k} d_{k+1,k} > 1$.*

The conclusion of Conj. 1.4 holds if and only if it holds with each $\mathcal{H}_k$ replaced by $\mathcal{H}_k^{\rm rd}$. Also, if $r = 4$, Conj. 1.4 holds unless for $k >> 0$, either the cover $\bar{\mathcal{H}}_{k+1}^{\rm rd} \to \bar{\mathcal{H}}_k^{\rm rd}$

- *doesn't ramify and each $\bar{\mathcal{H}}_k^{\rm rd}$ has genus 1,*
- *or it is equivalent to a degree p polynomial cover of $\mathbb{P}_w^1 \to \mathbb{P}_j^1$,*
- *or it is equivalent to a degree p rational (Redyi) function ramified (of order p) at two points.*

PROOF. Everything in this proposition is already in [**Fr06a**, §5] except the observation showing the impossibility of $d_{k+1,k} = 1$ for all large k. Assume it is 1 for all $k \geq k_0$. Then, let $\boldsymbol{p}^{\rm rd} \in \mathcal{H}_{k_0}^{\rm rd}$ be any point defined over some number field K that is the image of $\boldsymbol{p} \in \mathcal{H}_{k_0}(K)$ (a K point of the non-reduced space).

Since G is p-perfect and its center is p', [**BF02**, §2.2.2] shows all the G_k s have the same p' center. First assume (holding for all our examples): G has no center at all. Then, the non-reduced spaces, given as fiber products [**BF02**, §2.2.2], $\{\mathcal{H}_k = \mathcal{H}_k^{\rm rd} \times_{J_r} U_r\}_{k=0}^{\infty}$, all have fine moduli [**FV91**, Cor. 1]. The degree 1 maps between the reduced spaces induce degree 1 maps $\mathcal{H}_{k+1} \to \mathcal{H}_k$, identifying them as the same cover of U_r. So, the points on each space identified to $\boldsymbol{p}$ give a projective sequence of covers $X_k \to \mathbb{P}_z^1$ realizing each G_k as a Galois group over K. This contradicts [**BF02**, Prop. 6.8]: No such projective sequence can exist over a number field K. This implies $\limsup_{\leftarrow k} d_{k+1,k} > 1$ if G has no center.

Suppose, G_0 does have a nontrivial p' center Z. Then, we see we can't immediately assume the conclusion above. While $\boldsymbol{p}$ has coordinates in K, since the $\mathcal{H}_k$ don't have fine moduli, there may be no Galois cover of $\mathbb{P}_z^1$ associated to $\boldsymbol{p}$ defined (with its automorphisms) over K. App. §C completes this case. □

2.3.2. *p ramification growth with k.* Princ. 2.14 says the power of p manifesting a p-cusp increases with the level for each cusp over a given p-cusp. Then, Cor. 2.16 concludes the p contribution to ramification also grows with rising level for those same cusps. Denote the power of p dividing $(\boldsymbol{g})\mathbf{mp}$ by $\mu_p(\boldsymbol{g})$ (Def. 2.3).

PRINCIPLE 2.14 (F(rattini) Princ. 1). [**Fr06a**, Princ. 3.5]*: Let $\mathcal{O} = \{O_k\}_{k=0}^{\infty}$ be a projective system of braid orbits on* $\mathbf{MT}$*. Assume $\{{}_k\boldsymbol{g}\}_{k=0}^{\infty}$ is a projective system of Nielsen class elements on these orbits. If ${}_{k_0}\boldsymbol{g}$ represents a p-cusp, then*

$$\mu_p({}_k\boldsymbol{g}) = k - k_0 + \mu_p({}_{k_0}\boldsymbol{g}).$$

Assume $r = 4$ and a $\mathbf{MT}$ for inner, reduced equivalence. As in Princ. 2.14, let ${}_{k_0}\boldsymbol{g} = ({}_{k_0}g_1, \ldots, {}_{k_0}g_4)$ represent a p-cusp, with corresponding geometric cusp $\boldsymbol{p}_{k_0}$.

Denote the center of a group G by $Z(G)$. While the next lemma is easy, I thought to show a reader new to braidings how they work on projective systems.

Lemma 2.15. *The length, $\alpha({}_k\boldsymbol{g})$, of the K_4 orbits in* (2.10b) *on* $({}_k\boldsymbol{g})\langle q_2\rangle$ *depends only on the braid orbit O_k of ${}_k\boldsymbol{g}$. Its values (1,2, or 4) are nondecreasing in k.*

Proof. That the length of the orbits of K_4 on O_k is constant follows from K_4 being normal in $H_4/\langle \mathbf{sh}^2\rangle$. That is if $q \in H_4$, then the collections $(\boldsymbol{g})K_4$ and

$$((\boldsymbol{g})q)K_4 = ((\boldsymbol{g})q)K_4(q^{-1}q = ((\boldsymbol{g})K_4)q$$

have the same cardinality.

That the K_4 orbit lengths are nondecreasing with k follows from the map O_{k+1} to O_k having constant fibers and commuting with the K_4 action. □

Combine notation of Lem. 2.15 on Princ. 2.14. The following is a special case of [**BF02**, Lem. 8.2].

Corollary 2.16. *Suppose $Z({}_0g_2, {}_0g_3) \cap \langle {}_0g_2{}_0g_3\rangle$ is trivial. For p odd and $k \ge k_0$, $p^{\mu_p({}_k\boldsymbol{g})}$ exactly divides the ramification index of $\boldsymbol{p}_k$ over the j-line. For $p = 2$, replace $2^{\mu_2({}_k\boldsymbol{g})}$ by $2^{\mu_2({}_k\boldsymbol{g})}/\alpha({}_k\boldsymbol{g})$ for the exact 2-power index divisor.*

The following problem is very close to the Main Conjecture when $r = 4$. We suspect a more refined statement is close to the Geometric $\mathrm{MC}^{\mathrm{ab}}$ for all $r \ge 4$.

Problem 2.17 (Goal 2). Given a **MT**, $\{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, classify when there is a p-cusp on O_k for $k >> 0$.

Conjecture 2.18. p-Cusp Presence: If a **MT** $\mathcal{O}$ has no projective system of p-cusps (a *p-cusp branch*), then high levels are relatively unramified, and they have no uniform defining number field.

Remark 2.19. Conj. 2.18 implies a **MT**, $\mathcal{O}$, over a number field has p-cusps at high levels. Then, we expect, for $k >> 0$, the relative monodromy groups $G(O_{k+1}/O_k)$ to be p groups, generated by the relative p-cusp monodromy groups.

Remark 2.20. This section assumed a **MT** of inner (vs absolute) braid orbits. §B.1 uses modular curve cusps to show what happens with absolute classes.

2.3.3. *Finding p-cusps and using g-p' cusps.* Finding p-cusps, and labeling the components on which they lie is the main technical ingredient in this paper. §3.1.2 identifies certain cusps as H(arbater)-M(umford) because the **MT** braid orbits all have specific so-named Nielsen class representatives (§2.2.2).

That the same components also have 2-cusps at level $k \ge 1$ is a property shared with modular curves (as noted in §1.1.3), and from it we get the Main Conjecture. The complication is that o-p' cusps (as in (2.7b)) can occur, too. These have no analog on modular curves.

The proof combines works of the author, Serre [**Ser90**] and Weigel [**Wei05**], for which a special case is Prop. 2.26. Its special ingredient is that the Poincaré extension in (1.7d), denoted $M_{\boldsymbol{g}}$ instead of M_φ below, satisfies p-Poincaré duality.

It comes through interpreting the Main **MT** conjecture as a problem of computing braid equivalence classes of extensions of the groups $M_{\boldsymbol{g}}$. We make these definitions explicit in applying them to groups generated by odd pure-cycle Nielsen classes, to which results of Liu-Osserman [**LOs06**] apply.

We prove the **MT** Main Conjecture in many of their cases. Comparing their result with [**Fr11**, Thms. A and B] indicates ingredients for a likely proof of the Main Conjecture in general for $r = 4$. §6.3.3 uses the Spin-lift invariant of (2.14) for a conjectured umbrella to the combined Liu-Osserman and Fried results on connected pure-cycle Nielsen classes.

REMARK 2.21. The production of full (not abelianized) **MT**s, with all components over $\mathbb{Q}$, has only come about so far through Prop. 2.4 using g-p' cusps (see (2.7a)). There are many examples of abelianized **MT**s (like $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ of § E.3) with braid orbits having no g-p' cusps. We don't, however, know if their components are uniformly defined over a number field.

2.4. The Spin lift invariant. I'll denote the universal central extension of A_n by Spin_n. It happens that $\ker(\mathrm{Spin}_n \to A_n)$ is $\mathbb{Z}/2$ ($n \ge 4$). To present it, embed A_n in the determinant 1 elements $\mathrm{SO}_n(\mathbb{R})$ of the orthogonal group. The fundamental group of $\mathrm{SO}_n(\mathbb{R})$ ($n \ge 4$) is $\mathbb{Z}/2$, so $\mathrm{SO}_n(\mathbb{R})$ has a 2-sheeted cover, $\mathrm{Spin}_n(\mathbb{R})$. Then, Spin_n is the pullback of A_n in $\mathrm{Spin}_n(\mathbb{R})$. It arises in practice often. For example, $A_5 = \mathrm{PSL}_2(\mathbb{Z}/5)$ — 2×2 matrices of determinant 1, mod $\pm I_2$ — and Spin_5 is just $\mathrm{SL}_2(\mathbb{Z}/5)$.

2.4.1. *Universal central extensions.* Examples of Schur multipliers in this paper occur when G is a subgroup of A_n (often it will be A_n). Then, consider the pullback $\hat{G}$ to Spin_n. (It depends on the embedding in A_n, but that will be clear from the context.) If $\hat{G} \to G$ is a nonsplit extension, then $\ker(\hat{G} \to G)$ is a $\mathbb{Z}/2$ quotient of the Schur multiplier of G.

In this situation, assume $\mathbf{C}$ consisting of odd ($2'$) conjugacy classes in G. Then, we attach to $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ a *lift invariant*, $s_{\hat{G}/G}(\boldsymbol{g}) \mod 2$, defined from the following. Take the unique $2'$ lift $\hat{g}_i \in \hat{G}$ lying over g_i, $i = 1, \dots, r$. Then,

$$(-1)^{s_{\hat{G}/G}(\boldsymbol{g})} \overset{\text{def}}{=} \hat{g}_1, \dots, \hat{g}_r \in \{\pm 1\}. \tag{2.14}$$

These definitions easily generalize to arbitrary groups and arbitrary primes. This paper concentrates, in our special case, on the meaning of the lifting invariant and our ability (often) to compute it explicitly thanks to [**Ser90**] and [**Fr11**].

REMARK 2.22 (Well-definedness of the lift invariant). Though the covers in an absolute Nielsen class (§2.1.5) such as $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$ are not Galois, the lifting invariant still makes sense. The lift invariant is trivial if and only if there is an unramified cover of the Galois closure so that the total cover down to $\mathbb{P}^1_z$ has automorphism group Spin_n. The Galois closure cover is only canonical up to an inner isomorphism of the Galois group with G. So, that requires knowing the lifting invariant doesn't depend on changing that isomorphism. That is because any inner isomorphism of G lifts to a canonical inner isomorphism of $\hat{G} \to G$.

2.4.2. *Using the lift invariant.* It makes sense to replace $\mathrm{Ni}(G, \mathbf{C})$ under the hypotheses of §2.4 with $\mathrm{Ni}(\hat{G}, \mathbf{C})$: replace G by its nonsplit degree two to extension. This gives a natural one-one (often not onto) map $\mathrm{Ni}(\hat{G}, \mathbf{C}) \to \mathrm{Ni}(G, \mathbf{C})$. The lifting invariant is a braid invariant. Conway-Fried-Parker-Völklein (C-F-P-V, §E.2) here says that if each class in $\mathbf{C}$ appears "suitably often," then there are *exactly* two braid orbits on $\mathrm{Ni}(G, \mathbf{C})$. The values of the lift invariant separate them.

We often use Prop. 2.23 ([**Ser90**] or [**Fr11**, Cor. 2.3]). For odd order $g \in A_n$, let $w(g)$ count length l disjoint cycles in g with $(l-1)/2 \equiv 1$ or $2 \mod 4$.

PROPOSITION 2.23 (Invariance). *Let $n \geq 3$. If $\varphi : X \to \mathbb{P}^1$ is in the Nielsen class* $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$*, then* $\deg(\varphi) = n$*, X has genus 0, and* $s(\varphi) = n-1 \mod 2$.

Generally, for any genus 0 Nielsen class of odd order elements, and representing $\boldsymbol{g} = (g_1, \dots, g_r)$, $s(\boldsymbol{g})$ is constant, equal to $sum_{i=1}^r w(g_i)$ mod 2.

Examples 2.24 and 2.25 show the significance of the phrase "suitably often." In Ex. 2.24 the covers in the Nielsen class have genus 0. Prop. 2.23 shows they only achieve one value of the lift invariant. In fact, there is just one braid orbit [**LOs06**]. In Ex. 2.25 the covering group is $G_1(A_4)$, the 1st 2-Frattini extension of A_4. It has a Schur multiplier of order 4, and correspondingly, there are four braid orbits, each corresponding to a different value of the lift invariant.

2.4.3. *Pure cycles and the invariance Corollary.* Our chief source of examples is from *pure-cycle* Nielsen classes: $\mathbf{C}$ consists of pure-cycles (§1.3.2).

Let $d_1, \dots, d_r$ be the disjoint cycle lengths. We often use $d_1 \cdots d_r$, often with exponents to indicate repetitions. **R-H**: A cover in this Nielsen class has genus

(2.15) $\mathbf{g} = \mathbf{g}_{d_1 \cdots d_r} \stackrel{\text{def}}{=} \frac{\sum_{i=1}^r d_i - 1}{2} - (n-1)$, a non-negative integer.

Suppose $G \leq S_n$ and $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, a pure cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. Denote the image of $\mathbf{C}$ in S_n by $\mathbf{C}^{S_n} \stackrel{\text{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$. Choose branch points, and classical generators (§1.3), so $\varphi : X \to \mathbb{P}^1_z$ corresponds to $\boldsymbol{g}$ in this Nielsen class. Here is a case of computing the lift invariant that shows what we mean by "explicit."

EXAMPLE 2.24 (Genus 0 pure-cycles). When the Nielsen class is odd pure-cycle, and the genus is 0, the lift invariant is $\sum_{i=1}^r \frac{d_i^2 - 1}{8} \mod 2$. Example: $r = 3$, $n \equiv 1 \mod 4$, $d_1 = d_2 = \frac{n+1}{2}$ and $d_3 = n$. Then, $G = \langle g_1, g_2 \rangle = A_n$, and

$$(2.16) \qquad s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g}) = \frac{n^2 - 1}{8} \mod 2 = \begin{cases} 0 & \text{if } n \equiv 1 \mod 8 \\ 1 & \text{if } n \equiv 5 \mod 8 \end{cases}.$$

Example 2.25 reappears in §E.3. Details for it are in [**BF02**, Ex. 9.2]. It illustrates many of this section's points about braid orbits. It also gives expectations beyond the case $\mathbf{g}_{d_1 \cdots d_r} = 0$ in (2.15). As in Def. 2.1.2, $G_1(A_4)$ denotes (with $p = 2$ implicit) the universal exponent 2-Frattini cover of A_4.

EXAMPLE 2.25. Take $G = G_1(A_4)$, a centerless extension of A_4 with kernel of order 2^5. Its Schur multiplier (for $p = 2$) is $(\mathbb{Z}/2)^2$. Then, there are exactly six braid orbits on $\mathrm{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$. As with $n \equiv 1 \mod 8$ in Lem. 4.4, two of those are orbits of H-M reps. Each, however, of the other four orbits have a non-trivial lift invariant associated to at least one $Z/2$ quotient of the Schur multiplier.

2.4.4. *Inductive criterion for existence of a non-empty* **MT**. We can check that a braid orbit at level k has above it (a nonempty) braid orbit at level $k+1$. §1.3.1 says this is equivalent to extending a given $M_{\boldsymbol{g}} \to G_k$ to $M_{\boldsymbol{g}} \to G_{k+1}$.

In the abelianized case, the inductive procedure simplifies to just one test to see if $M_{\boldsymbol{g}} \to G$ extends to $M_{\boldsymbol{g}} \to G_{k,\mathrm{ab}}$ for all k. The first statement is from [**BF02**, Prop. 3.21]. The last two are from [**Fr06a**, Cor. 4.19] (using results of [**Wei05**]).

Recall (§2.1.2) the representation cover $R^*_{G,p} \to G$: Its kernel – a finite abelian group – is the maximal p quotient of the Schur multiplier of G. So $\ker(R^*_{G,p} \to G)$ has a finite exponent p^{u_0}. Drop p in the notation above, as in $R^*_{G,p} \mapsto R^*_G$.

We extract two groups from this for Prop. 2.26. Denote the smallest quotient of $R^*_{G_k}$ whose map to G_k has exponent p by R_k. Then, denote the $\ker(R^*_G \to G)$ subgroup generated by all elements of exponent larger than p^t $(t \leq u_0)$ by U_t.

PROPOSITION 2.26. *If G has p' center (as in (2.12)), then so does G_k, $k \geq 1$.*
A braid orbit $O \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$ is in the image of $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{in}}$ if and only if O is in the image of $\mathrm{Ni}(R_k, \mathbf{C})$.

Also, the braid orbit $O \subset \mathrm{Ni}(G, \mathbf{C})$ is in the image of $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})$ if and only if it is in the image of $\mathrm{Ni}(R^*_G/U_k, \mathbf{C})$. If $k \geq u_0$, the conclusion holds exactly when O is in the image of $\mathrm{Ni}(R^*_G, \mathbf{C})$.

The following example works because we know precisely the Schur multiplier of alternating groups. The idea works for other groups, though there is as yet no Invariance Cor. 2.23 to calculate the lift invariant for other simple groups.

EXAMPLE 2.27. Suppose $G = A_n$, $n \geq 4$, with $p = 2$ and $\mathbf{C}$ has only classes of odd order elements. Then, for $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, there is an extension of $\psi_{\bar{\boldsymbol{g}}} : M_{\boldsymbol{g}} \to G$ to ${}_p\tilde{G}_{\mathrm{ab}}$ if and only if $s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g})$ is trivial. If $p \neq 2$ (where $\mathbf{C}$ are p') then there is always an extension to ${}_p\tilde{G}_{\mathrm{ab}}$.

2.5. Our choice of MTs. Our examples show how connectedness helps us compute when **MT**s have p-cusps. Prop. 2.13 shows we have only to compute that the genus of a **MT** level exceeds 0, and if it is 1 then it has at least 1 p-cusp. In our examples we will precisely exceed these modest goals.

2.5.1. *Pure-cycle Nielsen classes.* Return to pure-cycle Nielsen classes (all conjugacy classes have one disjoint cycle; §2.4.2). We will do the case $p = 2$.

All the Liu-Osserman examples stand out because there are no 2-cusps at level 0, but they do appear at level 1. Reminder: When $r = 4$, then there are two places were genus 0 may come up: the inner reduced Hurwitz spaces may have genus 0; and for a given space, its points may represent $\mathbb{P}^1_z$ covers of genus 0.

Recall: As S_n is not 2-perfect, it doesn't enter into the Main Conjecture for $p = 2$. Here are our general assumptions (see §2.2.2 on the slight ambiguity in conjugacy classes when all pure-cycle cusps are odd, because $G = A_n$, not S_n).

(2.17a) We have a pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})$, $r \geq 3$, of odd order elements, $G \leq A_n$ a transitive subgroup and $p = 2$; and

(2.17b) covers in the absolute Nielsen class all have genus 0: $\sum_{i=1}^{r} d_i - 1 = n - 1$.

Since Nielsen classes can, significantly (as in the Prop. 3.11 criterion), be empty, for convenience we state separately this is not the case here.

LEMMA 2.28. *When (2.17) holds, then $G = A_n$ and $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})$ is nonempty.*

PROOF. For why $G = A_n$ see Prop. 3.11. For nonemptiness of $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})$, Princ. 3.1 gives them all for $r = 3$. Princ. 3.7 not only constructs them for $r = 4$, it notes their easy construction when the d_i s are equal in pairs through H-M reps., and outside that case it constructs split-cycle cusps. Inductive arguments are instructive in constructing special cusp types for higher values of r. □

We can often conclude genus growth of the reduced Hurwitz spaces in a **MT** (and the Main Conjecture) for other primes when (2.17) holds, not just $p = 2$.

2.5.2. *Cases to compare with Liu-Osserman.* A few genus 0 Nielsen class collections have long been known to give connected spaces. The first such is the space of simple-branched (2-cycle) covers. the argument for that is so simple, that Clebsch used it 140 years ago, for all genuses , to show the connectedness of the genus $\mathbf{g}$ moduli of curves. Less obvious, yet more relevant for today, are modular curves.

We assume p is odd. Recall: The dihedral group is $D_{p^{k+1}} = \mathbb{Z}/p^{k+1} \times^s (\mathbb{Z}/p^{k+1})^*$ and $\mathbf{C}_{2^4}$ is four repetitions of the involution conjugacy, represented by multiplication by -1 on $\mathbb{Z}/p$. We already have a **MT** with its kth level the Nielsen class $\text{Ni}(G_k = D_{p^{k+1}}, \mathbf{C}_{2^4}, p)$ with $G_k = G_k(D_p)$ in the notation of §2.1.2. We use the absolute Nielsen class: The representation of G_k has degree p^{k+1}.

Note: $(p^{k+1}-1)/2$ is the number of length two orbits from multiplying by -1 on $\mathbb{Z}/p^{k+1}$. From this, (2.17b) holds: **R-H** gives the genus of covers representing the absolute Nielsen classes as $\mathbf{g} = 0$ satisfying $2(p^{k+1}+\mathbf{g}-1) = 4(p^{k+1}-1)/2$. For $p \geq 5$, these conjugacy classes aren't pure-cycle. This is the elementary modular curve case identifying $\bar{\mathcal{H}}_0^{\text{abs,rd}}$ with $X_0(p^{k+1})$. Also, $\bar{\mathcal{H}}_0^{\text{in,rd}}$ identifies with $X_1(p^{k+1})$. Though [**Fr78**, §2] is now old, this is not the traditional look of these spaces.

Now consider cases that are like the Clebsch case. In Ex. 2.29 you fix a group G and one conjugacy class C within it. Then, you vary the multiplicity of that conjugacy class to consider different Nielsen classes. In both cases denote the collection of absolutely irreducible components by I, and consider the natural map from $i \in I$ to the conjugacy class collection for that component. The listing of components for absolute classes and inner classes is the same in these cases.

Example 2.29 (Dihedral and Alternating cases). If $G_k = D_{p^{k+1}}$ with p odd, and $\mathbf{C}^* = \{\text{C}_2\}$ (conjugacy class of an involution), then $i \mapsto \mathbf{C}_{2^{r_i}}$ is one-one and onto, with the r_i s running over all even integers ≥ 4. Also, H_i^{rd} identifies with the space of cyclic p^{k+1} covers of hyperelliptic jacobians of genus $\frac{r_i-2}{2}$ [**DFr94**, §5].

If $G = A_n$ with $\mathbf{C}^* = \{\text{C}_3\}$, class of a 3-cycle, then $i \mapsto \mathbf{C}_{3^{r_i}}$ with $r_i \geq n$ is two-one [**Fr11**, Main Result]. Denote indices mapping to r by $i_r^{\pm}$. Covers in $\mathcal{H}_{i_r^{\pm}}$ are Galois closures of degree n covers $\varphi : X \to \mathbb{P}_z^1$ with 3-cycles for local monodromy. Write the divisor $(d\varphi)$ of the differential of φ as $2D_\varphi$. Then, $\varphi \in \mathcal{H}_{i_r^+}$ (resp. $\mathcal{H}_{i_r^-}$) if the linear system of D_φ has even (resp. odd) dimension; it is an even (resp. odd) θ characteristic. For $r_i = n-1$ the map $i \mapsto \mathbf{C}_{3^{r_i}}$ is one-one.

3. Cusp Principles

We assume the Liu-Osserman conditions (2.17) hold from this point through §5. §3.1 nails the description of pure-cycle Nielsen classes when $r = 3$.

For $r = 4$, §3.2 is a basic tool kit for this the contribution of the cusps to the genus of the reduced spaces. Its three principles detect when we have p-cusps.

Princ. 3.7 says we get *only* pure-cycle cusps — $(\boldsymbol{g})$**mp** is pure-cycle (or trivial) for all $\boldsymbol{g}$ in the Nielsen class — precisely when the Nielsen class is $\text{Ni}_{(\frac{n+1}{2})^4}$. That implies there are no 2-cusps at level 0. Yet, applying [**Fr06a**, Fratt. Princ. 3], §3.3 gives some 2-cusps at level 1, proving the Main Conjecture.

3.1. Detecting p-cusps in the cusp tree.

For $r \geq 4$, all reduced spaces have well-defined cusps. Their combinatorial definition (§2.2.1) applied to an element $\boldsymbol{g}$ in the Nielsen class starts by imposing a grouping on the entries of $\boldsymbol{g}$. When $r = 4$, we inspect the ordered pairs (g_2, g_3) and (g_4, g_1) both for their products and the groups they generate. This coalescing definition forces the case $r = 3$ on us.

3.1.1. *The case* $r = 3$. Assume $r = 3$ and the genus of the cover genus in the Nielsen class is 0. By applying a braid from H_3, as previously (§1.2.1) assume $d_1 \leq d_2 \leq d_3$. Write g_1 as in (3.1) for some integer $1 \leq u \leq d_1-1$. With no loss, assume the segment x_{d_1-u+1,d_1} (§1.3.2) disappears in $g_1 g_2$. Use that all integers appear in the g_1 and g_2, and their product is a pure-cycle. Form:

$$\begin{aligned} g_1 = &\ (1\ \dots\ d_1{-}u\ d_1{-}u{+}1\ \dots\ d_1) \\ g_2 = &\ (d_1\ d_1{-}1\ \dots\ d_1{-}u{+}1\ n\ \dots\ d_1{+}1),\ \text{and} \\ g_3 = &\ (1\ \dots\ d_1{-}u{+}1\ n\ \dots\ d_1{+}1)^{-1}. \end{aligned} \tag{3.1}$$

Easily check: (g_1, g_2, g_3) has product-one and the genus, $\mathbf{g}_{\boldsymbol{g}}$ is 0. Up to conjugation by S_n and reordering, we have the unique element in the Nielsen class.

PRINCIPLE 3.1. *For $r = 3$ and $\mathbf{g}_{d_1\cdot d_2\cdot d_3} = 0$, there is a unique*

$$\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}_{d_1\cdot d_2\cdot d_3})^{\mathrm{abs}}\ \textit{with}\ \mathrm{ord}(g_i) = d_i, i = 1, 2, 3.$$

3.1.2. *p-cusps and Main* **MT** *Conjecture.* The projective system of cusp orbits forms a directed tree. A *cusp branch* is the Cu_4 orbit of a projective system of representatives $\tilde{\boldsymbol{g}} = \{{}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$. If all are H-M reps., call it an *H-M branch*. Its system of braid orbits defines an H-M **MT**, or an H-M *component branch*. Assume, with no loss (start of [**Fr06a**, §5]), that the p-part of the center of all the G_k s is trivial. So, the G_k centers identify as the same p' group.

It will simplify some expressions to use a short-hand, $[g_1, g_2]$, for the H-M rep. $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$.

PRINCIPLE 3.2. *We have the following formulas:*

$$([g_1, g_2])q_1 = [g_1^{-1}, g_2], ([g_1, g_2])q_3 = [g_1, g_2^{-1}]\ \textit{and}\ ([g_1, g_2])q_1q_2 = [g_1^{-1}, g_2^{-1}].$$

Among them is a p-cusp if and only if one of $p|\mathrm{ord}(g_1^{\pm 1}g_2)$.

PROOF. The display repeats the definition of q_1 and q_2 action. By definition, these give p-cusps if and only if p divides the middle product order of one of $g_1^{\pm 1}g_2^{\pm 1}$. Notice, the inner equivalent H-M rep. $g_1[g_1, g_2]g_1^{-1}$ has middle product

$$g_1^{-1}g_1g_2g_1^{-1} = g_2g_1^{-1} = (g_1g_2^{-1})^{-1}.$$

So, the middle product order of $g_1g_2^{-1}$ is among $\mathrm{ord}(g_1^{\pm 1}g_2)$, etc. □

Assume $\{{}_k\boldsymbol{g}\}_{k=0}^{\infty}$ defines an H-M cusp branch.

(3.2) Non-Weigel cusp branch: Suppose for $k >> 0$, $\{{}_k\boldsymbol{g}\}_{k=0}^{\infty}$ does not define an o-p' cusp branch (as in (2.7b)).

It is significant to figure out when condition (3.2) is automatic.

Assuming (3.2), I now show we can spin off low ramification p-cusps for general p. For notational simplicity (adjustments are easy), assume the cusp branch starts with an H-M rep. $(g_1, g_1^{-1}, g_2, g_2^{-1}) = [g_1, g_2] = \boldsymbol{g}_0$ with middle product $d\cdot p^u$, $u \geq 1$. Further simplify by taking $u = 1$. Now let ${}_{k+1}\boldsymbol{g}$ be the level $k+1$ representative in our projective sequence. With o_k the middle product order of ${}_k\boldsymbol{g}$,

$$\begin{aligned} \bar{c}_k = &\ ({}_{k+1}g_1^{-1}\,{}_{k+1}g_2)^{o_k} \in \ker(G_{k+1,\mathrm{ab}} \to G_{k,\mathrm{ab}}),\ \text{and, so} \\ {}_{k+1}\boldsymbol{g}' = &\ ({}_{k+1}\boldsymbol{g})q_2^{2o_k} = ({}_{k+1}g_1, \bar{c}_k({}_{k+1}g_1^{-1})\bar{c}_k^{-1}, \bar{c}_k({}_{k+1}g_2)\bar{c}_k^{-1}, {}_{k+1}g_2^{-1}). \end{aligned}$$

Further, for product-one to hold, $\bar{c}_k$ must centralize ${}_{k+1}g_1^{-1}\,{}_{k+1}g_2$. Now use that $\langle {}_{k+1}g_1, {}_{k+1}g_1\,{}_{k+1}g_2\rangle = G_{k+1,\mathrm{ab}}$ and this group's center has no p-part. As $\bar{c}_k$ commutes with the second generator, it can't commute with the first. Conclude:

$$ {}_{k+1}\boldsymbol{g}' \neq {}_{k+1}\boldsymbol{g}. \tag{3.3}$$

Now form $({}_{k+1}\boldsymbol{g}')\mathbf{sh}$, whose 2nd and 3rd entries are $(\bar{c}_k({}_{k+1}g_2)\bar{c}_k^{-1}, {}_{k+1}g_2^{-1})$. Exactly one power of p divides their product.

Principle 3.3. *As above,* $({}_{k+1}\boldsymbol{g}')\mathbf{sh}$ *is a new p-cusp. So, Princ. 3.2 combined with* (3.2) *for the* H-M *cusp branch of* $\tilde{\boldsymbol{g}}$ *implies the main conjecture for its* H-M *component branch.*

Proof. Prin. 2.14 gives k_0, so that for $k \geq k_0$, $p^{k-k_0+1}||({}_k\boldsymbol{g})\mathbf{mp}$. The first paragraph [**Fr06a**, Prop. 5.5] proof shows how spinning new p-cusps grows the number of p-cusps with k: p cusps at level k so produced have above them only cusps with middle product divisible by one more power of p. So, this new cusp cannot equal any p cusps from the induction production of previous cusps.

The Main Conjecture counterexample towers have at most two p-cusps at each level [**Fr06a**, Thm. 5.1] (or Prop. 2.13). This concludes the proposition. □

Example 3.4. In A_5, consider $g_1 = (1\,2\,3\,4\,5)$ and $g_2 = (1\,2\,3)$. They generate A_5, and $([g_1, g_2])\mathbf{mp}$ is $(5\,4\,3)$ while $([g_1, g_2^{-1}])\mathbf{mp}$ is $(3\,1\,2\,5\,4)$. This shows middle products of H-M reps. are not a braid orbit invariant.

3.2. Two cusp Principles. Princ. 3.5, a version of [**BF02**, Prop. 2.17], makes transparent the width of most cusps. Princ. 3.7 smooths the way between $r = 3$ and $r = 4$ for pure-cycle Nielsen classes. It is a version of [**LOs06**, §4], the hardest combinatorial part of their paper, where $r = 4$. Our simplification results from using cusps to improve the efficiency in computing braid orbits.

3.2.1. *The Twisting Principal.* For $(g, g') \in G \times G$, denote $(g, g') \mapsto (gg'g^{-1}, g)$ by tw. It is just the q_2 operator restricted to 2-tuples, instead of 4-tuples. As such the iterated action of tw starting from (g, g') has an orbit in $G \times G$.

Principle 3.5. *Given* (g, g')*, denote* gg' *by* g''*. Assume* $g^{-1} \neq g'$ *and for simplicity that* $Z(\langle g, g'\rangle) \cap \langle g''\rangle$ *is trivial (see Rem. 3.9). Then, the* tw *orbit length is* $2 \cdot \mathrm{ord}(g'')$ *unless*

(3.4) $\mathrm{ord}(g'') = o$ *is odd, and* $\mathrm{ord}((g'')^{\frac{o-1}{2}} g) = 2$.

In turn, (3.4) *is equivalent to*

$$(g, g')\mathrm{tw} = (g'')^{\frac{o+1}{2}} (g, g')(g'')^{-\frac{o+1}{2}}. \tag{3.5}$$

More generally, suppose $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ *and* $o_{\boldsymbol{g}}$ *is the order of* $(\boldsymbol{g})\mathbf{mp}$*. Then, the orbit of* q_2 *on* $\boldsymbol{g}^{\mathrm{in}}$ *has length* $2 \cdot o_{\boldsymbol{g}}$ *unless* $o_{\boldsymbol{g}}$ *is odd, and* $(g_2g_3)^{\frac{o_{\boldsymbol{g}}-1}{2}} g_2$ *has order 2 mod* $\mathrm{Cen}_G(\langle g_1, g_4\rangle)$*, in which case the orbit length is* $o_{\boldsymbol{g}}$*.*

Proof. The first paragraph is [**BF02**, Prop. 2.17]. Now assume o is odd and let $j = \frac{o+1}{2}$. If (3.5) holds, then $gg'g^{-1} = (g'')^j g (g'')^{-j}$, or $g' = (g'')^{\frac{o-1}{2}} g (g'')^{-\frac{o-1}{2}}$

$$\implies g'(g'')^{\frac{o-1}{2}} = (g'')^{\frac{o-1}{2}} g.$$

This is reversible; the last restates (3.4) that $(g'')^{\frac{o-1}{2}} g$ has order 2.

For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$, assume $\alpha \in \mathrm{Cen}_G(\langle g_1, g_4\rangle)$. So α commutes with $g'' = g_2g_3$ and satisfies

$$\alpha(g'')^{\frac{o-1}{2}} g (g'')^{-\frac{o-1}{2}} \alpha^{-1} = g'.$$

Then, the inner class of $(\boldsymbol{g})q_2$ contains conjugation of the middle pair by $(g'')^{\frac{o+1}{2}}$, and the argument above works again. □

3.2.2. *The Pure-Cycle Cusp Principle.* Princ. 3.7 shows exactly why, among the Liu-Osserman cases, the Nielsen classes $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ (§4.1) stand out. It is these for which all cusps are pure-cycle (or have trivial middle product; Def. 3.8). Ex. 3.10 does a split-cycle case in detail to assure the reader of the notation.

As previously consider an element $\boldsymbol{g} = (g_1, \dots, g_r)$ in a Liu-Osserman genus 0 pure cycle (genus 0) Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{d_1\cdots d_r}) \stackrel{\text{def}}{=} \mathrm{Ni}_{\boldsymbol{d}}$.

LEMMA 3.6. *Product one and transitivity (§1.3.1) imply each $j \in \{1, \dots, n\}$ is in the support of at least 2 of the g_i s. Let k_j be the number of g_i s containing j in their support, and let $k_i - 2 = k_i' \ge 0$. So, $\sum k_j' = r - 2$. If $r = 4$, conclude, each integer is in the support of exactly two of the g_i s modulo one of two possibilities:*

(3.6a) *Either there is an i_0 in the support of all four g_i s; or*

(3.6b) *There are two integers i_0 and k_0 in the support of exactly three g_i s.*

PROOF. Each support appearance of j adds 1 to $\sum_{i=1}^r d_i$ in **R-H** as in (2.15). From $\mathbf{g}_{\boldsymbol{g}} = 0$, there are only $n+r-2$ total appearances. For $r = 4$, there can be only two total, beyond 2, appearances of integers; (3.6) splits that into two cases. □

PRINCIPLE 3.7. *Consider the common support of (g_2, g_3). With no loss, unless it is empty, take it to be $\{1, \dots, k\}$. Then, consider the overlap, $U(\boldsymbol{g})$, of that with $(\boldsymbol{g})\mathbf{mp}$. This consists of at most two integers.*

If $|U(\boldsymbol{g})| = 1$ (with no loss take it to be k) then (g_2, g_3) has the form

$$((k \dots 1\,\boldsymbol{v}), (1 \dots k\,\boldsymbol{w})) \text{ with } \boldsymbol{v},\ \boldsymbol{w} \text{ and } \{1, \dots, k\} \text{ mutually disjoint.}$$

Here, $(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}\,\boldsymbol{v})$, is an odd pure-cycle.

If $|U(\boldsymbol{g})| = 2$, then there is no common support in the 3-tuple (g_4, g_1, g_4g_1). Further, (g_2, g_3) has the form

$$((k \dots i_0{+}1\,\boldsymbol{v}_1\,i_0 \dots 1\,\boldsymbol{v}_2), (1 \dots i_0\,\boldsymbol{w}_1\,i_0{+}1 \dots k\,\boldsymbol{w}_2)), \tag{3.7}$$

with the sets $\boldsymbol{v}_1$,$\boldsymbol{v}_2$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\{1, \dots, k\}$ pairwise disjoint and

$$(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}_2\,\boldsymbol{v}_2)(i_0\,\boldsymbol{w}_1\,\boldsymbol{v}_1), \text{ a split-cycle.} \tag{3.8}$$

The two disjoint cycles are the inverses of g_1 and g_4, giving conditions (see (3.9)) on the lengths (orders) of $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$, $i = 1, 2$, so the $\boldsymbol{g}$ entries have the right orders.

The condition $|U(\boldsymbol{g})| = 2$ happens for some rep. in each allowed Nielsen class if and only if it is not $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ for some $n \ge 4$.

PROOF. We characterize each case of (3.6). For a segment labeled $\boldsymbol{v}$ (or $\boldsymbol{w}$) in the calculations, compatible with previous notation, denote its length $o(\boldsymbol{v})$.

If $U(\boldsymbol{g})$ is empty, then g_2 and g_3 are disjoint. Otherwise, assume $k \in U(\boldsymbol{g})$. If no other letter is in $U(\boldsymbol{g})$, then consider the effect of g_2g_3 to see that by reordering $1, \dots, k$, we may assume g_3 maps $i \mapsto i{+}1$, and g_2 reverses this, for $i = 1, \dots, k{-}1$. So, these integers disappear in the support of the product, and (g_2, g_3) has the shape given in the proposition statement. The length of $(\boldsymbol{g})\mathbf{mp}$ is $1 + o(\boldsymbol{v}) + o(\boldsymbol{w})$, and $2k + o(\boldsymbol{v}) + o(\boldsymbol{w}) = d_2 + d_3$. Since d_2 and d_3 are both odd, conclude $o(\boldsymbol{v}) + o(\boldsymbol{w})$ is even, and the length of $(\boldsymbol{g})\mathbf{mp}$ is odd.

It is similar for $|U(\boldsymbol{g})| = 2$. Now consider, by cases, what happens with the complementary pair (g_4, g_1).

Suppose $|U(\boldsymbol{g})| = 2$. Then, two integers having three supports among the entries of $\boldsymbol{g}$ appear in $(\boldsymbol{g})\mathbf{mp}$. Apply the argument to $(g_4, g_1, g_4g_1 = (g_2g_3)^{-1})$ that we used on (g_2, g_3, g_2g_3). If there were further integers in the common support of (g_4, g_1, g_4g_1) and (g_4, g_1), that would give at least three integers appearing in the common support of three entries of $\boldsymbol{g}$. So, that can't happen. Similarly, if $|U(\boldsymbol{g})| = 1$, then the common support for $(g_4, g_1, (g_4g_1)^{-1})$ has also cardinality 1, different from the integer in $U(\boldsymbol{g})$.

For $\boldsymbol{g} \in \mathrm{Ni}_{(\frac{n+1}{2})^4}$, all pairs of $\boldsymbol{g}$ entries have overlapping support. So there can be no split-cycle cusps. Given $\boldsymbol{d} \neq (\frac{n+1}{2})^4$, we now produce split-cycle cusps.

With $d_1 \le d_2 \le d_3 \le d_4$, apply a braid to produce $\boldsymbol{g}'$ with $o(g'_i) = d_{i+1}$, $i = 1, \dots, 4 \mod 4$. This assures the two smallest lengths are at positions 1 and 4. From genus 0, $\sum_{i=1}^4 d_i = n{+}2$ implies $d_1 + d_2 \le n$. Here are equations expressing the respective segment lengths of g'_4, g'_1, g'_2, g'_3 using $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\begin{aligned} 1 + o(\boldsymbol{v}_1) + o(\boldsymbol{w}_1) &= d_1, \quad 1 + o(\boldsymbol{v}_2) + o(\boldsymbol{w}_2) = d_2, \\ k + o(\boldsymbol{v}_1) + o(\boldsymbol{v}_2) &= d_3, \quad k + o(\boldsymbol{w}_1) + o(\boldsymbol{w}_2) = d_4. \end{aligned} \tag{3.9}$$

Solve the equations, as in Ex. 3.10, to canonically, up to absolute equivalence, produce a split-cycle cusp. For example, $d_1 - 1 + d_2 - 1 = d_3 - k + d_4 - k$, determines k. This concludes the proposition. □

Suppose $\boldsymbol{g} \in \mathrm{Ni}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4}$. Then, consider the cusp $\mathrm{Cu}_4(\boldsymbol{g})$ it generates.

Definition 3.8. Assume $\boldsymbol{g}$ is not the shift of an H-M rep. (so $(\boldsymbol{g})\mathbf{mp}$ is not trivial). Corresponding to the cases in Princ. 3.7, $\mathrm{Cu}_4(\boldsymbol{g})$ is a *pure-cycle* (resp. *split-cycle*) cusp if $|U(\boldsymbol{g})| = 1$ (resp. $|U(\boldsymbol{g})| = 2$ or 0).

Remark 3.9. In our applications here the triviality of $H = Z(\langle g, g' \rangle) \cap \langle g'' \rangle$ assumption in Princ. 3.5 holds. As in [**BF02**, Prop. 2.17], modding out by H gives a completely general result.

Example 3.10 (Split-cycle cusp). Let $n = 9$ and $(d_1, d_2, d_3, d_4) = (3, 5, 5, 7)$, so $\mathrm{Ni}(A_9, \mathbf{C}_{3 \cdot 5^2 \cdot 7})$ satisfies the genus 0 assumption. Make a split-cycle cusp $\mathrm{Cu}_4(\boldsymbol{g})$ where $(o_1, o_2, o_3, o_4) = (3, 5, 7, 5)$ by assigning values to $i_0, k, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1 \boldsymbol{w}_2$ in the formula in Princ. 3.7. As in (3.9), $1 + o(\boldsymbol{v}_1) + o(\boldsymbol{w}_1) = 3$, $1 + o(\boldsymbol{v}_2) + o(\boldsymbol{w}_2) = 5$, $k + o(\boldsymbol{v}_1) + o(\boldsymbol{v}_2) = 5$ and $k + o(\boldsymbol{w}_1) + o(\boldsymbol{w}_2) = 7$.

So, $2 + 4 = 5 - k + 7 - k$, or $k = 3$ and $o(\boldsymbol{v}_1) = o(\boldsymbol{v}_2) = o(\boldsymbol{w}_1) = 1, o(\boldsymbol{w}_2) = 3$. With no loss:

$$\boldsymbol{v}_1 = |4|, \boldsymbol{v}_2 = |5|, \boldsymbol{w}_2 = |6\,7\,8|, \boldsymbol{w}_1 = |9|. \tag{3.10}$$

Fill in g'_2, g'_3 (resp. g'_4, q'_1) from (3.7) (resp. (3.8)). For $i_0 = 1$, $g'_2 = (3\,2\,4\,1\,5)$ and $g'_3 = (1\,9\,2\,3\,6\,7\,8)$ ($(\boldsymbol{g}')\mathbf{mp} = (3\,6\,7\,8\,5)(1\,9\,4)$);. For $i_0 = 2$ $g'_2 = (3\,4\,2\,1\,5)$ and $g'_3 = (1\,2\,9\,3\,6\,7\,8)$ (middle product $(3\,6\,7\,8\,5)(2\,9\,4)$).

3.3. 2-cusps and Liu-Osserman examples. Prop. 3.11 assumes the genus 0, pure-cycle hypotheses of (2.17). For $r \ge 3$ it characterizes when an abelianized $\mathbf{MT}$ lies over $\mathrm{Ni}(A_n, \mathbf{C}_{\boldsymbol{d}}) \stackrel{\text{def}}{=} \mathrm{Ni}_{\boldsymbol{d}}$. It says there are no 2-cusps at ($\mathbf{MT}$) level 0 for $r = 4$, yet shows how to find 2-cusps at level 1. Denote the order of $g \in G$ by $o(g)$. Cusp notation is from §2.2.2. Though Prop. 3.11 doesn't use it, [**LOs06**, Cor. 4.11] says there is only one braid orbit on $\mathrm{Ni}^{\mathrm{abs}}_{\boldsymbol{d}}$. Also, when there are two braid orbits on $\mathrm{Ni}^{\mathrm{in}}_{\boldsymbol{d}}$ (as in Prop. 4.1), any element of $S_n \setminus A_n$ conjugates between them.

Proposition 3.11. *Let $\boldsymbol{d} = d_1, \dots, d_r$, $r \ge 3$, with $\mathrm{Ni}^{\mathrm{abs}}_{\boldsymbol{d}}$ a Nielsen class of odd pure-cycle genus 0 covers. Then, $G = A_n$, $n \ge 4$. For $p = 2$ there is a (nonempty)* *abelianized* **MT** *above any component of* $\mathcal{H}(A_n, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{in}}$ *if and only if*

$$\sum_{i=1}^{r} \frac{o(g_i)^2 - 1}{8} \equiv 0 \mod 2. \tag{3.11}$$

For $p \neq 2$*, there is always an abelianized* **MT** *above any component of* $\mathcal{H}(A_n, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{in}}$*.*

If the d_i *s are equal in pairs, there is always (irrespective of* p*) a (full — not abelianized)* **MT** *over any component of* $\mathcal{H}(A_n, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{in}}$*.*

Proof. All appearances of alternating groups come directly from [**Wm73**], whose hypotheses (Rem. 2.5 and Rem. 3.14 add to [**LOs06**, Thm. 5.3]) imply a noncyclic, transitive subgroup G of A_n, generated by odd pure-cycles must be A_n, $n \geq 4$. If we can exclude that G is cyclic, then $G = A_n$, $n \geq 4$, in any such Nielsen class. If, however, $G = \langle h \rangle$, then transitivity implies h is an n-cycle. Apply the pure-cycle and genus 0 conditions. Conclude: all the g_i s are invertible powers of h. So, by **R-H**: $2(n-1) = r(n-1)$ and $r = 2$, contrary to hypothesis.

Lem. 2.28 notes the level 0 Nielsen classes are nonempty. Consider, then, (3.11). Inv. Prop. 2.23 says this is precise for when $\mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{\boldsymbol{d}})$ has a nonempty H_r orbit over any H_r orbit of $\mathrm{Ni}(A_n, \mathbf{C}_{\boldsymbol{d}})$. The representation cover of A_n is Spin_n. So, Prop. 2.26 says this is exactly when there is an abelianized **MT** for $p = 2$ over any braid orbit of $\mathrm{Ni}(A_n, \mathbf{C}_{\boldsymbol{d}})$. Also, this always holds when $p \neq 2$. □

From here on assume (3.11) holds (Ex. 3.13 has examples when it doesn't) so there is a **MT** over any braid orbit of $\mathrm{Ni}_{\boldsymbol{d}}$. §2.2.2 defines $U_{2,3}$ and $U_{1,4}$.

Proposition 3.12. *For* $r = 4$*, all cusps in* $\mathrm{Ni}_{\boldsymbol{d}}$ *are either g-*$2'$ *or o-*$2'$*. Also:*

(3.12a) *All g-2' cusps are shifts of* H-M *reps; and*
(3.12b) *all* H-M *reps are o-*$2'$ *cusps.*

Let $\mathrm{Cu}_4(\boldsymbol{g})$ *be an o-*$2'$ *pure-cycle cusp. Then,* $U_{2,3}(\boldsymbol{g}) = A_u$ *and* $U_{1,4}(\boldsymbol{g}) = A_v$ *for some* $u, v \geq 4$*. All level 1 cusps above it are 2-cusps if and only if*

$$\frac{o(g_2)^2 - 1}{8} + \frac{o(g_3)^2 - 1}{8} + \frac{o(g_2 g_3)^2 - 1}{8} \equiv 1 \mod 2. \tag{3.13}$$

Let $\mathrm{Cu}_4(\boldsymbol{g})$ *be a split-cycle (Def. 3.8) cusp with* $|U(\boldsymbol{g})| = 2$ *and* $\gcd(d_1, d_4) = d'$*. Then,* $U_{2,3}(\boldsymbol{g}) = A_n$ *and* $U_{1,4}(\boldsymbol{g}) = \mathbb{Z}/d_1 \times_{\mathbb{Z}/d'} \mathbb{Z}/d_4$*. There are both 2-cusps and o-*$2'$ *cusps at level 1 of any* **MT** *over* $\mathrm{Cu}_4(\boldsymbol{g})$*.*

Proof. Apply Princ. 3.7 to consider cases for $\mathrm{Cu}_4(\boldsymbol{g})$. First assume it is a g-$2'$ cusp: both $U_{2,3}(\boldsymbol{g})$ and $U_{1,4}(\boldsymbol{g})$, each generated by odd pure-cycles, have orders prime to 2. From Prop. 3.11 (see the 1st sentence of the proof), this implies they are both cyclic groups. Therefore, $(\boldsymbol{g})\mathbf{mp}$ generates a common normal subgroup of these two groups. This would be a normal subgroup of A_n, so is $n = 3$ or 4. In both cases, however, the entries of $\boldsymbol{g}$ would have to be 3-cycles. Apply **R-H** to see this gives $2(n-1) = 4 \cdot 2$, which works for neither $n = 3$ or 4. So, the middle product is trivial. Apply the shift to conclude $(\boldsymbol{g})\mathbf{sh}$ is an H-M rep.

Now suppose $\mathrm{Cu}_4(\boldsymbol{g})$ is a pure-cycle cusp. (This includes the H-M rep. case.). First exclude that $U_{2,3}(\boldsymbol{g})$ is cyclic. That would, however, imply that g_2 and g_3 commute, contrary to an explicit calculation from their shape given in Princ. 3.7. Thus, $U_{2,3}(\boldsymbol{g})$ and $U_{1,4}(\boldsymbol{g})$ are both alternating groups of degree exceeding 3.

Let G_1 be the 1st 2-Frattini extension of A_n (Def. 2.1.2). The exact condition that only 2-cusps in $\mathrm{Ni}(G_1, \mathbf{C}_{\boldsymbol{d}})$ lie over $\mathrm{Cu}_4(\boldsymbol{g})$ is that there are no o-$2'$ cusps over

$\mathrm{Cu}_4(\boldsymbol{g})$. First we show, if $\boldsymbol{g}' \in \mathrm{Ni}(G_1, \mathbf{C}_{\boldsymbol{d}})$ ($G_1 = G_1(A_n)$) lying over $\boldsymbol{g}$ gives an o-2′ cusp, then 3.13 does not hold.

Let $h = (\boldsymbol{g})\mathbf{mp}$, denoting its (2′) order by d. Then, the image $\boldsymbol{g}^* \in \mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{\boldsymbol{d}})$ is also o-2′. As $\boldsymbol{g}$ is o-2′, with $\langle g_2, g_3\rangle = U_{2,3}(\boldsymbol{g}) = A_u$, Invariance Prop. 2.23 applies to (g_2, g_3, h^{-1}), and the left side of (3.13) is $\equiv 0 \mod 2$.

Now consider the converse: Given that the left side of (3.13) is $\equiv 0 \mod 2$, we construct an o-2′ cusp $\boldsymbol{g}'$ over $\boldsymbol{g}$. Consider $\langle g_1, g_4\rangle = U_{1,4}(\boldsymbol{g}) = A_v$ and the analogous condition for (g_4, g_1, h) being lifted to an element of $\mathrm{Ni}(\mathrm{Spin}_v, \mathbf{C}_{d_1,d_4,d})$. Combine (3.11) and the negation of (3.13) to see this is now automatic. A special case of [**Fr06a**, Princ. 4.24] (F(rattini) Princ. 3), says these respective Nielsen class elements give an o-2′ cusp of $\mathrm{Ni}(G_1(A_n), \mathbf{C}_{\boldsymbol{d}})$ over $\mathrm{Cu}_4(\boldsymbol{g})$. This concludes the desired outcome of (3.13) not holding.

Now we establish the analog for a split-cycle cusp. Then, g_1 and g_4 have disjoint support and the analogous expression to the left side of (3.11) is just the left side of (3.13). Given that the former holds then, the latter reads as its negation. So, the previous argument —the lift invariant does not need pure-cycle elements to apply —gives that there are both 2-cusps and o-2′ cusps above $\boldsymbol{g}$. □

EXAMPLE 3.13 (Empty **MT**s over Liu-Osserman Nielsen classes). We check, for $r = 4$, how to get $\boldsymbol{d}$, satisfying $2(n-1) \equiv \sum_{i=1}^4 d_i - 1$ (genus zero), for which there is no abelianized **MT** over $\mathrm{Ni}_{\boldsymbol{d}}$. This is equivalent to the failure of (3.11) ⇔an odd number of d_i s are $\equiv \pm 3 \mod 8$.

If $G = A_n$, $n \equiv 1 \mod 4$, genus 0 gives $\sum_{i=1}^4 d_i - 1 \equiv 0 \mod 8$. This implies an even number of d_i s equal $+3$ or -3. So, (3.11) is automatic. For, however, $n \equiv 3 \mod 4$, (3.11) won't hold with these unordered mod 8 entries for $\boldsymbol{d}$:

$$1, 1, 1, -3; -1, -1, -1, 3; 1, -3, -3, -3.$$

For example, $\boldsymbol{d} = (5, 9, 9, 9)$ with $n = 15$. The case n is even can also happen.

REMARK 3.14. [Comments on [**Wm73**]] Excluding monodromy associated to dihedral, cyclic, alternating and symmetric groups, the remaining groups of degree n genus 0 covers form a finite set (the genus 0 program; see [**Fr05**, §7.2.3]). [**Wm73**] can neatly separate Nielsen classes $\mathrm{Ni}(G, \mathbf{C})$ where $G = A_n$ or $G = S_n$ —S_n/A_n-result —from others. Williamson's statement [**Wm73**]: The S_n/A_n-result holds if G is primitive, non-cyclic, and it contains a degree $d > 1$ pure-cycle with $d \le (n-d)!$. Note: $n = 5$ and $\mathbf{C}_{3^4}$ ($n = 5$ in §4.1) doesn't satisfy this. Yet, the S_n/A_n-result is well-known for G primitive having a 3-cycle. We easily detect conjugacy classes in A_n. So, one pure cycle class of length d may assure the S_n/A_n-conclusion. Example: $(d, n) = 1$ and $d > n/p$ for p the minimal prime dividing n.

4. The Liu-Osserman case $\mathrm{Ni}_{(\frac{n+1}{2})^4}$

This section shows the cusp structure of the reduced spaces in the subcase of Liu-Osserman from §2.5.1. We denote this case $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ with $G = A_n$: the conjugacy classes are four repetitions of $\frac{n+1}{2}$-cycles. The **sh**-incidence matrices memorably collects information coming from cusps. Most conjugations of pure-cycles by pure-cycles in this section fit the notation of *translations of a segment* from §1.3.2. Comparing Table 3 and Table 7 shows that, while the case $n = 5$ is overly simple, $n = 13$ captures almost all cusp phenomena of the general case.

4.1. $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ **cusps.** We list absolute and inner cusps using §1.3.2 notation. The proof of Prop. 4.1 takes up most of this section. Subsections do the respective cases of absolute and inner cusps.

4.1.1. *Fixing the 1st and 4th entries in pure-cycle reps.* Prop. 3.12 says all g-2′ cusps are **sh** applied to H-M cusps $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Also, all remaining cusps are pure-cycle. With $x_{i,j} = (i\, i{+}1 \cdots j)$, inner H-M class have one of two reps:

$$\text{H-M}_1 \stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$
$$\text{H-M}_2 = (\text{H-M}_1)q_1 \stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}})$$

PROPOSITION 4.1. *For* $n \equiv 5 \mod 8$, H-M_1 *and* H-M_2 *are not inner equivalent. So, there is one braid orbit on* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

For $n \equiv 1 \mod 8$, *if* $h \in S_n \setminus A_n$, *there is no braid between* $\boldsymbol{g}$ *and* $h\boldsymbol{g}h^{-1}$. *So, there are two braid orbits on* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

For $U \le \{1, \dots, n\}$, S_U is the set of permutations of U. We first show how **sh** applied to the cusp of H-M_1 gives representatives of all the absolute cusps. Table 1 consists of **sh** applied to elements of $\mathrm{Cu}_4(\text{H-M}_1)$, the cusp orbit of H-M_1:

$$\{\text{H-M}_{1,t} = (x_{\frac{n+1}{2},1}, x_{1+t,\frac{n+1}{2}+t}, x_{\frac{n+1}{2}+t,n+t}, x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}.$$

The expression $[k]_1$ or $[k]_2$ heading a Table 1 row indicates the order k of the middle product, and also the actual inner Nielsen class given by the representative.

TABLE 1. **sh** applied to elements of $\mathrm{Cu}_4(\text{H-M}_1)$

$[1]_1$	$(\text{H-M}_{1,0})\mathbf{sh} = (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
$[3]_1$	$(\text{H-M}_{1,1})\mathbf{sh} = (x_{2,\frac{n+3}{2}}, (x_{\frac{n+3}{2},n}\, 1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
$[5]_1$	$(\text{H-M}_{1,2})\mathbf{sh} = (x_{3,\frac{n+5}{2}}, (x_{\frac{n+5}{2},n}\, x_{1,2}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
	$\dots$
$[n]_1$	$(\text{H-M}_{1,\frac{n-1}{2}})\mathbf{sh} = (x_{\frac{n+1}{2},n}, (n\, x_{1,\frac{n-1}{2}}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
$[n]_2$	$(\text{H-M}_{1,\frac{n+1}{2}})\mathbf{sh} = ((x_{\frac{n+3}{2},n}\, 1), x_{1,\frac{n+1}{2}}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
	$\dots$
$[5]_2$	$(\text{H-M}_{1,n-2})\mathbf{sh} = ((x_{n-1,n}\, x_{1,\frac{n-3}{2}}), x_{\frac{n-3}{2},n-2}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$
$[3]_2$	$(\text{H-M}_{1,n-1})\mathbf{sh} = ((n\, x_{1,\frac{n-1}{2}}), x_{\frac{n-1}{2},n-1}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

Lem. 4.2 comes from noting the centralizer of an n-cycle $h \in S_n$ is just $\langle h \rangle$.

LEMMA 4.2. *For* $3 \le k \le n$ *(odd), let* g_1', g_4' *denote, respectively the 1st and 4th entries of* $[k]_j$ *in* Table 1. *Let* V *be the union of the supports of* g_1' *and* g_4'. *Then, the centralizer of* $\langle g_1', g_4' \rangle$ *in* S_V *is trivial. So, with* $h = ([k]_j)\mathbf{mp}$, *a complete list of elements in the Nielsen class having respective 1st and 4th entries* g_1', g_4' *is*

$$\{\beta^{-1}(g_1, h^{-u}g_2h^u, h^{-u}g_3h^u, g_4)\beta\}_{\beta \in S_{V'}},\ V' = \{1, \dots, n\} \setminus V.$$

In particular, if $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$ *has the same 1st and 4th entries as* $[n]_1$ *(resp.* $[n{-}2]_1$*), then* $\boldsymbol{g} \in \mathrm{Cusp}_{[n]_1}$ *(resp.* $\mathrm{Cusp}_{[n-2]_1}$*). Conjugation of* $[n]_2$ *by* $x^{-1}_{1,\frac{n+1}{2}}$ *and* $[n]_1$ *have the same 1st and 4th entries,* $\mathrm{Cusp}_{[n]_1} = \mathrm{Cusp}_{[n]_2}$.

PROOF. For the first part use $\langle g_1' \rangle \cap \langle g_4' \rangle = \{1\}$. For the rest apply Lem. 4.3 and that for, respectively, $[n]_1$ or $[n{-}2]_1$, the set $|V'|$ is 0 or 1. □

4.1.2. *Cusps of* $\mathrm{Ni}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$. First we take care of absolute cusps.

LEMMA 4.3. *For* $1 \le k \le n$ *odd,* $[k]_1$ *gives a complete list of reps for cusp orbits in* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$. *The middle product* $([k]_1)\mathbf{mp}$ *(resp.* $([k]_2)\mathbf{mp}$*) is*

$$\alpha_{k,1} = (x_{1,\frac{k-1}{2}}\, x_{\frac{n+k}{2},\frac{n+1}{2}}),$$
$$(\textit{resp.}\quad \alpha_{k,2} = (x_{\frac{n-k+2}{2},\frac{n+1}{2}}\, x_{n,n-\frac{k-3}{2}})).$$

The order of the $\mathbf{mp}$ *of the cusp rep. equals the cusp width. In particular,*

$$|\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}| = (\frac{n+1}{2})^2.$$

PROOF. We want to know when the list above gives a complete set of representatives of all absolute cusps. Given $\boldsymbol{g} = (g_1, g_2, g_3, g_4)$ in the Nielsen class, denote the centralizer (in S_n) of the pure cycle g_2g_3 (from Prop. 3.12) by $\mathrm{Cen}_{(\boldsymbol{g})\mathbf{mp}}$. Assume $\mathrm{ord}(g_2g_3) = k$. Princ. 3.1 shows anything in the cusp of $\boldsymbol{g}$ (since it has middle product k) has an absolute Nielsen class rep. in

$$T_{\boldsymbol{g}} \stackrel{\mathrm{def}}{=} \{\boldsymbol{g}_\alpha \stackrel{\mathrm{def}}{=} (g_1, \alpha g_2\alpha^{-1}, \alpha g_3\alpha^{-1}, g_4)\}_{\alpha\in \mathrm{Cen}_{(\boldsymbol{g})\mathbf{mp}},\ \text{with } \langle \boldsymbol{g}_\alpha\rangle \text{ transitive}}.$$

We show absolute equivalence classes of elements in $T_{\boldsymbol{g}}$ is just the unique absolute cusp with middle product k.

Look at $[k]_1 = \boldsymbol{g}$. Then, $\mathrm{Cen}_{([k]_1)\mathbf{mp}} = \langle \alpha_{k,1}, S_{\frac{k+1}{2},\frac{n-1}{2}}, S_{\frac{n+k+2}{2},n}\rangle$. Also, $S_{\frac{k+1}{2},\frac{n-1}{2}} = \mathrm{Cen}_{\langle g_2,g_3\rangle}$ and $S_{\frac{n+k+2}{2},n} = \mathrm{Cen}_{\langle g_1,g_4\rangle}$. Since g_2 and g_3 have no support in $\{\frac{k+1}{2}\dots\frac{n-1}{2}\}$, $\boldsymbol{g}_\alpha$ won't be transitive if α moves one of $\frac{n+k+2}{2},\dots,n$ to one of $\frac{k+1}{2},\dots,\frac{n-1}{2}$. Finally:

(4.1a) Given $\boldsymbol{g}_\alpha \in T_{\boldsymbol{g}}$, $u_{2,3} \in \mathrm{Cen}_{\langle g_2,g_3\rangle}$ and $u_{1,4} \in \mathrm{Cen}_{\langle g_1,g_4\rangle}$, then $\boldsymbol{g}_\alpha$ and $\boldsymbol{g}_{u_{1,4}\alpha u_{2,3}}$ represent the same Nielsen class.

(4.1b) For $\boldsymbol{g}_{\alpha'} \in T_{\boldsymbol{g}}$ the cusp $\mathrm{Cu}_4(\boldsymbol{g}_{\alpha'})$ consists of $\{\boldsymbol{g}^{\mathrm{abs}}_\alpha\}_{\alpha\in\langle\alpha_{k,1}\rangle\alpha'}$.

The cusp of ${[k]_1}^{\mathrm{abs}}$ in (4.1b) includes all elements with middle product k. □

4.1.3. *Cusps of* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$. Now we adjust Lem. 4.3 for inner cusps. There are either two inner cusps for each absolute cusp, or one inner cusp of twice the absolute cusp width. Lem. 4.4 inspects Table 1 to distinguish these cases.

LEMMA 4.4. *For* $n \equiv 5 \mod 8$, $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$ *has one braid orbit. We list the cusps in this orbit using a parameter* ℓ *in the range* $0 \le \ell \le \frac{n-1}{2}$.

(4.2a) *There is a just one cusp of width* $2\cdot(n-2\ell)$ *represented by* $[n-2\ell]_1$ *if both* ℓ *and* $\frac{\ell-1}{2}$ *are odd; or if* $2||\ell$.

(4.2b) *There are exactly two inner width* $n-2\ell$ *cusps if* ℓ *is odd and* $\frac{\ell-1}{2}$ *is even; or if* $4|\ell$. *In the former case* $[n-2\ell]_1$ *and* $[n-2\ell]_2$ *represent the two cusps, but in the latter case they represent the same cusp.*

For $n \equiv 1 \mod 8$, *there are two braid orbits* O_1 *and* O_2 *on* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$. *These have cusp width 1 representatives* H-M$_1$ *and* βH-M$_1\beta^{-1}$, *(any)* $\beta \in S_n \setminus A_n$. *For* $j = 1, 2$, *and odd* k, $1 \le k \le n$, *there is one inner cusp of width* k *in* O_j.

PROOF. Use [**BiF82**, Lem. 3.8]: For each $h \in G$ there is a braid that goes from $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ to $h\boldsymbol{g}h^{-1}$. When $G = A_n$, for any $h \in S_n$, there is a braid from $\boldsymbol{g}$ to $h\boldsymbol{g}h^{-1}$ if and only if there is such a braid for one case of $h \in S_n \setminus A_n$.

Now replace H-M$_1$ with H-M$_2$. The analogous table of absolute cusps for H-M$_2$ must, from above, be exactly the same as that for H-M$_1$. Apply q_1 to H-M$_2$ to get H-M$_1$. This braid is equivalent to conjugation by

$$\beta' = (2\ \tfrac{n+1}{2})(3\ \tfrac{n-1}{2})\cdots(\tfrac{n-1}{4}\ \tfrac{n+1}{4}).$$

We have listed every absolute class that comes from applying $\bar{M}_4$ to H-M$_1$ in Table 1. So, we get one (resp. 2) $\bar{M}_4$ orbits on $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$, if and only if β is (resp. is not) in $S_n \setminus A_n$. These cases occur, respectively, for $n \equiv 5 \mod 8$ (resp. $1 \mod 8$), and in these cases H-M$_1$ represents an inner cusp of width 2 (resp. 1).

Now consider $n \equiv 5 \mod 8$. The 2nd and 3rd entries of $[n{-}2\ell]_1$ are

$$g_2'' = (x_{n-\ell,n}\, x_{1,\frac{n-1}{2}-\ell}) \text{ and } g_3'' = x_{n,\frac{n+1}{2}}.$$

Also, $([n{-}2\ell]_1)\mathbf{mp} = \alpha_{n-2\ell,1} = (x_{1,\frac{n-1}{2}-u}\, x_{n-u,\frac{n+1}{2}})$. We determine the element γ for which $([n{-}2\ell]_1)q_2$ —whose 3rd term is g_2'' —has 2nd and 3rd terms equal to $\gamma(g_2'', g_3'')\gamma^{-1}$. From the proof of Lem. 4.3, this must be a power of $\alpha_{n-2\ell,1}$ times an element, γ', centralizing $\langle g_1'', g_4''\rangle$. We see this directly.

Determine $\gamma \in S_{\frac{n+3}{2},n}$ from $\gamma(g_3'')\gamma^{-1} = g_2''$. A power of $\alpha_{n-2\ell,1}$ translates $|x_{1,\frac{n-1}{2}-\ell}\, n{-}\ell|$ to $|x_{n-\ell,\frac{n+1}{2}}|$. So, γ' inverts $|x_{n-\ell+1,n}|$. Iterations of q_2 give further conjugations by γ. Therefore, the cusp width is $n{-}2\ell$ exactly when $(\gamma')^{n-2\ell} = \gamma'$ has parity 1. Otherwise, the cusp of $[n{-}2\ell]_1$ will have width $2\cdot(n{-}2\ell)$. The list of (4.2) translates the parity statement of Lem. 1.8 to conclude the result.

The computation for when $[n{-}2\ell]_j$ represent the same cusp —when ℓ is even —is similar, so we outline it. Let V be the support of the 1st and 4th entries of $[n{-}2\ell]_2$. A forced conjugation from S_V on $[n{-}2\ell]_2$ yields $[n{-}2\ell]_2'$ having its 1st and 4th entries the same as $[n{-}2\ell]_1$.

The conjugation is a composition of a power of $x_{1,\frac{n+1}{2}}$ and a permutation consisting of $\frac{n-2\ell-1}{2}$ disjoint cycles so of parity $(-1)^\ell$. Then, the cusp of $[n{-}2\ell]_2'$ has a representative conjugate by $(x_{n-\ell+1,n})^{\frac{n-2\ell-1}{2}}$ (of parity $(-1)^{(\ell-1)\cdot\ell} = 1$).

The case $n \equiv 1 \mod 8$ follows from the absolute pattern. □

Remark 4.5 ($k = n$ in Lem. 4.4). Though $k = n$ is not special, it has a special role and a special proof. Write out $([n]_1)^{\frac{n-1}{2}} x_{1,\frac{n+1}{2}}$. You get $(1\ \frac{n+3}{2})\cdots(\frac{n+1}{2}\ n)$, of order 2. Princ. 3.5 gives the width of the cusp of $[n]_1$ as n.

4.1.4. *Naming the cusps.* When $n \equiv 1 \mod 8$, Lem. 4.4 gives two braid orbits, so two components $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $i = 1, 2$. Each maps one-one to $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$, identifying its cusps with those on each $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$.

Denote the cusps simply as ${}_{\mathbf{c}}O_{n-2\ell} \stackrel{\text{def}}{=} \mathrm{Cusp}_{n-2\ell}$ for ℓ, with $0 \le l \le \frac{n-1}{2}$. Even as a moduli space (structure beyond being an algebraic variety; App. C) $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ has definition field $\mathbb{Q}$. Yet, $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, $i = 1, 2$, are conjugate over $\mathbb{Q}$ (Prop. 6.13).

For $n \equiv 5 \mod 8$, Lem. 4.4 says $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}} \to \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ is a degree 2 cover of absolutely irreducible $\mathbb{Q}$ varieties. Inner cusps need more intricate notation.

Definition 4.6. For $n \equiv 5 \mod 8$ denote the unique cusp of middle product 1 and width 2 by ${}_{\mathbf{c}}O_{1,2}$ ($\ell = \frac{n-1}{2}$ in Lem. 4.4 notation):

$${}_{\mathbf{c}}O_{1,2} = \mathrm{Cusp}_{(\text{H-M}_1)\mathbf{sh}} = \mathrm{Cusp}_{(\text{H-M}_2)\mathbf{sh}} = ((\text{H-M}_1)\mathbf{sh})^{\mathrm{in,rd}} \cup ((\text{H-M}_2)\mathbf{sh})^{\mathrm{in,rd}}.$$

Similarly, for the other values of k when there is one (inner) cusp of width $2k$ – say, when $n = 13$ and $k = 7, \ell = 3$ or $k = 9$, $\ell = 2$ – denote this ${}_{\mathbf{c}}O_{k,2k}$.

Finally, for those k with two cusps of width k, say for

$$n = 13 : (k, l) \text{ respectively } (3, 5), (5, 4), (11, 1), (13, 0),$$

use this notation. If $k \equiv 1 \mod 4$ denote the cusp represented by $[k]_1$ by

$${}_{\mathbf{c}}O'_{k;1} = \mathrm{Cusp}_{k,1} = \mathrm{Cusp}_{(\text{H-M}_{1,\frac{k-1}{2}})\mathbf{sh}}, \ ([k]_2 \text{ represents the same cusp}).$$

Similarly, $\mathrm{Cusp}_{(\text{H-M}_{2,\frac{k-1}{2}})\mathbf{sh}} = {}_{\mathbf{c}}O'_{k;2} = \mathrm{Cusp}_{k,2}$. For $k \equiv 3 \mod 4$ denote the cusp of $[k]_i$ by ${}_{\mathbf{c}}O_{k,k;i} \stackrel{\text{def}}{=} {}_{\mathbf{c}}O'_{k;i}$, $i = 1, 2$.

4.2. sh-incidence for $\mathrm{Ni}^{\text{in,rd}}_{(\frac{n+1}{2})^4}$**.** Def. 2.7 has the **sh**-incidence matrix. Combining the naming from §4.1.4 with Def. 4.6 fills in the pairings of ${}_{\mathbf{c}}O_{1,2}$ with all cusps (Table 2) from the sum of the entries in the column being the cusp width (2 in this case). Pairing of ${}_{\mathbf{c}}O_{1,2}$ with $\mathrm{Cusp}_{(\text{H-M}_i)}$ is 1, fulfilled by $((\text{H-M}_i)\mathbf{sh})^{\text{in,rd}}$.

Table 3 displays the matrix for $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$; $n = 5$, is easier having none of the cusps labeled $\mathcal{O}_{k,2k}$, $k > 1$ odd. §5 completes the sh-incidence matrix for all the $\mathrm{Ni}^{\text{in,rd}}_{(\frac{n+1}{2})^4}$, $n \equiv 5 \mod 8$. Identify ${}_{\mathbf{c}}O'_{n;1}$, represented by (both) $[n]_i$, $i = 1, 2$ (as in (4.2b), with $\mathrm{Cusp}_{\text{H-M}_2}$:

$$(\text{H-M}_2)\mathbf{sh}^2 = (x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) = ([n]_1)q_2^{-1}. \tag{4.3}$$

We augment Def. 4.6 for the cusps ${}_{\mathbf{c}}O_{k,2k}$, $k > 1$, by splitting those cusps in Def. 4.7 into two pieces. Consider again the entries $(g''_1, g''_2, g''_3, g''_4) = [k]_1$. The Lem. 4.4 proof has an odd parity element γ' commuting with $\alpha_{k,1} = (g''_2, g''_3)\mathbf{mp}$. Then, inner reps. of elements in ${}_{\mathbf{c}}O_{k,2k}$ have two forms:

$$\begin{aligned} R'_{k,1} = \quad & \{(g''_1, \alpha^t_{k,1} g''_2 \alpha^{-t}_{k,1}, \alpha^t_{k,1} g''_3 \alpha^{-t}_{k,1}, g''_4)\}^{k-1}_{t=0}; \text{ and;} \\ R'_{k,2} = \quad & \{(g''_1, \gamma'\alpha^t_{k,1} g''_2 \alpha^{-t}_{k,1}(\gamma')^{-1}, \gamma'\alpha^t_{k,1} g''_3 \alpha^{-t}_{k,1}(\gamma')^{-1}, g''_4)\}^{k-1}_{t=0}. \end{aligned} \tag{4.4}$$

Definition 4.7 (Splitting ${}_{\mathbf{c}}O_{k,2k}$). For $k = n - 2\ell$, ℓ as in (4.2a), let ${}_{\mathbf{c}}O'_{k;1}$ (resp. ${}_{\mathbf{c}}O'_{k;2}$) be all inner (reduced also from Lem. D.3) classes in $R'_{k,1}$ (resp. $R'_{k,2}$). We call these ${}_{''}$cusps. The concept p-${}_{''}$cusps (extending p-cusps) makes sense.

Lemma 4.8. *From* (4.3) *or* (4.2b)*, both* $[n]_i$ *s are in* $\mathrm{Cusp}_{\text{H-M}_2} = {}_{\mathbf{c}}O'_{n;1}$*. So, the sh-incidence entry for* $({}_{\mathbf{c}}O'_{n;1}, {}_{\mathbf{c}}O'_{n;2})$ *is 2. The n fulfilling elements for nontrivial intersections with* $({}_{\mathbf{c}}O'_{n;2})\mathbf{sh}$ *are in* Table 1*. With* $0 \le u \le \frac{n-3}{2}$, $[n-2u]_1 \in ({}_{\mathbf{c}}O'_{n;2})\mathbf{sh}$, *and* $[n-2u]_2 \in ({}_{\mathbf{c}}O'_{n;1})\mathbf{sh}$ *(resp.* $({}_{\mathbf{c}}O'_{n;2})\mathbf{sh}$*) for u odd (resp. even).*

Table 2 records each ${}_{\mathbf{c}}O_{k,2k}$ with two columns corresponding to ${}_{''}$cusps. For the **sh**-incidence matrix (so Lem. 2.11 applies, Rem. 4.10), sum the two row entries. Example: If $n = 13$, Table 2 has a 1×2 matrix entry $|0 \quad 2|$ (resp. $|1 \quad 1|$) for pairing ${}_{\mathbf{c}}O'_{13;1}$ with $|{}_{\mathbf{c}}O'_{9;1} \quad {}_{\mathbf{c}}O'_{9;2}|$ (resp. $|{}_{\mathbf{c}}O'_{7;1} \quad {}_{\mathbf{c}}O'_{7;2}|$). Instead, replace that by a 1×1 entry of 2 for the pairing of ${}_{\mathbf{c}}O'_{13;1}$ with ${}_{\mathbf{c}}O_{9,18}$ (resp. ${}_{\mathbf{c}}O_{7,14}$).

With the symmetry of Table 3, and that column entries sum to the cusp width, Prop. 5.1 finishes the table for $n = 5$ by concluding the $({}_{\mathbf{c}}O'_{3;1}, {}_{\mathbf{c}}O'_{3;2})$ entry is 1.

Proposition 4.9. *The genus* $\mathbf{g}_{3^4}$ *of* $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$ *is 0.*

There are no 2-cusps at level 0. Yet, every cusp at level 1 over each of the non H-M *cusps, is a 2-cusp. So, there are at least four 2-cusps on every level 1*

TABLE 2. ${}_{\eta}$cusp **sh**-incidence Matrix: Rows for ${}_{\mathbf{c}}O'_{n;j}$, $j = 1, 2$

cusp orbit	${}_{\mathbf{c}}O'_{n-2u;1}{}^{u}_{\text{even}}$	${}_{\mathbf{c}}O'_{n-2u;2}{}^{u}_{\text{even}}$	${}_{\mathbf{c}}O'_{n-2u;1}{}^{u}_{\text{odd}}$	${}_{\mathbf{c}}O'_{n-2u;2}{}^{u}_{\text{odd}}$	$\dots$	${}_{\mathbf{c}}O_{1,2}$
${}_{\mathbf{c}}O'_{n;1}$	0	2	1	1	$\dots$	1
${}_{\mathbf{c}}O'_{n;2}$	2	0	1	1	$\dots$	1

TABLE 3. **sh**-incidence Matrix: $r = 4$ and $\mathrm{Ni}^{\mathrm{in,rd}}_{3^4}$

Cusp orbit	${}_{\mathbf{c}}O'_{5;1}$	${}_{\mathbf{c}}O'_{5;2}$	${}_{\mathbf{c}}O'_{3;1}$	${}_{\mathbf{c}}O'_{3;2}$	${}_{\mathbf{c}}O_{1,2}$
${}_{\mathbf{c}}O'_{5;1}$	0	2	1	1	1
${}_{\mathbf{c}}O'_{5;2}$	2	0	1	1	1
${}_{\mathbf{c}}O'_{3;1}$	1	1	0	1	0
${}_{\mathbf{c}}O'_{3;2}$	1	1	1	0	0
${}_{\mathbf{c}}O_{1,2}$	1	1	0	0	0

component of the **MT** *for* $p = 2$*. In particular, the Main Conjecture holds for* $p = 2$ *and all component branches.*

The only other prime of consideration is $p = 5$*, and the Main Conjecture holds for this prime, too, for all* H-M *component branches.*

PROOF. Apply (2.11) to get $\gamma'_0, \gamma'_1, \gamma'_\infty$ acting on the unique braid orbit on $(A_5, \mathbf{C}_{3^4})$. The diagonal entries of the **sh**-incidence matrix has only 0's. Lem. 2.11 implies neither γ'_0 nor γ'_1 has a fixed point. So, we have a degree 18 cover of the j-line. **R-H** gives its genus $\mathbf{g}_{3^4}$ through

$$
\begin{aligned}
2(18 + \mathbf{g}_{3^4} - 1) = \quad & \mathrm{ind}(\gamma'_0) + \mathrm{ind}(\gamma'_1) + \mathrm{ind}(\gamma'_\infty) = \\
& 2(18/3) + 18/2 + (1 + 2(2 + 4)) : \mathbf{g}_{3^4} = 0.
\end{aligned}
\tag{4.5}
$$

Apply Prop. 3.12 to the cusps of width 3 and 5. Respectively, since $\frac{3^2-1}{8}$ and $\frac{5^2-1}{8}$ are $\equiv 1 \mod 2$, all cusps above these are 2-cusps. In particular, any level 1 component has at least four 2-cusps. Therefore, the Main Conjecture is automatic for any component branch for $p = 2$ (Prop. 2.13).

The two H-M rep. cusps are already 5-cusps at level 0. Princ. 3.3 therefore shows, there are at least three 5-cusps on every H-M level 1 component. So, the Main Conjecture holds for any component branch through a level 1 H-M component. Still, if a component is not H-M at level 1, we only can guarantee two 5-cusps. □

REMARK 4.10 (${}_{\mathbf{c}}O_{k,2k}$ cusps and Lem. 2.11). Suppose — Def. 4.7 — we have an ${}_{\mathbf{c}}O_{k,2k}$ cusp, $k > 1$. Applying q_2 shifts between its ${}_{\eta}$cusps, ${}_{\mathbf{c}}O'_{k;1}$ and ${}_{\mathbf{c}}O'_{k;2}$. So, the proof of Lem. 2.11 fails unless we combine the rows (and columns) for ${}_{\mathbf{c}}O'_{k;1}$ and ${}_{\mathbf{c}}O'_{k;2}$. This isn't, however, necessary for sh-incidence symmetry (when $r = 4$).

REMARK 4.11 (Distinction between non-H-M and H-M level 1 components). An H-M component — a Hurwitz component containing an H-M cusp (§3.1.2) — figures in properties of $G_{\mathbb{Q}}$ (as in §E.2). Even if level 0 of a **MT** over $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ is an H-M component, that may not hold for some level 1 components over it. For example, $\bar{\mathcal{H}}(G_1(A_4), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ (A_4, not A_5; §E.3.2) with $p = 2$ has six components. Three are level 1 of a **MT** over $\mathcal{H}(A_4, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$; but just two are H-M components.

Also, to show there are p-cusps at some level for all allowable primes in just those Nielsen classes of Prop. 4.9 requires new ideas to avoid detailed calculations about non-H-M components. §6.2.2 completes Prop. 4.9 by showing all level 1 components for a $p = 5$ **MT** over $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$ have at least three 5-cusps. A general braid orbit principle likely applies, though we used [**GAP00**].

5. $\text{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$ sh-incidence and genus computation

Prop. 5.1 reduces computing the inner, reduced sh-incidence matrix to deciding if conjugations between length $\frac{n+1}{2}$ pure-cycles are in $A_{\frac{n+1}{2}}$. §5.2 has the actual sh-incidence display. Table 3 ($n = 5$) doesn't tackle all cusp matters. So, Table 7 has $n = 13$, the next case for $n \equiv 5 \mod 8$, to show the concise abs-inn sh-incidence form. This includes how the genus of reduced spaces comes from these matters.

5.1. Organizing conjugation parities. §5.1.1 decorates the $x_{i,j}$ notation to precisely label the shift of an ${}_{\shortparallel}$cusp (Def. 4.7). §5.1.2 displays the shifted Nielsen class representatives (denoted ${}_{k,u}\boldsymbol{g}'$) that dominate the rest of the computation.

5.1.1. *Additions to $x_{i,j}$ notation.* We use special notation for Prop. 5.1. For $V = \{\frac{n+m}{2}, \dots, n\}$, $m \geq 3$ denote the alternating (resp. symmetric) group acting on V by $A_{\frac{n+m}{2},n}$ (resp. $S_{\frac{n+m}{2},n}$). Also, $k-1 - |2u - (k-1)| \stackrel{\text{def}}{=} m_{k,u}$ and

$$([k]_1)\mathbf{mp} = (x_{\frac{n+k}{2},\frac{n+1}{2}}\, x_{1,\frac{k-1}{2}}) \stackrel{\text{def}}{=} \alpha_{k,1} \text{ (as in Lem. 4.3).}$$

The cycle $\alpha_{k,1}$ maps $\frac{n+1}{2} \mapsto 1$. Yet, sometimes we group $\frac{n+1}{2}$ with $\{1, \dots, \frac{k-1}{2}\}$. As i runs from $\frac{n+1}{2}$ toward $\frac{k-1}{2}$, use $x'_{i,j}$ to mean the segment from i to j, including interpreting $x'_{\frac{n+1}{2},j}$ ($i = \frac{n+1}{2}$) as ($\frac{n+1}{2}\, 1 \dots j$). Example:

$$\alpha_{k,1} = (x_{\frac{n+k}{2},\frac{n+3}{2}}\, x'_{\frac{n+1}{2},\frac{k-1}{2}}).$$

Also, the end points of the list for ${}_{k,u}\boldsymbol{g}$ as u varies include cases where we mean to indicate $x_{i,j}$ is empty. Example: $x_{\frac{n+k-2u}{2},\frac{n+3}{2}}$ appears as the 2nd segment of the first entry of (5.2), with $0 \leq u \leq \frac{k-1}{2}$. In all terms except $u = \frac{k-1}{2}$, $\frac{n+k-2u}{2} \geq \frac{n+3}{2}$. So, denote it $x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}$ to mean $x''_{\frac{n+1}{2},\frac{n+3}{2}}$ is empty.

Similarly, for $x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}$ appearing in the 2nd segment of the 2nd entry of (5.2): When $u = 0$ take it to mean $x''_{\frac{n+k}{2},\frac{n+k+2}{2}}$ is empty.

5.1.2. *Finding conjugating elements.* Use the ${}_{\mathbf{c}}O'_{k;j}$ convention of §4.2, and the splitting of ${}_{\mathbf{c}}O_{k,2k}$ into ${}_{\shortparallel}$cusps (Def. 4.7), to include these, too. We could have denoted ${}_{\mathbf{c}}O'_{k;1}$ by ${}_{\mathbf{c}}O'_{[k]_1;1}$, and similarly formed ${}_{\mathbf{c}}O'_{\boldsymbol{g};1}$ for Nielsen class rep. $\boldsymbol{g}$.

Explicit asides on the sh-incidence column for the cusp of $[3]_1$ help follow notation. The $[3]_1$ row of Table 1 is $(x_{2,\frac{n+3}{2}}, (\frac{n+3}{2} \dots n\, 1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$, with middle product $\alpha_{3,1} = (1\, \frac{n+3}{2}\, \frac{n+1}{2})$. Conjugate the 2nd and 3rd positions of $[3]_1$ by $\alpha_{3,1}^u$, then shift. Here are the 1st and 2nd positions of the result for $u = 1, 2$:

$$(5.1) \qquad {}_{3,1}\boldsymbol{h} = ((x_{n,\frac{n+5}{2}}\, \tfrac{n+1}{2}\, 1), x_{\frac{n+1}{2},1}) \qquad {}_{3,2}\boldsymbol{h} = ((x_{n,\frac{n+5}{2}}\, 1\, \tfrac{n+3}{2}), x_{\frac{n+1}{2},1}).$$

The respective products of entries of ${}_{3,1}\boldsymbol{h}$ and ${}_{3,2}\boldsymbol{h}$ are

$$(x_{n,\frac{n+5}{2}}\, x_{\frac{n+1}{2},2}) \text{ and } (x_{n,\frac{n+5}{2}}\, 1\, x_{\frac{n+1}{2},2}\, \tfrac{n+3}{2}).$$

Key to Prop. 5.1: Apply $q_1^{-1}q_3$, leaving the reduced Nielsen class unchanged. Once ℓ is given in Prop. 5.1, denote the entries of $[\ell]_2$ by $(g''_1, g''_2, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$.

Proposition 5.1. *Each* ${}_{\mathbf{c}}O'_{n;1}$, ${}_{\mathbf{c}}O'_{n;2}$ *and* ${}_{\mathbf{c}}O'_{n-2;2}$ *intersect once with* $({}_{\mathbf{c}}O'_{3,1})\mathbf{sh}$ *giving all nonzero sh-incidence entries in the column (or row) of* ${}_{\mathbf{c}}O'_{3;j}$, $j=1,2$ *(*Table 5*). Elements of* $({}_{\mathbf{c}}O'_{k;1})\mathbf{sh}$ *have reps. of form*

$${}_{k,u}\boldsymbol{g} \stackrel{\text{def}}{=} (\alpha^u_{k,1}x_{n,\frac{n+1}{2}}\alpha^{-u}_{k,1}, \alpha^u_{k,1}(x'_{\frac{n+1}{2},\frac{k-1}{2}}\,x_{\frac{n+k+2}{2},n})\alpha^{-u}_{k,1}, \bullet, x_{\frac{n+1}{2},1})$$

in ${}_{\mathbf{c}}O'_{\ell;1}\cup{}_{\mathbf{c}}O'_{\ell;2}$, $\ell=n-m_{k,u}$ *with* $m_{k,u}=2u$ *(resp.* $2(k-1-u)$*) for* $0\le u\le\frac{k-1}{2}$ *(resp.* $\frac{k+1}{2}\le u\le k-1$*). For* $0\le u\le\frac{k-1}{2}$, ${}_{k,u}\boldsymbol{g}$ *is inner equivalent to some* ${}_{k,u}\boldsymbol{g}'$ *with 4th entry* $x_{\frac{n+1}{2},1}$ *and with respective 1st and 2nd entries*

$$(5.2)\qquad (x_{n,\frac{n+k+2}{2}}\,x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}\,x_{1,1+u}), (x_{1+u,\frac{k+1}{2}}\,x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\,x_{\frac{n+k+2}{2},n}).$$

Conclude: ${}_{k,u}\boldsymbol{g}\in{}_{\mathbf{c}}O'_{\ell;2}$ *(resp.* ${}_{\mathbf{c}}O'_{\ell;1}$*) if* $\frac{\ell-1}{2}$ *is odd (resp. even), if and only if for some* $\beta'\in A_{\frac{n+3}{2},n}$ *and* j,

$$(5.3)\qquad \beta' g'_2(\beta')^{-1}=\alpha^j_{\ell,2}g''_2\alpha^{-j}_{\ell,2},\ \text{with } g'_2 \text{ the 2nd entry of } {}_{k,u}\boldsymbol{g}'.$$

Here is the analog of expression (5.2) *for* $\frac{k+1}{2}\le u\le k-1$*, with* $k-1-u=u'$*:*

$$(5.4)\quad (x_{n,\frac{n+k+2}{2}}\,x_{1,1+u'}\,x_{\frac{n+k}{2},\frac{n+3+2u'}{2}}), (x_{\frac{n+3+2u'}{2},\frac{n+3}{2}}\,x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\,x_{\frac{n+k+2}{2},n}).$$

Proof. Three distinct sets of form $\boldsymbol{g}^{\text{in,rd}}$ comprise $({}_{\mathbf{c}}O'_{3;1})\mathbf{sh}$. From Table 2, ${}_{\mathbf{c}}O'_{n;j}$, $j=1,2$, gives two. Then, $\boldsymbol{g}^*=((x_{\frac{n+1}{2},n}\ \frac{n+3}{2}),\boldsymbol{h}^{3,1},x_{2,\frac{n+3}{2}})$ is the 3rd, and it is in the cusp of one of ${}_{\mathbf{c}}O'_{n-2;j}$, $j=1,2$. Apply $q_1^{-1}q_3$ to $\boldsymbol{g}^*$ to get the 4th entry $x_{\frac{n+1}{2},1}$. Conjugating (§1.3.2) by $x_{1,\frac{n+1}{2}}$ gives:

$${}_{3,1}\boldsymbol{g}' \stackrel{\text{def}}{=} ((x_{n,\frac{n+5}{2}}\,x_{1,2}),(2\,x_{\frac{n+3}{2},n}),\bullet,x_{\frac{n+1}{2},1}).$$

Now conjugate the first entry of ${}_{3,1}\boldsymbol{g}'$ to the first entry of

$$[n-2]_2=((x_{\frac{n+5}{2},n}\,x_{1,2}),x_{2,\frac{n+3}{2}},x_{n,\frac{n+1}{2}},x_{\frac{n+1}{2},1}).$$

The conjugation is by $\beta'_{3,1}$: It inverts $|x_{n,\frac{n+5}{2}}|$ (parity $+1$ from Lem. 1.8). The result $\boldsymbol{g}^\dagger$ has the same 1st and 4th entries as $[n-2]_2$. Lem. 4.2 now says $\text{Cusp}_{\boldsymbol{g}^*}={}_{\mathbf{c}}O'_{n-2;2}$. Rem. 5.2 shows $k=3$ matches the general case.

That accounts for the column in the ${}_{\prime\prime}$cusp sh-incidence (comment following Lem. 4.8) for the cusp of $[3]_1$. Analogously, for $[3]_2$: one intersection with each of the cusps for H-M$_1$ and H-M$_2$; one intersection with the cusp of $[n-2]_1$.

Now we extend the pattern above for computing into which ${}_{\prime\prime}$cusp does the shift of inner, reduced elements fall in the columns of the other ${}_{\mathbf{c}}O'_{k;1}$. (It is then automatic to deduce the same for ${}_{\mathbf{c}}O'_{k;2}$.) With $\alpha_k\stackrel{\text{def}}{=}\alpha_{k,1}$, as in the statement, apply Princ. 3.5 and $q_1^{-1}q_3$ as above to get

$${}_{k,u}\boldsymbol{g} \stackrel{\text{def}}{=} (\alpha^u_k x_{n,\frac{n+1}{2}}\alpha_k^{-u}, \alpha^u_k(\tfrac{n+1}{2}\,x_{1,\frac{k-1}{2}}\,x_{\frac{n+k+2}{2},n})\alpha_k^{-u},\bullet,x_{\frac{n+1}{2},1}).$$

The following steps give the sh-incidence matrix:

(5.5a) As (k,u) varies, $3\le k\le n$ odd and $0\le u\le k-1$, compute

$$\ell=\text{ord}(({}_{k,u}\boldsymbol{g})\mathbf{mp})=\text{ord}(x_{\frac{n+1}{2},1}\cdot\alpha^u_k x_{n,\frac{n+1}{2}}\alpha^{-u}_k).$$

(5.5b) Conjugate ${}_{k,u}\boldsymbol{g}$ by some power $x_{1,\frac{n+1}{2}}$ to get ${}_{k,u}\boldsymbol{g}'$ with 1st entry having the same segment from $\{1,\dots,\frac{n+1}{2}\}$ as does g''_1.

(5.5c) Conjugate ${}_{k,u}\boldsymbol{g}'$ by $\beta_{k,u}\in S_{\frac{n+3}{2},n}$ to get $\boldsymbol{g}^\dagger$ with 1st entry g''_1.

(5.5d) Find $\beta'_{k,u}$ centralizing $\langle g''_1, x_{\frac{n+1}{2},1}\rangle$ so $\beta' g^\dagger (\beta')^{-1} \in {}_{\mathbf{c}}O'_{[\ell]_2;1}$.

To help with notation (5.6) gives an example of "translate segment" §1.3.2. For $k=7$ and $0 \le u \le 6$, it has the 1st (col. 1) and 2nd (col. 2) entries of ${}_{k,u}\boldsymbol{g}$: That is of the shift of the cusp ${}_{\mathbf{c}}O_{7,1}$. Here $\alpha_7 = (\frac{n+7}{2}\ \frac{n+5}{2}\ \frac{n+3}{2}\ \frac{n+1}{2}\ 1\,2\,3)$ $(k=7)$.

The value of $\ell = \mathrm{ord}(({}_{k,u}\boldsymbol{g})\mathbf{mp})$ heads each row. Here ℓ is n minus two numbers: the cardinality of the subset of $\{\frac{n+k}{2}, \dots, \frac{n+3}{2}\}$ (resp. of $\{1, 2, \dots, \frac{n+1}{2}\}$) missing from $\alpha_k^u x_{n,\frac{n+1}{2}} \alpha_k^{-u}$ (resp. moved by $\alpha_k^u x_{n,\frac{n+1}{2}} \alpha_k^{-u}$ as in $x_{1,\frac{n+1}{2}}$).

$$
(5.6)\qquad
\begin{array}{rll}
n: & x_{n,\frac{n+1}{2}} & (\frac{n+1}{2}\ x_{1,3}\ x_{\frac{n+9}{2},n}) \\
n-2: & (x_{n,\frac{n+9}{2}}\ x_{\frac{n+5}{2},\frac{n+1}{2}}\ 1) & (x_{1,3}\ \frac{n+7}{2}\ x_{\frac{n+9}{2},n}) \\
n-4: & (x_{n,\frac{n+9}{2}}\ x_{\frac{n+3}{2},\frac{n+1}{2}}\ x_{1,2}) & (x_{2,3}\ x_{\frac{n+7}{2},\frac{n+5}{2}}\ x_{\frac{n+9}{2},n}) \\
n-6: & (x_{n,\frac{n+9}{2}}\ \frac{n+1}{2}\ x_{1,3}) & (3\ x_{\frac{n+7}{2},\frac{n+3}{2}}\ x_{\frac{n+9}{2},n}) \\
n-4: & (x_{n,\frac{n+9}{2}}\ x_{1,3}\ \frac{n+7}{2}) & (x_{\frac{n+7}{2},\frac{n+1}{2}}\ x_{\frac{n+9}{2},n}) \\
n-2: & (x_{n,\frac{n+9}{2}}\ x_{2,3}\ x_{\frac{n+7}{2},\frac{n+5}{2}}) & (x_{\frac{n+5}{2},\frac{n+1}{2}}\ 1\ x_{\frac{n+9}{2},n}) \\
n: & (x_{n,\frac{n+9}{2}}\ 3\ x_{\frac{n+7}{2},\frac{n+3}{2}}) & (x_{\frac{n+3}{2},\frac{n+1}{2}}\ x_{1,2}\ x_{\frac{n+9}{2},n}).
\end{array}
$$

The pattern is clear. For a general k the g_1 term has $x_{n,\frac{n+k+2}{2}}$ on its left side, and the g_2 term has $x_{\frac{n+k+2}{2},n}$ on its right side. For each i in the support of α_k, going in the direction $\frac{n+1}{2} \to 1$, start with $\frac{n+1}{2}$. Then, the rest of g_1 (resp. g_2) is the $\frac{k+1}{2}$ integers including i and immediately to the left (resp. right) of i in α_k. Then, $u=0 \leftrightarrow i = \frac{n+1}{2}$, $u=1 \leftrightarrow i=1$, etc. Also, for u from 0 to $\frac{k-1}{2}$ (resp. $\frac{k+1}{2}$ to $k-1$), $m_{k,u}$ is u (resp. $k-1-u$). This concludes step (5.5a).

Here is the analog for $0 \le u \le \frac{k-1}{2}$ for u (or i) of (5.6) for a general (n,k):

$$
(5.7)\qquad (x_{n,\frac{n+k+2}{2}}\ x''_{\frac{n+k-2u}{2},\frac{n+3}{2}}\ x'_{\frac{n+1}{2},u}), (x'_{u,\frac{k-1}{2}}\ x''_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\ x_{\frac{n+k+2}{2},n}).
$$

Conjugating ${}_{k,u}\boldsymbol{g}$ by $x_{\frac{n+1}{2},1}$ (leaving its 4th entry unchanged) shows ${}_{k,u}\boldsymbol{g}'$, with entries expressed by (5.2), is inner equivalent to ${}_{k,u}\boldsymbol{g}$; (5.4) similarly.

Now turn to the proposition's last paragraph. The first entries of both $[\ell]_2$ and ${}_{u,k}\boldsymbol{g}'$ (as in (5.2)) each have as support the segment $|x_{1,1+u}|$, and no other integers of $\{1, \dots, \frac{n+1}{2}\}$. This concludes showing that ${}_{k,u}\boldsymbol{g}'$ satisfies (5.5a) and (5.5b). The conclusion is an interpretation of (5.5c) and (5.5d). □

REMARK 5.2. Prop. 5.1 starts with $k=3$. Lem. 4.2 says the conclusion about $[n-2]_2$ didn't need the 2nd term in $\boldsymbol{g}^\dagger = \beta'_{3,1}({}_{3,1}\boldsymbol{g}')(\beta'_{3,1})^{-1}$. Still, a pattern emerges. Conjugating by $\langle([n-2]_2)\mathbf{mp}\rangle \overset{\text{def}}{=} \langle(x_{2,\frac{n+1}{2}}\ x_{n,\frac{n+5}{2}})\rangle$ on the 2nd term $g_2^\dagger = (2\ \frac{n+3}{2}\ x_{n,\frac{n+5}{2}})$ of $\boldsymbol{g}^\dagger$ contains $x_{2,\frac{n+3}{2}}$, the second entry of $[n-2]_2$.

5.1.3. *sh-incidence for absolute Liu-Osserman spaces.* The 2nd paragraph of Prop. 5.1 lets us fill in the absolute sh-incidence table for all $n \equiv 1 \mod 4$.

(5.8a) For odd k, $1 \le k \le u$ and $1 \le u < \frac{k-1}{2}$, $({}_{\mathbf{c}}O_k, {}_{\mathbf{c}}O_{n-2u}) = 2$.
(5.8b) $({}_{\mathbf{c}}O_k, {}_{\mathbf{c}}O_{n-(k-1)}) = 1$ and, modulo symmetry, all other entries are 0.

We list cusps in descending width along the rows and columns, as in Table 4. The case $n=13$ (Table 7) shows how we build the *abs-inn* form of the inner sh-incidence matrix along a(nti)-(sub)d(iagonal)s. A less intricate version is in the absolute sh-incidence matrix: the 1-1 a-d is $\frac{n+1}{2}$ 1's, the 3-3 a-d is $\frac{n-1}{2}$ 2's, etc.

TABLE 4. **sh**-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{3^4}^{\mathrm{abs,rd}}$

Cusp orbit	${}_{\mathbf{c}}O_5$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_5$	2	2	1
${}_{\mathbf{c}}O_3$	2	1	0
${}_{\mathbf{c}}O_1$	1	0	0

Prop. 4.9 says $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ has genus $\mathbf{g}_{3^4}^{\mathrm{in,rd}} = 0$. So, its degree 9 image over $\mathbb{P}^1_j$, $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{abs,rd}}$, does too. Then, γ_0' has at most 3 orbits of length 3, and γ_1' has at most 4 orbits of length 2. Apply **R-H**: the maxima are necessary so that

$$2(9 + \mathbf{g}_{3^4}^{\mathrm{abs,rd}} - 1) = \mathrm{ind}(\gamma_0') + \mathrm{ind}(\gamma_1') + \mathrm{ind}(\gamma_\infty') \text{ gives } \mathbf{g}_{3^4}^{\mathrm{abs,rd}} = 0$$

since $\mathrm{ind}(\gamma_\infty') = (1-1) + (3-1) + (5-1) = 6$. This shows $\mathrm{ind}(\gamma_0') = 6$, and $\mathrm{ind}(\gamma_1') = 4$. So, γ_1' has one fixed point, γ_0' none.

In general, then, the sh-incidence matrix for $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ has 1's along the anti-diagonal, 2's above that, and 0's below that.

REMARK 5.3. Prop. 4.9 shows that for $n = 5$, Lem. 5.12 accounts for all the ramification from $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ to $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$.

REMARK 5.4. To save space, Prop. 5.1 left a $\bullet$ for the 3rd term of ${}_{k,u}\boldsymbol{g}$ and ${}_{k,u}\boldsymbol{g}'$. We fill them for Prop. 5.15: $(x_{\frac{k-1}{2},\frac{n-1}{2}}\, x_{\frac{n+3}{2},\frac{n+k}{2}})$ and $x_{\frac{k+1}{2},\frac{n+k}{2}}$ in (5.2).

5.2. Completing inner sh-incidence entries. The absolute sh-incidence entries are in (5.8). This section produces the inner ${}_{\mathrm{r}}$cusp sh-incidence matrix for which we have only to find $\beta_{k,u}$ (resp. $\beta'_{k,u}$) from (5.5c) (resp. (5.5d)).

§5.2.1 does this when $u \le \frac{k-1}{2}$. §5.2.2 completes the rest and §5.2.3 combines them for the inner ${}_{\mathrm{r}}$cusp sh-incidence display using the §5.1.3 absolute display.

5.2.1. *sh-incidence parities — 1st half.* Start with $0 < u < \frac{k-1}{2}$: Table 2 handled $u = 0$ and $u = \frac{k-1}{2}$ appears in Cor. 5.6. This gives the 1st (resp. 2nd) entry of (5.2) as $g_1' \stackrel{\mathrm{def}}{=} {}_{k,u}g_1' = (x_{n,\frac{n+k+2}{2}}\, x_{\frac{n+k-2u}{2},\frac{n+3}{2}}\, x_{1,1+u})$ (resp. g_2').

As in (5.5c), for such (k,u) (with $\ell = n-2u$), find $\beta_{k,u} \in S_{\frac{n+3}{2},n}$ conjugating g_1' to $(x_{\frac{n+2u+3}{2},n}\, x_{1,1+u})$, the 1st entry g_1'' of $[\ell]_2$.

The element $\beta_{k,u} = \beta_{k,u,1}\beta_{k,u,2}^u$ that works comes from these two (Lem. 5.5):

(5.9) $\beta_{k,u,2} = x_{\frac{n+3}{2},\frac{n+k}{2}}$ (shifts $|x_{\frac{n+k-2u}{2},\frac{n+3}{2}}|$ to $|x_{\frac{n+k-2(u-1)}{2},\frac{n+5}{2}}|$); and $\beta_{k,u,1}$ inverting $|x_{\frac{n+2u+3}{2},n}|$ (Lem. 1.8).

Finally, consider $u = \frac{k-1}{2}$. In the previous notation, we drop the x'' segment in $g_1' = (x_{n,\frac{n+k+2}{2}}|x_{1,\frac{k+1}{2}})$, while $g_2' = (\frac{k+1}{2}\, x_{\frac{n+k}{2},\frac{n+3}{2}}|x_{\frac{n+k+2}{2},n})$ retains its previous form. Then, the resulting $\beta_{k,\frac{k-1}{2}}$ is just $\beta_{k,\frac{k-1}{2},1}$: invert $|x_{\frac{n+k+2}{2},n}|$.

Take $a_2 = (-1)^{u\cdot(\frac{k-3}{2})}$. With $u = \frac{k-1}{2}$, a_2 is $(-1)^{\frac{k-1}{2}\cdot(\frac{k-3}{2})} = 1$ shows the general computation for $\beta_{k,u}$ even works for $u = \frac{k-1}{2}$.

LEMMA 5.5. *If $\beta = (i_1\, i_2\, \dots\, i_k)$ (parity $(-1)^{k-1}$) and $U = |i_1\, i_2\, \dots\, i_j|$, $1 \le j < k$, is in pure-cycle α, then, $\beta\alpha\beta^{-1}$ substitutes $|i_2\, i_3 \dots\, i_{j+1}|$ (right β-shift) for U. So, $\beta_{k,u,2}^u$ has parity a_2. Also, $\beta_{k,u,1}$ has parity $(-1)^{\frac{n-2u-3}{4}} = (-1)^{\frac{1-u}{2}}$ (resp. $(-1)^{\frac{n-2u-1}{4}} = (-1)^{1-\frac{u}{2}}$) if u is odd (resp. even).*

Conclude: $\beta_{k,u}$ has parity $(-1)^{\frac{1-u}{2}+\frac{k-3}{2}}$ (resp. $(-1)^{1-\frac{u}{2}}$) if u is odd (resp. even).

With ${}_{k,u}\boldsymbol{g}^\dagger \overset{\text{def}}{=} \beta_{k,u}({}_{k,u}\boldsymbol{g}')\beta_{k,u}^{-1}$ (even for $u=\frac{k-1}{2}$), consider $g_2^\dagger$

$$\begin{aligned} &= \beta_{k,u}(x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+k}{2},\frac{n+k+2-2u}{2}}\, x_{\frac{n+k+2}{2},n})\beta_{k,u}^{-1}\\ &= \beta_{k,u,1}(x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+2u+1}{2},\frac{n+3}{2}}\, x_{\frac{n+k+2}{2},n})\beta_{k,u,1}^{-1}\\ &= \qquad (x_{1+u,\frac{k+1}{2}}\, x_{\frac{n+2u+1}{2},\frac{n+3}{2}}\, x_{n-\frac{k-1-2u}{2},\frac{n+2u+3}{2}}). \end{aligned} \tag{5.10}$$

Recall $\alpha_{n-2u,2}=(x_{1+u,\frac{n+1}{2}}\, x_{n,\frac{n+2u+3}{2}})$ (Lem. 4.3), and $g_2''=(x_{1+u,\frac{n+2u+1}{2}})$, the 2nd entry of $[n{-}2u]_2$. By inspection, $\beta'_{k,u}\overset{\text{def}}{=}\beta'$ that inverts $|x_{\frac{n+2u+1}{2},\frac{n+3}{2}}|$ conjugates the orbit of $\langle\alpha_{n-2u,2}\rangle$ to contain g_2''. Combine this with Lem. 5.5 for $1\le u\le\frac{k-1}{2}$.

Corollary 5.6. *For u odd (resp. even), β' has parity $(-1)^{\frac{u-1}{2}}$ (resp. $(-1)^{\frac{u}{2}}$). So, for u odd,* (5.2) *is in ${}_{\mathbf{c}}O'_{n-2u,2}$ if and only if* $k\equiv 3 \mod 4$. *Otherwise, it is in ${}_{\mathbf{c}}O'_{n-2u,1}$. For u even,* (5.2) *is in ${}_{\mathbf{c}}O'_{n-2u,2}$.*

Proof. Apply Lem. 1.8 for the 1st sentence parities. If u is odd (resp. even), then Lem. 4.4 says ${}_{\mathbf{c}}O'_{[n-2u]_2}={}_{\mathbf{c}}O'_{n-2,2}$ (resp. ${}_{\mathbf{c}}O'_{n-2,1}$). Then, from Lem. 5.5, the parity of $\beta_{k,u}\beta'$ is $+1$ iff either u is even or u is odd and $k\equiv 3 \mod 4$. □

5.2.2. *sh-incidence parities — 2nd half.* Now consider $\frac{k+1}{2}\le u\le k{-}1$. With $u'=k{-}1{-}u$, compare 1st and 2nd entries of $[n{-}2u']_2$ with corresponding entries g_1',g_2' of (5.4). 1st conjugate (5.4) by $\beta_{k,u'}\in S_{\frac{n+3}{2},n}\times\langle x_{\frac{n+1}{2},1}\rangle$ to get ${}_{k,u'}\boldsymbol{g}^\dagger$ whose 1st and 4th entries match those of $[n{-}2u']_2$. A $\beta_{k,u'}$ inverting both segments $|x_{\frac{n+k}{2},\frac{n+3+2u'}{2}}|$ and $|x_{n,\frac{n+k+2}{2}}|$ does it. An induction computes its parity.

Lemma 5.7. *With t odd, the conjugation $\alpha_{t,j}\in S_t$ that inverts both segments $|x_{1,j}|$ and $|x_{j+1,t}|$, $1\le j\le t{-}1$ has — independent of j — parity $(-1)^{\frac{t-1}{2}}$. With t even, the analogous result is that $\alpha_{t,j}$ has parity $(-1)^{\frac{t-2j}{2}}$.*

Apply $t=\frac{n-1-2u'}{2}$ to $\beta_{k,u'}$: For u' odd (resp. even), $\beta_{k,u'}$ has parity

$(-1)^{\frac{n-3-2u'}{4}}=(-1)^{\frac{1-u'}{2}}$ *(resp.* $(-1)^{1-\frac{u'}{2}}$ *if* $k\equiv 1 \mod 4, (-1)^{\frac{u'}{2}}$ *if* $k\equiv 3 \mod 4$).

Proof. The 1st paragraph is easy. For the 2nd, we do the toughest case, when u', so t, is even. The lengths of the two segments are $\frac{n-k}{2}$ and $\frac{k-1-2u'}{2}$. These are even (resp. odd) if $k\equiv 1 \mod 4$ (resp. $k\equiv 3 \mod 4$). According to the 1st paragraph, the parity in each of these cases will be the same as for the case when the seqments have length 2 and $\frac{n-5-2u'}{2}$ (resp. 1 and $\frac{n-3-2u'}{2}$). □

We insert dividers in the relevant permutations to see the effect of a middle product translation. Denote the 2nd entry (as previously) of $[n{-}2u']_2$ by

$$g_2''=x_{1+u',\frac{n+1}{2}+u'}=(x_{1+u',\frac{n+1}{2}}|x_{\frac{n+3}{2},\frac{n+1}{2}+u'}).$$

Denote the parity for (k,u') in Lem. 5.7 by $b_{k,u',1}$. Note: $\alpha_{n-2u',2}$ translates the segment $|x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\, x_{n,\frac{n+k}{2}}|$ of $g_2^\dagger$ in (5.11) into $|x_{1+u',\frac{n+1}{2}}|$ of g_2''.

Lemma 5.8. *Assume $u'>0$. So, for β' inverting $x_{\frac{n+3}{2},\frac{n+1}{2}+u'}$ and some j:*

$$\begin{aligned} \text{with } g_2^\dagger &= (x_{\frac{n+1}{2}-(\frac{k-3-2u'}{2}),\frac{n+1}{2}}\, x_{n,\frac{n+k}{2}}|x_{\frac{n+1}{2}+u',\frac{n+3}{2}}),\\ (\beta')g_2^\dagger(\beta')^{-1} &= \alpha^j_{n-2u',2}g_2''\alpha^{-j}_{n-2u',2} \end{aligned} \tag{5.11}$$

Then, for u' odd (resp. even), β' has parity $b_{k,u',2}=(-1)^{\frac{u'-1}{2}}$ (resp. $(-1)^{\frac{u'}{2}}$).

Conclude: (5.4) *is in* ${}_{\mathbf{c}}O'_{[n-2u']_2,1}$ *if and only if* $b_{k,u',1}b_{k,u',2}=1$. *For* u' *odd,* (5.4) *is in* ${}_{\mathbf{c}}O'_{n-2u',2}$. *For* u' *even,* (5.4) *is in* ${}_{\mathbf{c}}O'_{n-2u',2}$ *if and only if* $k\equiv 1 \mod 4$.

Now combine Cor. 5.6 and Lem. 5.8, with $0<u=u'<\frac{k-1}{2}$, where u (resp. u') corresponds to ${}_{k,u}\boldsymbol{g}'\in({}_{\mathbf{c}}O'_{k,1})\mathbf{sh}$ (resp. ${}_{k,u}\boldsymbol{g}^*\in({}_{\mathbf{c}}O'_{k,1})\mathbf{sh}$) in (5.2) (resp. (5.4)).

Corollary 5.9. *For* u *even:* ${}_{k,u}\boldsymbol{g}'\in{}_{\mathbf{c}}O'_{n-2u,2}$; *and* ${}_{k,u}\boldsymbol{g}^*\in{}_{\mathbf{c}}O'_{n-2u,2}$ *(resp.* ${}_{\mathbf{c}}O'_{n-2u,1}$*) if and only if* $k\equiv 1 \mod 4$ *(resp.* $k\equiv 3 \mod 4$*).*

For u *odd:* ${}_{k,u}\boldsymbol{g}^*\in \mathrm{Cusp}_{n-2u,2}$; *and* ${}_{k,u}\boldsymbol{g}'\in{}_{\mathbf{c}}O'_{n-2u,2}$ *(resp.* ${}_{\mathbf{c}}O'_{n-2u,1}$*) if and only if* $k\equiv 3 \mod 4$ *(resp.* $k\equiv 1 \mod 4$*). Always:* ${}_{k,\frac{k-1}{2}}\boldsymbol{g}'\in{}_{\mathbf{c}}O'_{n-k+1,2}$.

Remark 5.10 ($u'=0$ in Lem. 5.8). The case $u'=0$ for inverting $x_{\frac{x+3}{2},1+u'}$ differs from the others: the range is descending, not ascending. The final parity value $(-1)^{u'}$ is also wrong, but Table 2 already handled that case.

5.2.3. **sh**-*incidence display.* Use Defs. 4.6 and 4.7 (as in Table 2), for $n\equiv 5 \mod 8$ to label columns/rows of the general ${}_{\prime\prime}$cusp sh-incidence matrix as follows:

$${}_{\mathbf{c}}O'_{n;1},\ {}_{\mathbf{c}}O'_{n;2},\ {}_{\mathbf{c}}O'_{n-2;1},\ {}_{\mathbf{c}}O'_{n-2;2},\dots,{}_{\mathbf{c}}O'_{3;1},\ {}_{\mathbf{c}}O'_{3;2},\ {}_{\mathbf{c}}O_{1,2}.$$

Use the subscripts of the symbols to label its entries as $((\ell;j),(\ell';j'))$. By shortening this to $(\bar\ell,\bar\ell')$, we can throw in the (1,2) subscript as another $\bar\ell$.

§4.1.4 lists cusps and their widths. Conclude: $2\cdot(\frac{n+1}{2})^2$ is the cover degree of

$$\bar\psi_{(\frac{n+1}{2})^4}:\bar{\mathcal{H}}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}\stackrel{\mathrm{def}}{=}\mathcal{H}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}\to\mathbb{P}^1_j.$$

Principle 5.11. *For* $n\equiv 5 \mod 8$, *the* $(\bar\ell,\bar\ell')$ *are all 0, 1 or 2. Also:*

(5.12a) *Symmetry:* $(\bar\ell,\bar\ell')=(\bar\ell',\bar\ell)$.

(5.12b) *Width sum: Entries in the row for* ${}_{\mathbf{c}}O'_{\ell;j}$ *(resp.* ${}_{\mathbf{c}}O_{1,2}$*) sum to* ℓ *(resp. 2).*

(5.12c) *With* $0\le u<\frac{k-1}{2}$ *or (resp.* $u=\frac{k-1}{2}$*),* $((\ell;1),(\ell';1))+((\ell;1),(\ell';2))=2$ *(resp. 1) if* $\ell'=n-2u$*; and 0 otherwise.*

To describe the ${}_{\prime\prime}$cusp sh-incidence matrix of $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$ use §5.1.3 for $\mathrm{Ni}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$. You get the former from the latter using symmetry and these two rules.

Rule 1: At an entry (ℓ,ℓ') (odd integers) with $3\le\ell\le\ell'\le n$: Replace 2 (resp. 1; resp. 0) by one of these matrices

$$\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix};(\text{ resp. }\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix};\text{ resp. }\begin{pmatrix}0&0\\0&0\end{pmatrix}).$$

In each case, you determine the replacing matrix from just one of its entries.

Rule 2: At the entry $(1,n)$ (resp. $(1,\ell)$, ℓ odd, between 3 and $n-2$; resp. $(1,1)$): Replace 1 by (1 1) (resp. (0 0); resp. 0).

Use symbols such as $\frac{2|0}{0|2},\frac{0|2}{2|0},\frac{1|1}{1|1}$, etc. to substitute for the 2×2 matrices, and $1|1, 0|0$ (resp. $\frac{1}{1},\frac{0}{0}$) for the 2×1 (resp. 1×2) matrices. Refer to the matrix of such substitutions as *abs-inn* form.

We already know the last three sh-incidence rows (and from Table 2 the first two), including all the data of Rule 2. The first two rows of Table 5 are from the opening paragraph of Prop. 5.1, with the last from the first line of Prop. 4.1. Table 6 renders this with even more detail in the abbreviated abs-inn form:

Use the anti-subdiagonal notation of §5.1.3. For $n=13$, Table 7 gives the *abs-inn* form of the matrix. For example, the 1-1 a-d (read from lower left to upper right) consists of $1|1$, then five $\frac{0|1}{1|0}$'s, followed by $\frac{1}{1}$. Below the 1-1 a-d are $\frac{0|0}{0|0}$'s

TABLE 5. Rows for ${}_{\mathbf{c}}O'_{3,1}$, ${}_{\mathbf{c}}O'_{3,2}$ and ${}_{\mathbf{c}}O_{1,2}$

Cusp orbit	${}_{\mathbf{c}}O'_{n;1}$	${}_{\mathbf{c}}O'_{n;2}$	${}_{\mathbf{c}}O'_{n-2;1}$	${}_{\mathbf{c}}O'_{n-2;2}$	$\dots$	${}_{\mathbf{c}}O'_{3;1}$	${}_{\mathbf{c}}O'_{3;2}$	${}_{\mathbf{c}}O_{1,2}$
${}_{\mathbf{c}}O'_{3;1}$	1	1	0	1	$\dots$	0	0	0
${}_{\mathbf{c}}O'_{3;2}$	1	1	1	0	$\dots$	0	0	0
${}_{\mathbf{c}}O_{1,2}$	1	1	0	0	$\dots$	0	0	0

TABLE 6. Abs-inn form of the rows ${}_{\mathbf{c}}O_3$ and ${}_{\mathbf{c}}O_1$

Cusp orbit	${}_{\mathbf{c}}O_n$	${}_{\mathbf{c}}O_{n-2}$	${}_{\mathbf{c}}O_{n-4}$	$\dots$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_3$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\dots$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_1$	$1\mid 1$	$0\mid 0$	$0\mid 0$	$\dots$	$0\mid 0$	0

TABLE 7. Abs-inn ${}_{\eta}$cusp form for $n = 13$

Cusp orbit	${}_{\mathbf{c}}O_{13}$	${}_{\mathbf{c}}O_{11}$	${}_{\mathbf{c}}O_9$	${}_{\mathbf{c}}O_7$	${}_{\mathbf{c}}O_5$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_{13}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{1}{1}$
${}_{\mathbf{c}}O_{11}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_9$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}^{0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_7$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}^{1}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_5$	$\frac{0\mid 2}{2\mid 0}$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_3$	$\frac{1\mid 1}{1\mid 1}$	$\frac{0\mid 1}{1\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0\mid 0}{0\mid 0}$	$\frac{0}{0}$
${}_{\mathbf{c}}O_1$	$1\mid 1$	$0\mid 0$	$0\mid 0$	$0\mid 0$	$0\mid 0$	$0\mid 0$	0

TABLE 8. Abs-inn ${}_{\eta}$cusp form for $n = 13$

Cusp orbit	${}_{\mathbf{c}}O_{13}$	${}_{\mathbf{c}}O_{11}$	${}_{\mathbf{c}}O_9$	${}_{\mathbf{c}}O_7$	${}_{\mathbf{c}}O_5$	${}_{\mathbf{c}}O_3$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_9^-$	$2\mid 2$	$2\mid 2$	4	$2\mid 2$	$1\mid 1$	$0\mid 0$	0
${}_{\mathbf{c}}O_7^-$	$2\mid 2$	$2\mid 2$	$2\mid 2$	2	$0\mid 0$	$0\mid 0$	0

except at the right edge ($\frac{0}{0}$), bottom edge ($0\mid 0$), or lower right corner (0). Table 2 says column ${}_{\mathbf{c}}O_n$ fills down as $\frac{0\mid 2}{2\mid 0} \to \frac{n-3}{2}\ \frac{1\mid 1}{1\mid 1}$'s $\to 1\mid 1$. Superscripts 0 and 1 along the diagonal indicate fixed points for γ'_0 and γ'_1 a la Prop. 5.12 and Prop. 5.15.

The distinction for the sh-incidence matrix for $n = 13$ requires replacing the rows and columns respectively for ${}_{\mathbf{c}}O_7$ and ${}_{\mathbf{c}}O_9$. The process, say in the columns is to add the contributions for ${}_{\mathbf{c}}O'_{k,1}$ and ${}_{\mathbf{c}}O'_{k,2}$ across the rows, but since you also do this with rows replacing columns, that adds extra at the diagonal terms. Table 8 gives the row result, using the notation ${}_{\mathbf{c}}O_k^-$ for the collapsed form:

5.3. Elliptic fixed points. Thm. 2.9 outlines the absolute and inner sh-incidence matrices for the Nielsen classes $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$. While related through Lem. 2.11, we need more to complete finding the fixed points of γ'_0 and γ'_1 on Nielsen classes. We do that here to compute the absolute and reduced space genuses.

5.3.1. *An absolute γ'_1 fixed point.* Lem. 5.12 locates a γ'_1 fixed point that ramifies in the cover from the absolute space to the inner space.

LEMMA 5.12. *For $n \equiv 1 \mod 4$, the 1-1* a-d *diagonal term in the absolute sh-incidence corresponds to $({}_{\mathbf{c}}O_{\frac{n+1}{2}}, {}_{\mathbf{c}}O_{\frac{n+1}{2}})$. À la Lem. 2.11, this arises from a fixed* *point* $\boldsymbol{p}'_1 \in \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$ *of* γ'_1 *(lying over* $j=1$*). The only* n*-*n a-d *term (a 2, on the diagonal) does not correspond to points fixed by* γ'_v*,* $v=0$ *or 1.*

For $n \equiv 5 \mod 8$*, the inner sh-incidence diagonal terms above the 1 in the absolute sh-incidence are both 0. This means* γ'_1 *fixes no point of* $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$ *over* $\boldsymbol{p}'_1$*. So, this and the width 1 cusp give two known points of* $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$ *that ramify to* $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$*.*

Proof. For any $n \equiv 1 \mod 4$, Prop. 5.1 says a unique absolute (reduced) class (represented by ${}_{\frac{n+1}{2},\frac{n-1}{4}}\boldsymbol{g}'$) of the cusp ${}_{\mathbf{c}}O_{\frac{n+1}{2}}$ shifts into ${}_{\mathbf{c}}O_{\frac{n+1}{2}}$. Since the sh-incidence entry is 1, Lem. 2.11 says this corresponds to a fixed point of γ'_1.

Now consider the n-n a-d. Table 1 explains the 2: $[n]_j$, $j=1,2$ represent the two reduced classes in ${}_{\mathbf{c}}O_n \cap ({}_{\mathbf{c}}O_n)\mathbf{sh}$. We show neither γ_0 nor γ_1 fixes either. In $\bar{M}_4$, the shift represents γ_1, and that sends $[n]_1^{\rm abs,rd}$ to $\text{H-M}^{\rm abs,rd}_{1,\frac{n-1}{2}}$.

Lem. 2.11 says this either fixes $[n]_1^{\rm abs,rd}$ or it sends it to $[n]_2^{\rm abs,rd}$. It is the latter: Conjugate $([n]_1)\mathbf{sh}$ by $(n\,1)(n\,2)\cdots(\frac{n+3}{2}\ \frac{n-1}{2})$ followed by $(n\ \frac{n+3}{2})(n{-}1\ \frac{n+3}{2})\cdots$. The resulting 3rd and 4th terms are $x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}$ (same as $[n]_2$). Check: The resulting 1st term is also the same as that of $[n]_2$.

Characterize that γ_0 fixes $[n]_1$: $(([n]_1)q_2^{-1})\mathbf{sh} \in [n]_1^{\rm abs,rd}$. From Prin. 3.5, q_2^{-1} has the effect of conjugating the 2nd and 3rd terms of $[n]_1$ by $([n]_1)\mathbf{mp}^{\frac{n-1}{2}}$. The combined effect is to send $[n]_1$ back to the shift of the H-M rep. it came from in Table 1. So, it has the wrong middle product to be fixed by γ_0.

For $n \equiv 5 \mod 8$, the symbol (as in §5.2.3) for what lies above the diagonal 1-1 a-d in the inner sh-incidence matrix is $\frac{0|1}{1|0}$, the last sentence of Cor. 5.9. At the diagonal position, it means there are no fixed points of γ'_1 (or γ'_0) over $\boldsymbol{p}'_1$. □

5.3.2. γ'_0 *fixed points.* For $n \equiv 5 \mod 8$, the degree 2 cover (Prop. 4.4)

$$\bar{\Psi}_n^{\rm in,abs} : \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd} \to \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd}$$

maps γ'_0 fixed points on the upper surface 2-1 (on)to γ'_0 fixed points to the lower. Consider the reduced class of ${}_{k,u}\boldsymbol{g} = (g_1,g_2,g_3,g_4)$ (from Prop. 5.1), and denote the common support cardinality of the pair $(g_1g_2g_1^{-1}, g_4)$ by $\nu_{k,u}$. Recall (Lem. 4.3): The degree of $\bar{\Psi}_n^{\rm abs} : \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm abs,rd} \to \mathbb{P}^1_j$ is $(\frac{n+1}{2})^2$.

Proposition 5.13. *Inner sh-incidence diagonal positions correspond to* (k,u) *with* $u = \frac{n-k}{2}$ *or* $\frac{3k-n-2}{2}$ *(subject to the latter being* ≥ 0 *and* $0 \le u \le k$*). For such a* (k,u)*, if the reduced absolute class of* ${}_{k,u}\boldsymbol{g} = (g_1,g_2,g_3,g_4)$ *is a fixed point of* γ'_0*,* $\nu_{k,u}$ *equals the common support cardinality of the pair* (g_1,g_4)*. Then:*

(5.13a) *For* $0 \le u \le \frac{k-1}{2}$*:* $\nu_{k,u} = \frac{k-1}{2} - u$.

(5.13b) *For* $\frac{k+1}{2} \le u \le k{-}1$ *and* $u' = k{-}1{-}u$*:* $\nu_{k,u} = \frac{k-1-2u'}{2} + 1$.

If $3|n$ *(resp.* $n \equiv 1 \mod 3$*), then* ${}_{k,\frac{n-k}{2}}\boldsymbol{g}$ *(resp.* ${}_{k,\frac{3k-n-2}{2}}\boldsymbol{g}$*) with* $3k = 2n+3$ *(resp.* $2n+1 = 3k$*) represents the only possible absolute diagonal fixed point of* γ'_0*. If* $n \equiv 0$ *or* $1 \mod 3$*, then* $\deg(\bar{\Psi}_n^{\rm abs}) \equiv 1 \mod 3$*, and this is the one absolute fixed point of* γ'_0*. There are none if* $n \equiv -1 \mod 3$*.*

Proof. From Prop. 5.1, among values of $u \le \frac{k-1}{2}$ one diagonal position is from (k,u_0) with $n{-}2u_0 = k$, or $u_0 = \frac{n-k}{2}$. A 2nd from $u_1 = u_0 + 2(\frac{k-1}{2}{-}u_0)$, giving the other u value in the lemma statement.

For $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$, Princ. 3.7 says if there are t integers of common support in g_i and g_j, $1 \le i \ne j \le 4$, then $\mathrm{ord}(g_i g_j) = 2(\frac{n+1}{2}) - 2t + 1$. That is, $\mathrm{ord}(g_i g_j)$ determines the common support of g_i, g_j.

Now suppose the reduced (absolute or inner) class of ${}_{k,u}\boldsymbol{g}$ is fixed by $q_1 q_2$ (that is by γ_0). Then, $({}_{k,u}\boldsymbol{g})\mathbf{mp} = (({}_{k,u}\boldsymbol{g})q_1q_2)\mathbf{mp}$. Applying q_2 doesn't change the middle product. So, $\mathrm{ord}(({}_{k,u}\boldsymbol{g})q_1)\mathbf{mp}) = \mathrm{ord}(g_1 g_3) = \mathrm{ord}(g_4 g_1 g_2 g_1^{-1})$, computed again by a cardinality of overlap. Since the common support of (g_1, g_4) is the same as the common support of (g_2, g_3), that completes the statement before (5.13).

We have fixed g_4 to be $x_{\frac{n+1}{2},1}$. As in Prop. 5.1, denote $k{-}1{-}u$ by u'. With $\alpha_{k,1} = (x_{1,\frac{k-1}{2}}\, x_{\frac{n+k}{2},\frac{n+1}{2}})$ (Lem. 4.3), for a given (k,u) we figure the cardinality, $v_{k,u}$, of the support in $\{1, \dots, \frac{n+1}{2}\}$ of $g_1 g_2 g_1^{-1} \stackrel{\text{def}}{=}$

$$(5.14)\quad \begin{aligned} &\alpha_{k,1}^u (x_{n,\frac{n+1}{2}} (x'_{\frac{n+1}{2},\frac{k-1}{2}}\, x_{\frac{n+k+2}{2},n}) x_{\frac{n+1}{2},n}) \alpha_{k,1}^{-u} = \alpha_{k,1}^u (x_{\frac{n+k}{2},n}\, x_{1,\frac{k-1}{2}}) \alpha_{k,1}^{-u} \\ &= \begin{cases} (x_{\frac{n+k}{2},\frac{n+k-2u}{2}}\, x_{\frac{n+k+2}{2},n}\, x_{1+u,\frac{k-1}{2}}) \text{ for } 0 \le u < \frac{k-1}{2} \\ (x_{\frac{n+2u'+1}{2},\frac{n+3}{2}}\, x'_{\frac{n+1}{2},\frac{k-1-2u'}{2}}\, x_{\frac{n+k+2}{2},n}) \text{ for } \frac{k+1}{2} \le u \le k{-}1. \end{cases} \end{aligned}$$

To see the 2nd case, substitute $\alpha_{k,1}^{u'+1}$ for $\alpha_{k,1}^{-u}$. Clearly, $v_{k,u}$ has the values in (5.13).

The conclusion on the possible representatives of fixed points follows by equating k and $2(\frac{n+1}{2}) - 2\nu_{k,u} + 1$ for values of u at the diagonal positions. For (5.13a), $\nu_{k,\frac{n-k}{2}} = \frac{k-1}{2} - \frac{n-k}{2}$. For (5.13b), with $u' = (k{-}1) - \frac{3k-n-2}{2}$,

$$\nu_{k,\frac{3k-n-2}{2}} = 1 + \tfrac{k-1-2u'}{2} = \tfrac{2k-n+1}{2}.$$

Then, $k = 2(\frac{n+1}{2}) - (2k{-}n{+}1) + 1$ completes the condition in the statement. □

Example 5.14 (γ_0 fixed point, $n = 13$). Use Cor. 5.9 notation. Prop. 5.13 says γ_0 fixes $({}_{9,2}\boldsymbol{g}^*)^{\mathrm{in}}$ when $n = 13$. Apply $q_1 q_2$, and conjugate by $x_{7,1}^2$ to get

$$((x_{9,8}\, x_{2,4}\, x_{12,13}), (x_{10,11}\, x'_{7,1}\, x_{8,9}\, 13), (x_{13,12}\, x_{2,4}\, x_{11,10}), x_{7,1}).$$

Conjugate in order by $\beta_3 = x_{11,8}^2$, $\beta_2 = x_{12,13}$ and $\beta_1 = x_{13,10}^2$ to get

$$((x_{13,12}\, x_{2,4}\, x_{11,10}), (x_{8,9}\, x'_{7,1}\, x_{12,13}\, 10), \bullet, x_{7,1})$$

Then, conjugate by $\beta' = (8\,9)$, with $\beta'\beta_1\beta_2\beta_3$ of parity $+1$ to get back to ${}_{9,2}\boldsymbol{g}^*$. This shows γ_0 fixes $({}_{9,2}\boldsymbol{g}^*)^{\mathrm{in}}$. It also fixes $(\beta'{}_{9,2}\boldsymbol{g}^*(\beta')^{-1})^{\mathrm{in}}$. This works on γ_0' fixed points for all $n \equiv 5 \mod 8$.

5.3.3. *Genuses of the inner and absolute spaces.* Prop. 5.15 finishes the elliptic fixed point analysis, giving absolute/inner reduced space genuses for $n \equiv 5 \mod 8$. (Also for $n \equiv 1 \mod 8$ for the absolute case; as in Lem. 4.4, inner components have the same genus.) Note: For t an integer, $[(3t+m)/3]$ is t, for $m = 0$ or 1.

Proposition 5.15. *Let $v = [n/5]$. Then, γ_1' has only one absolute fixed point and no inner fixed points. Exactly $2v + 2$ points of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ ramify in $\bar{\Psi}_n^{\mathrm{in,abs}}$: The width 1 cusp, the point over $j = 1$ indicated in Lem. 5.12, and the $2v$ cusps ${}_{\mathbf{c}}O_{7+8m}, {}_{\mathbf{c}}O_{9+8m}$, $m = 0, \dots, v{-}1$.*

Conclude the following formulas for the respective genuses, $\mathbf{g}^{\mathrm{abs,rd}}$ and $\mathbf{g}^{\mathrm{in,rd}}$, of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ and $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$.

$$(5.15)\quad \begin{aligned} 2((\tfrac{n+1}{2})^2 + \mathbf{g}^{\mathrm{abs,rd}} - 1) &= ((\tfrac{n+1}{2})^2 - 1)/2 + 2[(\tfrac{n+1}{2})^2/3] + (\tfrac{n-1}{2})(\tfrac{n+1}{2}) \\ 2(2(\tfrac{n+1}{2})^2 + \mathbf{g}^{\mathrm{in,rd}} - 1) &= (\tfrac{n+1}{2})^2 + 4[(\tfrac{n+1}{2})^2/3] + 2(\tfrac{n-1}{2})(\tfrac{n+1}{2}) + 1 + v. \end{aligned}$$

PROOF. To check for fixed points of γ'_1 consider first the absolute case. For each odd k between 3 and $n-2$ (1 and n are already done), and $u = \frac{n-k}{2} < \frac{k-1}{2}$, we are asking if ${}_{k,u}\boldsymbol{g}^\dagger \stackrel{\text{def}}{=} (({}_{k,u}\boldsymbol{g}')\mathbf{sh})q_1^{-1}q_3$ (as in (5.2)) is conjugate by some $\beta \in S_{\frac{n+3}{2},n} \times \langle x_{\frac{n+1}{2},1}\rangle$ to ${}_{k,u}\boldsymbol{g}'$. The point of applying $q_1^{-1}q_3$ is to assure – without changing the reduced class – that they both have $x_{\frac{n+1}{2},1}$ as their 4th entry.

With $g_1^\dagger = (x_{\frac{2k-n-1}{2},\frac{k-1}{2}}\, x_{\frac{n+3}{2},\frac{n+k}{2}})$, use Rem. (5.4) to fill in ${}_{k,\frac{n-k}{2}}\boldsymbol{g}^\dagger$:

$$(g_1^\dagger, (g_1^\dagger)^{-1}(x'_{\frac{n+1}{2},\frac{2k-n-1}{2}}\, x_{\frac{n+k}{2},k+1}\, x_{\frac{n+k+2}{2},n})g_1^\dagger, \bullet, x_{\frac{n+1}{2},1}).$$

The assumption on β means it conjugates $g_1^\dagger$ to g_1'. So, the power of $x_{\frac{n+1}{2},1}$ in it would translate the segment $|x_{\frac{2k-n-1}{2},\frac{k-1}{2}}|$ to $|x_{\frac{k+1}{2},\frac{n+1}{2}}|$. That is, you add $\frac{n-k+2}{2}$ to the subscripts. That means a conjugation in the second position of the form $g''' = (x_{\frac{k+1}{2},\frac{n+1}{2}}\dots)^{-1}(x_{\frac{n-k+2}{2},\frac{k+1}{2}}\dots)(x_{\frac{k+1}{2},\frac{n+1}{2}}\dots)$ would end up as $(x'_{\frac{n+1}{2},\frac{2k-n-1}{2}}\dots)$. In each case "$\dots$" means integers in $\{\frac{n+3}{2},\dots,n\}$.

So, it seems g''' maps $\frac{n+1}{2} \mapsto 1$ for k in the allowed range. Yet, the constituents of g''' don't even have 1 in their supports. Conclude: no appropriate β gives even an absolute fixed point.

Now we compute the genuses. Lem. 4.3 gives $(\frac{n+1}{2})^2$ as the degree of

$$\bar\Psi_n^{\text{abs}} : \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}} \to \mathbb{P}^1_j$$

and the lengths of the disjoint cycles of the absolute γ'_0 as all the odd integers from 1 to n with multiplicity 1. So γ'_∞ has index $\frac{n+1}{2}(\frac{n+1}{2}-1)$. Write $1+4t=n$. According to Prop. 5.15, $\text{ind}(\gamma'_0)$ is $2\cdot((\frac{n+1}{2})^2-1)/3$ (resp. $2\cdot(\frac{n+1}{2})^2/3$) if $t \equiv 0$ or $-1 \mod 3$ (resp. $1 \mod 3$). Similarly, from the above $\text{ind}(\gamma'_1)$ is $((\frac{n+1}{2})^2-1)/2$ (indicating one fixed point).

The degree doubles in going to the inner case (Lem. 4.4). Above we've computed the indices from the contributions of γ'_0 and γ'_1 in the inner case, leaving the contribution of γ'_∞ (in (2.11)). The cusps $\mathcal{O}_1, \mathcal{O}_{7+m8}, \mathcal{O}_{9+m8}$ ramify of index 2 in the cover. So, if we denote the width of one of these by k, then its index is $k-1$ and the index of the cusp above it is $2k-1 = 2(k-1)+1$. For all the other absolute cusps, there there are exactly two inner cusps of the same width above it. The calculations come directly from this. □

EXAMPLE 5.16 (No converse to Lem. 2.11). Assume $n \equiv 1 \mod 4$; as in (5.8) or Table 4. Then, sh-incidence matrices for $\text{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$ have a 1 and $\frac{n-1}{2}$ 2's as diagonal entries. Lem. 5.12 shows the diagonal term 1 in the 1-1 a(nti)-(sub)d(iagonal) corresponds to a fixed point of γ'_1. While the 2 in the n-n a-d corresponds to no fixed points of either γ'_0 or γ'_1.

In the inner case: Table 8, for the **sh**-incidence matrix labeled Ni_0^+, has a nonzero diagonal entry, though neither γ'_0 nor γ'_1 has a fixed point.

6. Modular towers over $\text{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$

Shimura varieties are moduli spaces renown for having attached ℓ-adic representations to each of their points. There are – no matter how you do the division – two very different types of Shimura varieties. The type we relate to here are those modeled on Siegel space generalizations of modular curve towers.

Abelianized Modular Towers give a related generalization – we refer to these as *ℓ-adic support spaces* – of modular curves towers. §6.1 gives the general context for these particular **MT**s, especially the two properties that we expect of them appropriate for generalizing Serre's Open Image Theorem. We refer to the two types coming from our A_n examples as *pure-split* and *pure-Frattini* in (6.2). Our ℓ-adic support spaces arise from the results of the previous section. The examples are so sweeping, yet structured, I decided not to make the treatment axiomatic.

The ℓ-adic support spaces have dimension 1 (upper half-plane quotients, and j-line covers). Yet they are non-trivial generalizations of the modular curve case. That is, they aren't pullback of some modular curve tower over a single upper-half plane cover $X \to \mathbb{P}^1_j$ of the j-line with X as a parameter space (Prop. 6.4).

§6.1.3 gets explicit on the pure-Frattini type by considering $G_{1,2}(A_n)$, the 1st universal 2-Frattini extension of A_n – a centerless 2-perfect group – replacing A_n. The main result here is that there is a modular curve-like structure to the tree of cusps in this tower, despite level 0 having most of its cusps quite alien to those of modular curves. §6.1.4 produces the simplest modular curve system – involving all primes p. By combining §6.1.3 and §6.1.4 is now natural to see the scope of simply stated modular curve generalizing problems, some solved by the previous results. We especially call attention to a problem generalizing one from modular curves; in the ball-park of Mazur-Meryl, (Prob. 6.10) but not answered by them.

§6.2 considers what we need to generalize $p = 2$ in §6.1.3. Then, §6.3 finishes some topics related to the Liu-Osserman examples. For example, the definition field of the components attached to $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$ for $n \equiv 1 \mod 8$. Also, formulating the "umbrella"result over all results of [**Fr11**, Main Thms.] and the remaining Liu-Osserman examples (not included in the previous sections).

6.1. Production of ℓ-adic support families. Suppose we have an absolutely irreducible quasi-projective algebraic variety P defined over a number field K. Typically such a parameter space might produce a family of ℓ-adic representations if P is the parameters for a smooth, projective, family of algebraic varieties. Then, the ℓ-adic representations might come from locally constant variation of some 'piece' of an ℓ-adic cohomology group of the vareties in the fibers of the family. We'll refer to the family as having rank u if the ℓ-adic modules have rank u.

6.1.1. *Frattini monodromy.* Typically for Shimura varieties it is the 1st ℓ-adic cohomology of a classical family of abelian varieties. This defines a sequence of varieties $\mathcal{P} = \{P_k\}_{k=0}^{\infty}$ starting from $P_0 = P$ with corresponding maps between them. For the Shimura variety case, the initial data right up front gives the geometric monodromy of the cover $P_k \to \mathcal{P}_0$, identifying it with the quotient of a classical arithmetic group (like $\mathrm{Sp}_{2n}(\mathbb{Z})$) by a congruence subgroup.

Among difficulties in analyzing data from cusps is that P (excluding the modular curve case) has dimension exceeding 1.

The cases of **MT**s derived from our previous sections have a dimension 1 parameter space and cusps we understand very well. (Rem. 6.9 notes we also get higher dimensional parameter spaces by increasing the number conjugacy classes. Yet, here our dimension 1 cases suffice to give myriad significantly new examples.) Our goal is just to list some illustrative **MT**s, along with what properties we need for them. The monodromy action is through a quotient of $\bar{M}_4$. So, it is extremely explicit. Here are the properties we need.

(6.1a) The geometric monodromy group U_k of $P_k \to P_0$ is transitive in the natural permutation representation, for each $k \geq 0$.

(6.1b) For some k_0 and $k \geq k_0$, $U_k \to U_{k_0}$ is an ℓ-Frattini extension (in particular, the kernel is an ℓ-group).

The U_k s, however, appear directly, not by a classical congruence subgroup. Example: That (6.1a) holds for modular curve towers (or Siegel towers) is immediate from their direct definition as upper-half plane quotients by a congruence subgroup. That makes the space the image of a connected space. In general, we only know that each component of spaces like those in our previous sections – say of the two components in Lem. 4.4 for $n \equiv 1 \mod 8$ – are images of the upper half-plane.

Is that a game-ender? Mathematically it is not, though it changes the way you play the game. You will have more direct control on the ℓ-adic module. For example, you should (conjecturally as in [**Fr06a**,]) have the ability to detect when – on $\mathcal{P}$ – there are dense sets of points akin to complex multiplication points on the j-line. Such points would correspond to (conjugacy classes of) Frattini extensions in G_k of subgroups $H \leq G_0$ following [**Fr06a**, Lem. 2.5].

[**Fr06a**, Result 5.6] notes that you may require $k_0 > 0$ in (6.1b). Even for modular curve towers, for the primes $p = 2$ and 3, $k_0 = 1$ (all other primes have $k_0 = 0$). Denote the projective limit of the U_k s by $\mathcal{U}$, regarded as an ℓ-adic group through condition (6.1b). There is also the collection of arithmetic monodromy groups ${}^{\mathrm{ar}}U_k$, and their projective limit ${}^{\mathrm{ar}}\mathcal{U}$. For Serre's O(pen) I(mage) T(heorem), the version considered in [**Se68**] – the special modular curve case of the reference in Rem. 6.5 – suffices.

6.1.2. *Specific A_n ℓ-adic support families.* As above, we stick with the Nielsen classes from Prop. 5.15 because that proposition gives something substantive about them. It also simplifies details to be able to use our present notation.

We have two very different types of families. The pure-split resemble modular curves in construction. The Pure-Frattini cases resemble nothing you've seen before unless you've followed the history of **MT**s. For both we can immediately phrase problems that challenge the regular version of the Inverse Galois Problem, and that generalize the famous results on modular curves.

To give these families, and their ℓ-adic towers we have only to give the Nielsen classes that produce their reduced inner Hurwitz spaces. Recall the standard $n{-}1$-dimensional A_n module. Take the standard degree n representation of S_n, mod out by the trivial representation generated by the sum x_n of its standard basis vectors. Denote the restriction of A_n to this by V_n. We will abbreviate $V_n/p^{k+1}V_n$ to V_n/p^{k+1}. Let V be an $\mathbb{Z}/p[A_n]$ module. We say it is *irreducible* if it has no proper submodules. Since A_n, for $n \geq 4$ acts doubly transitively on the standard representation, it is well-known that we can write V_n/p as a direct sum of the identity representation and an (absolutely) irreducible representation if $(p, n!) = 1$. This practical application of Mashke's Theorem [**Is94**, p. 214] can be done explicitly. The next lemma improves on this.

LEMMA 6.1. *As above $n \geq 4$. If $p \mid n$, then V_n/p has a unique – it is codimension 1– irreducible $\mathbb{Z}/p[A_n]$ submodule. If $p \nmid n$, then V_n/p is irreducible.*

PROOF. The case where $p|n$ is the more difficult, and is easily modified to give the case $p \nmid n$. Denote by $(V_n/p)^0 \stackrel{\mathrm{def}}{=} V_{\boldsymbol{a}}^0$ the submodule generated by elements

$(a_1, \dots, a_n) \in V_n/p$ whose entries sum to 0 mod p. Since x_n has its element sum divisible by p, when $p|n$, $(V_n/p)^0$ is well-defined.

If $(1, -1, 0, \dots, 0) \stackrel{\text{def}}{=} \boldsymbol{w}_1$ is in $V^0_{\boldsymbol{a}}$, since it clearlys generate V^0 as a $\mathbb{Z}/p[A_n]$ module, then $V^0_{\boldsymbol{a}} = V^0$. By application of an element of A_n and multiplication by an element of $(\mathbb{Z}/p)^*$ we have only to aim for finding $\boldsymbol{a}'$ representing an element of $V^0_{\boldsymbol{a}}$ with exactly two nonzero entries.

Consider any $(a_1, \dots, a_n)$ representative of $V^0 \setminus \{\mathbf{0}\}$. Denote the $\mathbb{Z}/p[A_n]$ submodule it generates by $V^0_{\boldsymbol{a}}$. Take a representative, $\boldsymbol{a}' = (a'_1, \dots, a'_k, 0, \dots, 0)$, of an element of $V^0_{\boldsymbol{a}}$ with all the a'_is nonzero, $4 \ge k < n$ having the minimal number of nonzero entries of all such elements in $V^0_{\boldsymbol{a}}$. This is possible using just the action of A_n. We do three cases, in each forming $\boldsymbol{a}''$ so that $\boldsymbol{a}' - \boldsymbol{a}''$ is nonzero and has fewer nonzero entries to conclude the induction.

First: Assume two a'_is are the same and one is different from these. With no loss take these as $(a_1, a_2, a_3$ and apply the 3-cycle $(1\,2\,3)$ to $\boldsymbol{a}'$ to get $\boldsymbol{a}''$.

Second: Assume all the a'_is are the same. Then, shift all entries of $\boldsymbol{a}'$ to the right by 1: $\boldsymbol{a}' - \boldsymbol{a}''$ has exactly two nonzero entries.

Third: The only other possibility remaining is that all entries are distinct, with 1st entry 1. Apply $(2\,3\,4)$ to get $\boldsymbol{a}''$. Finally, if V' is another proper submodule of (V_n/p), then its intersection with V^0 would be a nontrivial submodule of V^0 unless V' has dimension 1, a case easily eliminated. □

In each case, consider $k \ge 0$ as generating a series of Nielsen classes and so of spaces. Unlike §2.1.2, we don't drop the p notation here.

For each $n \equiv 5 \mod 8$, and classes $\mathbf{C}_{(\frac{n+1}{2})^4}$ consider these Nielsen classes.

(6.2a) Pure-Frattini: With $G_{k,p}(A_n) = G_{p,k,\text{ab}}$, and for p prime dividing $n!$, with $(p, \frac{n+1}{2}) = 1$,

$$\{\text{Ni}(G_{p,k,\text{ab}}, \mathbf{C}_{(\frac{n+1}{2})^4}) \stackrel{\text{def}}{=} \text{Ni}_{n,p,k,\mathbf{fr}}\}_{k=0}^{\infty}.$$

(6.2b) Pure-split: For each fixed prime p,

$$\{\text{Ni}(V_n/p^{k+1} \times^s A_n, \mathbf{C}_{(\frac{n+1}{2})^4}) \stackrel{\text{def}}{=} \text{Ni}_{n,p,k,\mathbf{spl}}\}_{k=0}^{\infty}.$$

In Lem. 6.2 the phrase *non-trivial sequence of covers* means that the moduli spaces form a sequence of increasingly higher degrees.

LEMMA 6.2. *For fixed n and p, in each case of* (6.2)*, there is a projective sequence, $\{\mathcal{H}'_{\text{H-M},k}\}_{k=0}^{\infty}$, of* H-M *components (§2.2.2) of the reduced Hurwitz spaces corresponding to the Nielsen classes as a function of k. They form a nontrivial sequence of j-line covers, with relative degrees exceeding 1. For each n, this produces a natural family of ℓ-adic representations.*

PROOF. Prop. 2.13 First we show that the Nielsen classes in each example are non-empty. For (6.2a), §2.4.4 (especially Prop. 2.26) has an if and only criterion for nonemptiness at all levels. But non-emptiness is especially easy here. The single component at level 0 contains an H-M cusp (§2.2.2) with representative $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Since $(p, \frac{n+1}{2}) = 1$, we can lift each g_i to a same order element ${}_kg_i$ in each $G_{p,k.\text{ab}}$, $i = 1, 2$. If $({}_kg_1, {}_kg_1^{-1}, {}_kg_2, {}_kg_2^{-1})$ satisfies *genertion* (§1.3.1), then it is in the level k Nielsen class. But, generation is automatic because $G_{p,k.\text{ab}} \to G_{p,0,\text{ab}}$ is a Frattini cover.

For (6.2b), again use the starting H-M rep, and that the higher levels are given by Frattini covers of the level 0 case. Except here, level 0 is given by $V_n/p \times^s A_n$, which is a split (definitely, not Frattini) extension of A_n. So, it is not automatic that level 0 is nonempty. Use the notation above.

We look for an H-M rep. in the Nielsen class for this group from $g_i^* = (v_i, g_i)$, $i = 1, 2$, with the v_i s in V_n/p. We are done so long as these g_1^* and g_2^* generate $V_n/p \times^s A_n$. Then, nonemptiness at all levels follows.

Denote the standard vectors in V_n/p by $x_1, \dots, x_n$ (with $\sum_{i=1}^n x_i = 0$). We now use that

$$(v_i, g_i)^{\mathrm{ord} g_i} = ((v_i) \sum_{j=0}^{\mathrm{ord}(g_i)-1} g_i^j, 1) \stackrel{\mathrm{def}}{=} (v, 1).$$

Assume with no loss that g_1 is the $\frac{n+1}{2}$-cycle $(1 \dots \frac{n+1}{2})$. If you take $v_1 = x_1$, then the resulting v is $x_1 + \cdots + x_{\frac{n+1}{2}}$. We show that the nonzero v is contained in no proper submodule of V_n/p. From Lem. 6.1 this is immediate, unless $p|n$ where there is a proper submodule to consider. Then, however, $(p, \frac{n+1}{2}) = 1$, so $p \not| \frac{n+1}{2}$, and v is not in that proper submodule. Therefore, the group generated by the g_i^* s will contain V_n/p. It must be $V_n/p \times^s A_n$, since the g_i s generate A_n.

Now we have the H-M rep. $(g_1^*, (g_1^*)^{-1}, g_2^*, (g_2^*)^{-1})$ in the Nielsen class $\mathrm{Ni}_{n,0,k,\mathbf{spl}}$. □

Remark 6.3. Level 1 comps in pure-Frattini case

Proposition 6.4. *Suppose a* **MT** *is a quotient of a pullback of a modular curve tower for the prime ℓ over $f : X \to \mathbb{P}^1_j$. Then the geometric monodromy groups of the tower levels are a quotient of the push-forward of G_f and* $\mathrm{PSL}_2(\mathbb{Z}/\ell^{k+1})$.

Suppose further, that G_f has no quotient that maps surjectively to $\mathrm{PSL}_2(\mathbb{Z}/\ell)$. *Then, G_f acts trivially on U_1/U_0. RETURN*

Recall: For each cover $X \to Z$, defined over a perfect field K, and for each $\boldsymbol{p} \in Z$, there is naturally attached a conjugacy class of subgroups of the full arithmetic covering group. Any group in that class is a *decomposition group* attached to $\boldsymbol{p}$.

Remark 6.5. Conditions (6.1) are equivalent to a weak form of the OIT [**Fr11b**,]. That says, for a dense set of $\boldsymbol{p} \in \mathcal{P}(\bar{\mathbb{Q}})$, for $K_{\boldsymbol{p}} = K(\boldsymbol{p})$: The action of the absolute Galois group, $G_{K_{\boldsymbol{p}}}$ (of $K(\boldsymbol{p})$) on any orbit of a projective system of points is an open subset of ${}^{\mathrm{ar}}\mathcal{U}$. In fact, it holds for any $\boldsymbol{p}$ for which the decomposition group attached to $P_1 \to P_0$ is the full arithmetic monodromy of the cover. *Hilbert's Irreducibility Theorem* says this holds for s dense set of $\boldsymbol{p}$. See [,] for details on that action.

Remark 6.6. Rem. 6.5 is a weak form because the modular curve version of the OIT describes the arithmetic monodromy at the exceptional – complex multiplication points – as well. [,] formulates the generalization of this for **MT**s. RETURN as happening at those points for which

6.1.3. *Spin assures 2-cusps at level 1.* Let $\bar{\mathcal{H}}'$ be a reduced Hurwitz space component corresponding to a braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$. For any odd integer k, $(k^2-1)/8$ is an integer. For any $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$, the left side of (3.11) is $\frac{(\frac{n+1}{2})^2-1}{2}$, so Prop. 3.11 says $\mathrm{Ni}(G_1(A_n), \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ is nonempty.

Prop. 3.12, condition (3.13), tells precisely when all cusps (in all components at level 1) above a specific cusp at level 0 must be 2-cusps. Indeed, for $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$ representing the cusp, the condition is that $(m^2-1)/8 \equiv 1 \mod 2$ with $m = \mathrm{ord}((\boldsymbol{g})\mathbf{mp})$. Conclusion: This holds precisely for those cusps that don't have the form $\mathcal{O}_{k,2k}$ for some odd k.

Corollary 6.7. *For $n = 5 + v8$, there are at least $4v + 4$ 2-cusps on $\bar{\mathcal{H}}'$.*

Corollary 6.8. *That for the case $n \equiv 5 \mod 8$, level one has a 2-Spire.*

6.1.4. *A_n analogy with modular curves.* Considering when there is a p-Spire is meaningful for any $r \geq 4$. Considering when there is a p-Spire is meaningful for any allowable primes p.

Remark 6.9 (Increase branch cycles).

6.1.5. *$G_{k,p}(A_n)$ analogy with modular curves.*

Problem 6.10.

6.2. How to approach primes different from 2. §6.3 considers the rest of the odd order pure-cycles cases of Liu-Osserman. Finally, §E.3 gives one example from the list of Ex. 2.29. This shows issues involved in dropping the condition that the absolute spaces represent genus 0 covers in (2.17b). Considering it may seem slight, since $G = A_4$ is such an "easy" group. Yet, it is our most important example for using this paper to head toward a general proof of the Main Conjecture.

Much of the idea of this section is general. The missing general ingredient is a purely modular representation step. We consider if there are non-H-M braid orbits on $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4}, p = 5)$. The Main Conjecture holds for any component branch through them if there are at least three 5-cusps at level 1.

To prove the following result we look carefully at how to write out the level 1 sh-incidence matrix for $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4}, p = 5)^{\mathrm{in,rd}}$, recognizing the Hurwitz components for this cover the unique component for $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$, and the level 1 components all factor through some one of these components.

Proposition 6.11. *Each braid orbit on $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ has two representives over* H-M_1*. Therefore, the Main Conjecture holds for all* **MT***s for $(A_5, \mathbf{C}_{3^4}, p = 5)$ and therefore for all* **MT***s with level 0 equal to $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$.*

§6.2.1 shows why it suffices to consider $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ to conclude Prop. 6.11. §6.2.2 considers how to compute the cusps of $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$, and the corresponding sh-incidence matrix.

6.2.1. *Relation of* $\mathrm{Ni}(G_1(A_5), \mathbf{C}_{3^4}, p = 5)$ *to* $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})$*.* Make use of $\mathrm{PSL}_2(\mathbb{Z}/5) = A_5$ using the notation 0_2 (resp. I_2) for the 2×2 zero (resp. identity) matrix. Then, $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has order 2 $\mod \{\pm I_2\}$. We can see A_4 in A_5 as a Klein 4-group with a $\mathbb{Z}/3$ action. Nonzero representatives of the Klein 4-group are order 2 matrices commuting with $A_1 \mod \pm I_2$: $A_2 = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ are representatives of the two non-identity classes. Note: Traces of the involution conjugacy class are 0.

A generator $\alpha \in \mathbb{Z}/3$ in A_4 conjugates A_1 to A_2: $\alpha A_1 = A_2 \alpha$: $\alpha = \left(\begin{smallmatrix} 2 & -3 \\ 1 & -1 \end{smallmatrix}\right)$ is a trace 1 representative. So, ± 1 is the trace of all elements in the order 3 conjugacy

$\mathrm{PSL}_2(\mathbb{Z}/5)$ class. To get A_5, throw into this copy of A_4 an element of order 5 by finding a representative $\beta \in \mathrm{SL}_2(\mathbb{Z}/5)$ of trace 2 or 3: $\beta = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ will do. Note: These representatives canonically lift to have determinant 1 in $\mathrm{SL}_2(\mathbb{Z}/5^2)$.

From [**Fr95b**, Rem. 2.10]:

(6.3a) $\ker(G_1(A_5) \to A_5)$ is a module with Loewy display $U_5 \to U_5$ with U_5 the trace 0 matrices in $\mathbb{M}_2(\mathbb{Z}/5)$; and
(6.3b) $G_1(A_5) \to A_5$ factors through $\mathrm{PSL}_2(\mathbb{Z}/5^2) \to \mathrm{PSL}_2(\mathbb{Z}/5)$.

Finally, we find in $\mathrm{PSL}_2(\mathbb{Z}/5)$ two H-M reps. with middle product order 5. As $\alpha\beta = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \gamma$ has trace 1, take $\text{H-M}_1 = (\gamma^{-1}, \gamma, \alpha, \alpha^{-1})$ as one H-M rep. and $\text{H-M}_2 = (\gamma, \gamma^{-1}, \alpha, \alpha^{-1})$ as the other. Use the same integer entries of α and γ to give representatives of all lifts of H-M_1 to $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$:

$$\boldsymbol{g}_{A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}}} \stackrel{\text{def}}{=} (\gamma^{-1}(I_2 + 5A_{\gamma^{-1}}), \gamma(I_2 + 5A_\gamma), \alpha(I_2 + 5A_\alpha), \alpha^{-1}(I_2 + 5A_{\alpha^{-1}})),$$

modulo conjugation by $\ker(\mathrm{PSL}_2(\mathbb{Z}/5^2) \to \mathrm{PSL}_2(\mathbb{Z}/5))$ subject to these conditions.

(6.4a) Entries in $\mathrm{PSL}_2(\mathbb{Z}/5^2)$: Entries of $(A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}})$ have trace 0.
(6.4b) Product-one: $\gamma^{-1} A_{\gamma^{-1}} \gamma + A_\gamma + \alpha A_\alpha \alpha^{-1} + A_{\alpha^{-1}} = 0_2$.

The effect of conjugation of U by $I_2 + 5B$ sends the former to

$$(I_2 + 5B)(U)(I_2 - 5B) = U + 5([B, U]),$$

with $[B, U] = BU - UB$.

With no loss, assume $A_\gamma = 0_2$, and consider the case A_α also is 0_2. Write $A_{\gamma^{-1}} = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, so $\gamma^{-1} A_{\gamma^{-1}} \gamma = -A_{\alpha^{-1}}$. With $\gamma^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$,

$$A_{\alpha^{-1}} = -\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} a-c & c \\ b-c+2a & c-a \end{smallmatrix}\right).$$

It is meaningful to have $q \in \bar{M}_4$ act on $\boldsymbol{g}_{A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}}}$, by acting on its 4-tuple. The Main Conjecture follows if for each q, so that its induced action mod 5 leaves $(\gamma^{-1}, \gamma, \alpha, \alpha^{-1})$ invariant, while not leaving $\boldsymbol{g}_{A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}}}$ invariant.

6.2.2. *sh-incidence matrix for* $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})$. §4.2 has the level 0 sh-incidence matrix. For $\boldsymbol{g}$ in some Nielsen class $\mathrm{Ni}(G, \mathbf{C})$, denote the full collection of elements in its reduced Nielsen class (its orbit under $\langle G, \langle \mathbf{sh}, q_1 q_3^{-1} \rangle\rangle$) by $\boldsymbol{g}^{\mathrm{in,rd}}$. The cusp containing $\boldsymbol{g}$ (as a subset of $\mathrm{Ni}(G, \mathbf{C})$) is the union of $\{((\boldsymbol{g})q_2^j)^{\mathrm{in,rd}}\}$ running over all integers j. Of course you only need at most the first $2 \cdot (\boldsymbol{g})\mathbf{mp}$ values of j. We denote this set by $\mathrm{Cu}_{\boldsymbol{g}}$, the cusp of $\boldsymbol{g}$.

Using this notation, Lem. 4.3 gives the five cusps of $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$ as $\mathrm{Cu}_{(\text{H-M}_1)q_2^j\mathbf{sh}}$, $j = 0, 1, 2, 3, 4$, with $j = 0$ the unique cusp of width 2, $j = 1, 4$ the cusps of width 3, and $j = 2, 3$ the cusps of width 5. Now suppose $\text{H-M}_1'$ lies over H-M_1 in $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})$. Then, we get the complete set of representatives of cusps for the spaces corresponding to braid orbits on $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})$ by considering the collection $\mathrm{Cu}_{(\text{H-M}_1')q_2^j)\mathbf{sh}}$, $j = 0, 1, 2, 3, 4$.

Let $\text{H-M}_1''$ denote another representative over H-M_1. A contribution to the sh-incidence matrix of $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ over the level 0 position of (i, j) comes from $(\text{H-M}_1')q_2^j\mathbf{sh})^{\mathrm{in,rd}} = (\text{H-M}_1')q_2^i)^{\mathrm{in,rd}}$ for some $\text{H-M}_1'$ and $\text{H-M}_1''$. So, to find such contributions requires only looking at the cases where RETURN

6.3. More on the Liu-Osserman Examples. Throughout this subsection assume we are given an odd-cycle Liu-Osserman Nielsen class $\mathrm{Ni}(A_n, \mathbf{C})$.

6.3.1. *Remaining odd-cycle Liu-Osserman examples for $r = 4$.* Give What changes if we don't have the $(\frac{n+1}{2})^4$ case?

6.3.2. *What about general $r \geq 3$?*

EXAMPLE 6.12. Give the alternating group obstructed components here.

6.3.3. *Umbrella result.*

6.3.4. *Rational functions representing elements of* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs}}$. For $n \equiv 1 \mod 4$ (and especially for $n \equiv 1 \mod 8$) we consider rational functions representing $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4}$. Here's the rubric.

Rational functions in $\mathbb{Q}$ with branch points in $\mathbb{Q}$, of which (with no loss) we take three to be $\{0, 1, \infty\}$ and the other as z'. So, we can write such an $f \stackrel{\mathrm{def}}{=} f_{x'}(x) : \mathbb{P}^1_x \to \mathbb{P}^1_z$ as $h_1(x)x^{\frac{n+1}{2}}/h_2(x)$ with h_1, h_2 of degree $\frac{n-1}{2}$. This automatically puts 0 (resp. ∞) as the ramified point over 0 (resp. ∞). The following equations encode the rest of the conditions at the branch points. These make $1 \in \mathbb{P}^1_x$ the ramified point over $z = 1$, to determine $f \stackrel{\mathrm{def}}{=} f_{x'}(x)$ with x' the ramified point over z':

$$\begin{aligned} h_1(x)x^{\frac{n+1}{2}} - h_2(x) &= (x-1)^{\frac{n+1}{2}} m_1(x) \\ h_1(x)x^{\frac{n+1}{2}} - z'h_2(x) &= (x-x')^{\frac{n+1}{2}} m_2(x). \end{aligned} \tag{6.5}$$

We can solve for h_1, h_2 as a function of m_1 and m_2:

$$\begin{aligned} &(a) & (z'-1)h_2 &= (x-1)^{\frac{n+1}{2}} m_1 - (x-x')^{\frac{n+1}{2}} m_2 \\ &(b) & (z'-1)x^{\frac{n+1}{2}} h_1 &= z'(x-1)^{\frac{n+1}{2}} m_1 - (x-x')^{\frac{n+1}{2}} m_2. \end{aligned} \tag{6.6}$$

So, we want coefficients (total of $n+1$ coefficients) on the degree $\frac{n-1}{2}$ polynomials m_1, m_2 polynomials so that h_1 and h_2 both have degree $\frac{n-1}{2}$, simultaneously figuring x' as a function of z'.

PROPOSITION 6.13. *As x' varies in $\mathbb{P}^1_x \setminus \{0, 1, \infty\}$, we run over the connected set of $f_{x'}(x)$ in the Nielsen class of covers with ordered branch points by solving the equations* (6.6) *according to the stipulations above. For x' lying in a field K, the solution for $f_{x'}(x)$ has coefficients also lying in K.*

Consider the cover $\Psi_n^{\mathrm{in,abs}} : \mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}} \to \mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs}}$. The top space has one (resp. 2) components, defined over $\mathbb{Q}$ (resp. the unique quadratic extension K_n of $\mathbb{Q}$ in $\mathbb{Q}(e^{\frac{n+1}{2}})$) when $n \equiv 5 \mod 8$ (resp. $n \equiv 1 \mod 8$).

PROOF. What we actually need to know, as x' runs over $\mathbb{Q}$ is is that the discriminant of the cover $f_{x'} : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is not locally a square in $\mathbb{Q}\{\{x'\}$. We will show for $n \equiv 5 \mod 8$, it has expression in the square root of x', while for $n \equiv 1 \mod 8$ you must extend the constants by

Expand the zeros of $f_{x'}(x) = z$ about z'. □

Appendix A. Riemann-Hurwitz and Classical $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$ Generators

Consider compact Riemann surface covers, $\varphi_i : X_i \to \mathbb{P}^1_z$, $i = 1, 2$. They are in the same *absolute* equivalence class if there is a continuous $\psi : X_1 \to X_2$ with $\varphi_1 = \varphi_2 \circ \psi$. Then ψ is automatically analytic.

If the covers are Galois (with respective monodromy groups G_i, $i = 1, 2$), assume both G_i s come with an isomorphism to $\mu_i : G_i \to G$. Then, the φ_i s are *inner equivalent* if they are absolute equivalent, and in addition, the induced map $\mu_1 \circ \psi^* \circ \mu_2^{-1} : G \to G$ is an inner automorphism.

Finally, if there is $\alpha \in \mathrm{PGL}_2(\mathbb{C})$, the Möbius transformations, so that $\alpha \circ \varphi_1$ is absolute (resp. inner) equivalent to φ_1, we say the covers are absolute (resp. inner) reduced equivalent.

Suppose given any Nielsen class $\mathrm{Ni}(G, \mathbf{C})^*$ where $*$ indicates absolute equivalence. That corresponds to $G \le S_n$ for some integer n: a faithful transitive permutation representation of G (the regular representation of G when $*$ is inner equivalence). Then, given classical generators $\mathcal{P}^0 = \{P_1^0, \dots, P_r^0\}$ of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$, gives a homorphism $\mu_{\mathcal{P},\boldsymbol{g}} : \pi_1(\mathbb{P}^1_z, \setminus \boldsymbol{z}^0, z_0) \to G$ by mapping P_i to g_i, $i = 1, \dots, r$. For any $g \in S_n$ denote its index by $\mathrm{ind}(g) \stackrel{\mathrm{def}}{=}$ to n minus the number of orbits of g on $\{1, \dots, n\}$. The homomorphism $\mu_{\mathcal{P},\boldsymbol{g}}$ corresponds to a cover $X^0 \to \mathbb{P}^1_z \setminus \boldsymbol{z}^0$.

One direction of R(iemann's)E(xistence)T(heorem) completes X^0 uniquely to a compact surface X covering $\mathbb{P}^1_z$. Conclusion: A set of classical generators uniquely corresponds covers of $\mathbb{P}^1_z$ (ramified over $\boldsymbol{z}^0$; up to $*$-equivalence) with elements of $\mathrm{Ni}(G, \mathbf{C})^*$. Indeed, RET goes in both directions. A cover of compact Riemann surfaces $\varphi : X \to \mathbb{P}^1_z$ *is* an analytic function on X. So, the only extra element under consideration is that X is algebraic (embeddable in some projective space). It is elementary, too, to show that comes from producing one other analytic function $\varphi' : X \to \mathbb{P}^1_w$, with φ' restricted to $\varphi^{-1}(z_0)$ having n distinct values. The one hard point is the production of that second function w.

It is not even elementary when you know the genus of X is zero, unless you accept the Riemann-Roch theorem for compact surfaces. The complete and elementary treatment of [**Fr08a**, Chap. 4] discusses all these points, though the last hard point is relevant to this paper only for applications. Further, since the index of an element depends only its conjugacy class in S_n, all covers in a given Nielsen class have the same genus. That genus $\mathbf{g}$ is expressed as $2(n+\mathbf{g}-1) = \sum_{i=1}^r \mathrm{ind}(g_i)$, which is elementary, too, in that it expresses that the degree of the differential $d\varphi$ is $2 \cdot \mathbf{g} - 2$ and is independent of the choice of analytic function on X chosen to compute this.

If you move along any path Γ in U_r from $\boldsymbol{z}^0$ to $\boldsymbol{z}' \in U_r$, you can deform the given classical generators $\mathcal{P}^0$ to some others $\mathcal{P}'$ based around $\boldsymbol{z}'$ by following Γ. This gives a homomorphism of $\pi_1(U_r)$ into the Hurwitz monodromy group (§**??**), canonically defined by $\mathcal{P}^0$. This, too, is explicit, and elementary, and it has been explained several times in the literature starting with [**Fr77**, §4], [**MM99**], [**Vo96**] and in complete detail in [**Fr08a**, Chap. 5].

Again, the one non-elementary aspect is that the cover of U_r so produced — the space $\mathcal{H}(G, \mathbf{C})^*$ —is algebraic (actually, in this case an affine variety). It's completion is not unique, true, though it has a unique normalization as a cover of $\mathcal{P}^r$. It is easy case from elimination theory, given affineness of $\mathcal{H}(G, \mathbf{C})^*$, that the reduced space $\mathcal{H}(G, \mathbf{C})^*/\mathrm{PGL}_2(\mathbb{C})$ —where you equivalence one cover φ with any other cover $\beta \circ \varphi$ for each $\beta \in \mathrm{PGL}_2$ —is also affine. [**BF02**, §2] explains this, and how this produces $\mathcal{H}(G, \mathbf{C})^{*,\mathrm{rd}}$ as a cover of U_r/PGL_2. When $r = 4$, it explicitly identifies a quotient of $H_4 = \langle q_1, q_2, q_3 \rangle$ with a group $\bar{M}_4$, via generators that reveal the latter to be $\mathrm{PSL}_2(\mathbb{Z})$, generated by the elements γ_0, γ_1 as indicated in §2.11. The particular result, then shows that $\gamma_0', \gamma_1', \gamma_\infty'$ acting on the reduced Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{*,\mathrm{rd}}$ are branch cycles for the cover $\bar{\mathcal{H}}(G, \mathbf{C})^{*,\mathrm{rd}} \to \mathbb{P}^1_j$.

This allows computing the genus of $\bar{\mathcal{H}}(G,\mathbf{C})^{*,\mathrm{rd}}$ just as for covers in the Nielsen class as it is done in Prop. 5.15.

Appendix B. Modular curve towers and g-p' cusps for all r

Consider an upper half plane quotient $\mathbb{H}$ by $\Gamma \le \mathrm{PSL}_2(\mathbb{Z}) = \langle \gamma_0, \gamma_1\rangle$ of finite index; γ_0 and γ_1 have respective orders 3 and 2. Then, $\mathbb{H}/\Gamma$ is an affine curve X^0_Γ covering $U_j = \mathbb{P}^1_j \setminus \{\infty\}$, branched over $j = 0$ and 1, ramified over with branch points 0 and 1 in a normalized j-variable. Orbits of $\gamma_\infty = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ on cosets of Γ correspond to *cusps* (points over ∞). The **MT** approach sees $\mathrm{PSL}_2(\mathbb{Z})$ as $\bar{M}_4$ (§2.3.1), to show how the §2.2.2 cusp classification works in the modular curve case (§B.1). Then, §B.2 reminds of general g-p' cusps, showing why connectedness results seek their presence.

B.1. Modular curve cusps. The classical count of cusps for

$$\Gamma = \Gamma_0(p^{k+1}) \stackrel{\text{def}}{=} \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \mod p^{k+1}$$

starts by selecting good coset representatives. Then compute γ_∞ orbits on them. We contrast this with representing $X_0(p^{k+1})$ cusps the **MT** way using .

B.1.1. *Nielsen class description and q_2 action.* Assume p is odd and continue the Nielsen class notation from §2.5.2. Write the order $2\cdot p^{k+1}$ dihedral group (resp. its normalizer, $N_{S_{p^{k+1}}}(G_k) \stackrel{\text{def}}{=} N_k$, in $S_{p^{k+1}}$) as

$$G_k = D_{p^{k+1}} \stackrel{\text{def}}{=} \{\left(\begin{smallmatrix} \pm 1 & b \\ 0 & 1\end{smallmatrix}\right)\}_{b\in\mathbb{Z}/p^{k+1}} (\text{resp. } \{\left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)\}_{a\in(\mathbb{Z}/p^{k+1})^*, b\in\mathbb{Z}/p^{k+1}}).$$

It acts on $\{(b',1) \mid b' \in \mathbb{Z}/p^{k+1}\}$ by $\left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right) : (b',1) \mapsto (a\cdot b'+b, 1)$. Use $b \Leftrightarrow \left(\begin{smallmatrix} -1 & b \\ 0 & 1\end{smallmatrix}\right) \in D_{p^{k+1}}$. With $\mathrm{C}_2 = \{\left(\begin{smallmatrix} -1 & b \\ 0 & 1\end{smallmatrix}\right)\}_{b\in\mathbb{Z}/p^{k+1}}$, use this [] notation for elements in the Nielsen class: $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C}_{2^4}) \mapsto [b_1,\dots,b_4] \in (\mathbb{Z}/p^{k+1})^4$.

Absolute Nielsen classes (§1.3.1), $\mathrm{Ni}(G_k,\mathbf{C}_{2^4})^{\mathrm{abs}}$:

$$\text{(B.1)} \quad \begin{aligned} &\{\boldsymbol{g} = (g_1,\dots,g_4) \in \mathbf{C}_{2^4} \mid b_i \ne b_j \mod p \text{ for some } i,j \text{ (generation)};\\ &\text{and } b_1 - b_2 + b_3 - b_4 \equiv 0 \mod p^{k+1} \text{ (product-one)}\}/N_k.\end{aligned}$$

For *inner* classes mod out by G_k instead of N_k. Generation (§1.3.1) is a Frattini property: It holds in G_0, so it holds in the Frattini cover G_k for any k.

Conjugate by a power of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ to assume $b_1 = 0$. Write $b_2 - b_3 = ap^u$ $(u \ge 0)$, $a \in \mathbb{Z}/p^{k+1-u}$ and $(a,p) = 1$. For $\mathrm{Ni}^{\mathrm{abs}}$, conjugate by $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right) \in N_k$ so $a = 1$. This allows further conjugation by

$$H_u = \{\alpha = 1 + bp^{k+1-u} \in \mathbb{Z}/p^{k+1} \mod p^u, b \in \mathbb{Z}/p^u\}.$$

Now write, $c = b_2$, $b_3 = c - p^u$ (u is a parameter). For $u = 0$: $(b_2, b_3) = (c, c-1)$ has q_2 orbit of width p^{k+1} containing $\boldsymbol{g} = \boldsymbol{g}_{\text{H-M}} = [0,0,1,1]$: a rep. for the unique Harbater-Mumford absolute class.

TABLE 9. sh-incidence for $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ ($k=1$ above)

Cusp orbit	${}_{\mathbf{c}}O_{p^2}$	${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	${}_{\mathbf{c}}O_1$
${}_{\mathbf{c}}O_{p^2}$	$p(p-1)$	1	1
${}_{\mathbf{c}}O_{a,p}, a \in (\mathbb{Z}/p)^*$	1	0	0
${}_{\mathbf{c}}O_1$	1	0	0

Apply Lem. D.3. Clearly $q_1q_3^{-1}$ is trivial on $\boldsymbol{g}_{\text{H-M}}$, and so is $\mathbf{sh}^2$ which gives $[1,1,0,0]$, conjugate by $\left(\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ to $\boldsymbol{g}_{\text{H-M}}$. So, K_4 is trivial on all inner Nielsen classes once we see — below — there is just one braid orbit.

B.1.2. *One braid orbit of cusps.* An odd (resp. even) number of elements from C_2 has product in C_2 (resp. a translation by some $b \in \mathbb{Z}/p^{k+1}$). So, this Nielsen class satisfies Princ. 3.5. For $\boldsymbol{g} \mapsto [b_1,b_2,b_3,b_4]$, $o_{\boldsymbol{g}} = \mathrm{ord}((\boldsymbol{g})\mathbf{mp})$ (Def. 2.3) is the order of $b' \stackrel{\text{def}}{=} b_3 - b_2$ in $\mathbb{Z}/p^{k+1}$. Here is a list of inner class reps. for the cusp of $\boldsymbol{g}$:

$$\{[b_1, b_2+mb', b_3+mb', b_4]\}_{m=0}^{o_{\boldsymbol{g}}-1}.$$

First we list absolute cusps. For $u > 0$,
$\langle \boldsymbol{g} \rangle = D_{p^{k+1}}$ requires $(c,p) = 1$. Conjugate by H_u to assume $c \in \mathbb{Z}/p^{k+1-u}$ is p'. The width of the cusp of $\boldsymbol{g}$ is

$$|\text{residues} \mod p^{k+1-u} \text{ differing by multiples of } p^u|.$$

Conclude: $\varphi(p^{k+1-u})$ Nielsen class elements fall in Cu_4 orbits of width p^{k+1-2u} (resp. 1) if $k+1-2u \ge 0$ (resp. $k+1-2u < 0$). Note the other extreme, $u = k+1$: $(b_2, b_3) = (1,1)$, the shift of an H-M rep. (orbit width 1).

To see there is one braid orbit, apply the *shift*: $(\boldsymbol{g})\mathbf{sh} \mapsto [c, c-p^u, -p^u, 0]$. From the above, in the cusp of this element is $[c,c,0,0]$. The sh-incidence pairing shows the H-M cusp intersects every other cusp, so the matrix has one block.

Conclude: $\mathrm{Ni}^{\mathrm{abs}}$ gives a degree $p^{k+1} + p^k$ cover $\bar{\mathcal{H}}(D_{p^{k+1}}, \mathbf{C}_{2^4})^{\mathrm{abs,rd}} \to \mathbb{P}^1_j$. The respective result for $\mathrm{Ni}^{\mathrm{in}}$ is degree $\frac{\varphi(p^{k+1})}{2}(p^{k+1}+p^k)$.

From the absolute case, there is one inner braid orbit if we can braid between elements of $\{[0,0,c,c]\}_{c \in (\mathbb{Z}/p^{k+1})^*}$, the collection of inner classes of H-M reps. Shift $[0,1,1+c,c]$, an element in the cusp of $[0,0,c,c]$ to get $[1,1+c,c,0] = \boldsymbol{g}'$. Now again, in the cusp of $\boldsymbol{g}'$ there is $[1,1,0,0]$ which we noted above is inner equivalent to $\boldsymbol{g}_{\text{H-M}}$. So, there is one inner braid orbit.

B.1.3. *Summary of modular curve cusps.* In either the inner or absolute case, any cusp for which $u \ne k+1$ is a p-cusp — p divides the middle product order. For $u = k+1$, $\boldsymbol{g} \mapsto [0,c,c,0]$, the shift of an H-M rep. and a width one cusp. In these cases $\langle g_2, g_3 \rangle = H_{2,3}(\boldsymbol{g})$ and $\langle g_1, g_4 \rangle = H_{1,4}(\boldsymbol{g})$ are p' groups; they are g-p' cusps.

Finally: *o(nly)-p'* is the phrase for those cusps neither p nor g-p'. Modular curves have none.
$u = 0 \leftrightarrow$ width p^2 H-M rep. cusp, ${}_{\mathbf{c}}O_{p^2}$;
$u = 1 \leftrightarrow$ cusps ${}_{\mathbf{c}}O_{a,p}$, $a \in (\mathbb{Z}/p)^*$ of width 1; and
$u = 2 \leftrightarrow$ width 1 cusp ${}_{\mathbf{c}}O_1$ of the shift of the H-M rep.

Adding to this data the fixed points of $\gamma_0 = q_1q_2$ and $\gamma_1 = \mathbf{sh}$ gives the genus of the space (App. C_1).

Adjustments for $\mathrm{Ni}(D_{p^2}, \mathbf{C}_{2^4})^{\mathrm{in,rd}}$:

$u = 0 \leftrightarrow \varphi(p^2)/2 = \frac{p(p-1)}{2}$ H-M inner cusps over the unique absolute H-M cusp.
$u = 2 \leftrightarrow$ Story the same as for $u = 0$, for shifts of H-M cusps (width 1).
$u = 1 \leftrightarrow$ over each such absolute (width 1) cusp are $\varphi(p)/2 = \frac{p-1}{2}$ width p cusps.

Since the sh-incidence matrix remains the same if we replace $\gamma_1 = \mathbf{sh}$ by γ_0, fixed points of either are represented on the diagonal.

Problem B.1. Compute which elements in the H-M absolute (resp. inner) cusp(s) are fixed points of γ_i, $i = 0, 1$.

B.2. Seeking g-p' cusps.

Appendix C. A p' moduli argument

Prop. 2.13 considered the possibility that we have a projective sequence of points $\{\boldsymbol{p}_k\}_{k=0}^{\infty}$ on an abelianized **MT** all defined over a number field K and it showed this was impossible if G_0 was centerless. It also reduced the general case to showing it impossible if the center Z of G_0 is p', but nontrivial. [**Fr06a**, Rem. 3.4] notes the universal p-Frattini cover ${}_p\tilde{G}$ of G identifies with the fiber product over G/Z of G and the universal p-Frattini cover of G/Z. Thus, Z is the exact center of ${}_p\tilde{G}$ and therefore also of the abelianized **MT**. Conclude: The (one-one) image of Z by the map ${}_p\tilde{G} \to G_{k,\mathrm{ab}}$ then identifies with the center of $G_{k,\mathrm{ab}}$.

With no loss assume K large enough that some $\varphi_0 : X_0 \to \mathbb{P}^1_z$ in $\mathrm{Ni}(G_0, \mathbf{C})^{\mathrm{in}}$ defined over K (with all its automorphisms also over K) represents $\boldsymbol{p}_0$.

Proposition C.1. *Under the assumptions above, the point $\boldsymbol{p}_{k+1}$ has a representative over K in $\mathrm{Ni}(G_{k+1,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$ that factors through a given representative over K in $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$. That is, there is a projective system of representatives over K in $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$, $k \ge 0$. This, however, contradicts* [**BF02**, Prop. 6.8].

Proof. Assume for a given k, we have found $\varphi_k : X_k \to \mathbb{P}^1_z$ over K in the Nielsen class $\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$ representing $\boldsymbol{p}_k$. Over some Galois extension L/K, there is $\varphi'_{k+1} : X'_{k+1} \to \mathbb{P}^1_z$ representing $\boldsymbol{p}_{k+1}$, in $\mathrm{Ni}(G_{k+1,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$ and mapping through $\varphi_k : X_0 \to \mathbb{P}^1_z$. Since $\boldsymbol{p}_{k+1}$ has coordinates in K, that means for each $\sigma \in G(L/K)$ there is $\psi_\sigma : X_{k+1} \to {}^{\sigma}X_{k+1}$ commuting with the maps to $\mathbb{P}^1_z$ and inducing by its action on $\mathrm{Aut}(X_{k+1}/\mathbb{P}^1_z)$ an isomorphism of $\mathrm{Aut}({}^{\sigma}X_{k+1}/\mathbb{P}^1_z)$ with G_{k+1}.

Recall Weil's cocycle condition as formulated in [**Fr77**, §2]. From it, for the first statement we have only to show that $\psi^{-1}_{\tau\sigma}{}^{\sigma}\psi_\tau\psi_\sigma = \beta_{\sigma,\tau}$, forced to be in Z, is the identity on X_{k+1}. Apply these expressions after modding out by $\ker(G_{k+1,\mathrm{ab}} \to G_{k,\mathrm{ab}})$ to induce similar expressions on X_k. Since X_k is already a representative of the Nielsen class over K, the existence of such a representative forces the image of $\beta_{\sigma,\tau} \in \mathrm{Aut}(X_k/\mathbb{P}^1_z)$ to be the identity. As Z from $\mathrm{Aut}(X_{k+1}/\mathbb{P}^1_z)$ to $\mathrm{Aut}(X_k/\mathbb{P}^1_z)$, conclude $\beta_{\sigma,\tau}$ is also the identity. So, the cocycle condition holds and the obstruction to giving a Nielsen class representative in $\boldsymbol{p}_{k+1}$ over K vanishes. Conclude there is $\varphi'_{k+1} : X'_{k+1} \to \mathbb{P}^1_z$ in the Nielsen class $\mathrm{Ni}(G_{k+1,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$ over K representing $\boldsymbol{p}_{k+1}$. Then, φ_{k+1} induces a cover $\varphi'_k : X'_{k+1}/\ker(G_{k+1,\mathrm{ab}} \to G_{k,\mathrm{ab}}) \to \mathbb{P}^1_z$ in $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$ over K.

Since we don't have fine moduli, we can't assert φ'_k is K equivalent to φ_k. Still, the full set of representatives in the Nielsen class over K representing $\boldsymbol{p}_k$ is then in the cohomology set $H^1(G_K, Z)$ (with trivial G_K action on Z). This is a pointed set for which we apply the cocycles to one representative (say, φ'_k) to get any others. If the cocycle $\alpha \in H^1(G_K, Z)$ applied to φ'_k gives φ_k. Then, the same cocycle applied to φ'_{k+} gives a new cover φ_{k+1} over K in $\mathrm{Ni}(G_{k+1,\mathrm{ab}}, \mathbf{C})^{\mathrm{in}}$ that factors through φ_k.

This establishes an induction on k as in the statement of the proposition, to conclude the final contradiction. □

Appendix D. Level 1 of an $(A_n, \mathbf{C}_{(\frac{n+1}{2})^4}, p = 2)$ tower

For $n \equiv 5 \mod 8$ (ditto $n \equiv 1 \mod 8$, but we stick to the former), there are many resemblances between all the **MT**s. This section, emulates [**BF02**] which used the case $n = 5$ to show the theory in action. We list properties that generalize those from $n = 5$.

D.1. Some help on the p-Frattini module in general. The prime here is 2. §2.1.2 has the definition of an abelianized **MT** over $\mathrm{Ni}(A_n, \mathbf{C})^{\mathrm{in,rd}}$: A projective system of braid orbits on $\{\mathrm{Ni}(G_{k,\mathrm{ab}}(A_n), \mathbf{C})^{\mathrm{in,ab}}\}_{k=0}^{\infty}$. Given $G_{1,\mathrm{ab}}(A_n) = G_1(A_n)$, then $\ker(G_{k+1,\mathrm{ab}}(A_n) \to G_{k,\mathrm{ab}}(A_n)) = M_{k,\mathrm{ab}}$ is the same A_n module as $\ker(G_1(A_n) \to A_n) = M_0$. As a function of n, there is no easy formula for precisely what is $\ker(G_1(A_n) \to A_n)$. Still, from [**Fr02**, Thm. 2.8], we get much information on the module and its rank from the following procedure for computing the 0th p-Frattini module $M_{G,0}$ in the general case. Recall: The Frattini subgroup of G is the intersection of all maximal proper (closed) subgroups of G. Denote it $\Phi(G)$.

Let P be a p-Sylow of G, and $M_{P,0}$ the characteristic p-Frattini module of P. If $P*$ is a pro-free group of the same rank, rk_P as P. Denote by $\psi_P : P* \to P$ a corresponding surjective homomorphism. Then, the kernel, $\ker(\psi_P)$ is also pro-free. Its rank, by the Schreier Thm., is $\mathrm{rk}_{P,0} = |P|(rkP-1)+1$. This is also the rank of $M_{P,0} = ker(\psi_P)/\Phi(ker(\psi_P))$, a $\mathbb{Z}/p[P]$ module. Then, $G_1(P) = P*/\Phi(ker(\psi_P))$ is the universal exponent p-extension of P. It has the natural cover $\psi_{P,0} : G_1(P) to P$.

Now for the general finite group case. Let N_P be the normalizer in G of P. Then, consider the module induced from $M_{P,0}$ in going from N_P to G, $\mathrm{Ind}_{N_P}^G(M_{P,0})$. [**Fr02**, Thm. 2.8] identifies $M_{G,0}$ as the direct summand of this module whose restriction to P contains $M_{P,0}$.

Recall the main properties of $M_{G,0}$:

(D.1a) It is an indecomposable $\mathbb{Z}/p[G]$ module (no nontrivial $\mathbb{Z}/p[G]$ module summand).

(D.1b) It is the kernel of a covering group $\psi_{G,0} : G_1 \to G$ versal for all covers $H \to G$ with kernel a $\mathbb{Z}/p[G]$ module.

(D.1c) The lift of any element of order p in G to G_1 has order p^2.

Given a clear understanding of P, this construction gives a handle on $M_{G,0}$, reasonably putting bounds on its rank. We can refine this construction of $M_{G,0}$: For each p-Sylow P, we have defined $G_1(P) \to P \subset G$. Further, we can pull P back in $G_1(G) = G_1$, to get $\psi_{G,0}^{-1}(P)$ covering P. Applying the universal extension property then induces an injection $\alpha : M_{P,0} \to M_{G,0}$.

More generally, any conjugation by $g \in G$ will map P to gPg^{-1}, and induce another map $\alpha_g : M_{P,0} \to M_{G,0}$. It often occurs that a subgroup of G, N_P^h, properly larger than N_P, acts to preserve $M_{P,0}$.

Lemma D.1. *The images of $\{\alpha_g\}_{g\in G}$ generate $M_{G,0}$. Define N_P^h to be those $g \in G$ with the same image as α. If we replace N_P in $Ind_{N_P}^G(M_{P,0})$ by N_P^h, the induced module gives precisely the module $M_{P,0}$.*

D.2. Monodromy of $\bar\Psi_n^*: \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{*,\mathrm{rd}} \to \mathbb{P}^1_j$. Use the notation $G_{\bar\Psi_n^{\mathrm{abs}}}$ (resp. $\hat G_{\bar\Psi_n^{\mathrm{in}}}$) for the geometric (resp. arithmetic) monodromy of the cover $\bar\Psi_n^{\mathrm{abs}}$, and the analogous notation for inner equivalence.

When $*$ is absolute (reduced) equivalence, the geometric monodromy $G_{\bar\Psi_n^{\mathrm{abs}}}$ has degree $N = (\frac{n+1}{2})^2$ generators γ_0' (of order 3, so in A_N) and γ_∞' given in Lem. 4.3 as a product of disjoint cycles, one of each odd length from 1 to n. The following generalizes the case $n = 5$ in [**BF02**, Lem. 7.1] (see it for extra details). Recall: The wreath product of two groups $G \le S_n$ and H —denote this $H \wr G$ is $(H)^n \times^s G$, where the action of G on $(H)^n$ is just to permute the n coordinates.

Proposition D.2. *For $n \equiv 5 \mod 8$ we have the following identification of groups $G_{\bar\Psi_n^{\mathrm{abs}}} = A_N$, $\hat G_{\bar\Psi_n^{\mathrm{abs}}} = A_N$, $G_{\bar\Psi_n^{\mathrm{in}}} = \mathbb{Z}/2 \wr A_N$ and $\hat G_{\bar\Psi_n^{\mathrm{in}}} = \mathbb{Z}/2 \wr S_N$. In the case of the arithmetic monodromy groups, the constant extension of $\mathbb{Q}$ in which the Galois closures have their definition field in a nontrivial quadratic extension $L_n/\mathbb{Q}$ in $\mathbb{Q}(e^{2\pi i/M})$ where $M = \mathrm{lcm}(3, 5, \dots, n-2, n)$.*

Proof. Again, use [**Wm73**], whose hypotheses are a noncyclic, transitive subgroup of A_n, generated by odd pure-cycles.

By Wilson's Theorem on primes, there is a prime q between $\frac{n+1}{2}$ and $n-1$, including the end points. By putting γ_∞' to an appropriate power there is a q-cycle $g(q)$ in $G_{\bar\Psi_n^{\mathrm{abs}}}$. Use transitivity of $G_{\bar\Psi_n^{\mathrm{abs}}}$ to conjugate $g(q)$ to a q-cycle containing any particular letter of the representation. Since any two q-cycles will have overlapping support, conclude that the set of q-cycles generates a non-cyclic transitive subgroup of A_N. So, Williamson's Th. says $G_{\bar\Psi_n^{\mathrm{abs}}} = A_N$.

Now we apply the B(ranch) C(ycle) L(emma) ([**Fr77**, §5] or [**Fr08b**, Item #1]; also find it in the books of Matzat-Malle and Völklein on the RIGP) to the cover $\bar\Psi_n^{\mathrm{abs}}$ to conclude $\hat G_{\bar\Psi_n^{\mathrm{abs}}} = S_N$. Recall: This arithmetic monodromy is the group of the minimal Galois closure cover (over $\mathbb{Q}$, not over $\bar{\mathbb{Q}}$) of $\bar\Psi_n^{\mathrm{abs}}$. It is automatic that $\hat G_{\bar\Psi_n^{\mathrm{abs}}}$ is in the normalizer of $G_{\bar\Psi_n^{\mathrm{abs}}}$ in S_N, so it is either A_N or S_N.

The BCL says that a *necessary* condition for $G_{\bar\Psi_n^{\mathrm{abs}}} = \hat G_{\bar\Psi_n^{\mathrm{abs}}}$ is that the conjugacy classes of $\gamma_0', \gamma_1', \gamma_\infty'$ form a rational union. Since the conjugacy classes are distinct, this means each is a rational conjugacy class. That means $(\gamma_\infty')^u$ is conjugate to γ_∞' for all u prime to $\mathrm{ord}(\gamma_\infty')$. The following facts come from [**Fr95a**, Prop. 2.1] and the later *Irrational Cycle Lemma* in the same paper applied to the disjoint cycle of γ_∞', whose cycle lengths in that notation are the distinct odd integers $m_0 = 1, \dots, m_u = 2u+1, \dots, m_{\frac{n-1}{2}} = n$.

For each odd integer $2u + 1$ between 1 and n, let $(-1)^{t_u}$ be the parity of the inversion given by Lem. For γ_∞' to be conjugate to $(\gamma_\infty')^{-1}$ the inversions (as in Lem. 1.8) of parity $(-1)^u$ must multiply to 1. Their product, however, is -1 to the exponent $(\frac{n-1}{2})(\frac{n+1}{2})/2$. For $n \equiv 5 \mod 8$, this is odd, and so γ_∞' is not conjugate to its inverse. So, the extension of constants $L_n/\mathbb{Q}$ in the proposition is nontrivial.

The wreath product statements for the inner covers are almost identical with the argument of [**BF02**, Lem. 7.1]. The composition $\bar\Psi_n^{\mathrm{abs}} \circ \bar\Psi_n^{\mathrm{abs,in}}$ automatically implies the monodromy group of the inner cover is in the wreath product, and the critical fact is to produce an element $(1, 0, \dots, 0) \in (\mathbb{Z}/2)^n = \ker(\mathbb{Z}/2 \wr S_n \to S_n)$. As for $n = 5$, the order two ramification in the inner cover of the width 1 cusp of the absolute cover produces such an element. That concludes the proof.l □

D.3. No fine moduli for Liu-Osserman. Take $n \equiv 5 \mod 8$. Fine moduli for an equivalence class of covers in a given Nielsen class means that there is a universal family of such covers by which all such families are obtained by pullback from the universal family. Given any such family of covers $\mathcal{F} \to \mathcal{P} \times \mathbb{P}^1_z$, there is a natural map to $\mathcal{P} \to J_r$. The case $r = 4$ is special. [**BF02**, Prop. 4.7] gives if and only if criteria for covers in a reduced Nielsen class to have either of two types of fine moduli properties.

B(irational)-fine moduli means there is such a universal family for all families where the natural map to $J_4 = \mathbb{P}^1_j$ misses the elliptic points 0 and 1. Let $q_1 q_3^{-1} = \alpha_1$ and $\mathbf{sh}^2 = \alpha_2$. The criterion for this is that $K_4 = \langle \alpha_1, \alpha_2 \rangle / \langle \mathbf{sh}^4 \rangle \triangleleft H_4 / \langle \mathbf{sh}^4 \rangle$ acts faithfully. In our main examples, it is just the opposite.

LEMMA D.3. *Denote the braid orbit of $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^*$ by $O_{\boldsymbol{g}}$. If α_1 and α_2 fix the Nielsen class of $\boldsymbol{g}$, then K_4 fixes the Nielsen class of any $\boldsymbol{g}' \in O_{\boldsymbol{g}}$.*

For $n \equiv 1 \mod 4$, K_4 is trivial on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$ (so on absolute Nielsen classes, too). Thus, $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ does not (even) have b-fine moduli.

PROOF. The essence of the first sentence is that K_4 is a normal subgroup of $H_4/\langle \mathbf{sh}^4 \rangle$ [**BF02**, §2.6]. To complete the first sentence, consider any $\alpha \in K_4$, and $\boldsymbol{g}' = (\boldsymbol{g})q$. Write $q\alpha q^{-1} \stackrel{\mathrm{def}}{=} \alpha' \in K_4$. Compute:

$$(\boldsymbol{g})\alpha = ((\text{H-M}_2)q)\alpha(q^{-1})q = ((\text{H-M}_2)\alpha')q = h'(\text{H-M}_2)q)(h')^{-1} \in \text{ class of } \boldsymbol{g}.$$

[**BF02**, (2.17) on p. 104] does the case $n = 5$ by directly showing $q_1 q_3^{-1}$ acts trivially on $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{in}}$. The general case works best by showing both α_i s act trivially on inner class of $\text{H-M}_2 = (g_1, g_1^{-1}, g_2, g_2^{-1})$ with $g_1 = x_{1,\frac{n+1}{2}}$, $g_2 = x_{\frac{n+1}{2},n}$. Indeed, $(\text{H-M}_2)\alpha_i = h_i \text{H-M}_2 h_i^{-1}$, $i = 1, 2$, with

$$\begin{aligned} h_1 = &\quad (1\, \tfrac{n-1}{2})(2\, \tfrac{n-3}{2}) \dots (\tfrac{n+3}{2}\, n)(\tfrac{n+5}{2}\, n{-}1) \dots \\ h_2 = &\quad (1\, \tfrac{n+3}{2})(2\, \tfrac{n+5}{2}) \dots (\tfrac{n-1}{2}\, n). \end{aligned}$$

Clearly, h_1 is even. Since $n \equiv 1 \mod 4$, so is h_2. As $K_4 = \langle \alpha_1, \alpha_2 \rangle$, K_4 fixes $\text{H-M}_2^{\mathrm{in}}$.

For $n \equiv 5 \mod 8$, Lem. 4.4 says $\{(\text{H-M}_2)q\}_{q \in H_4} = \mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

For $n \equiv 1 \mod 8$, use that each of the two braid orbits contains an H-M rep. and apply the same argument to each. □

LEMMA D.4. *But the space $\mathcal{H}(G_1(A_n), \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ does.* [**BF02**, Lem. 7.5] *does this for level $k \geq 1$ for $n = 5$.*

D.4. Level 1 for Liu-Osserman.

REMARK D.5 ($n = 5$).

REMARK D.6 ($n = 13$).

Appendix E. Connectedness Applications

Restricting to covers of the sphere by a compact Riemann surface of a *given type*, do all such compose one connected family? Or failing that, do they fall into easily discerned components? The answer has often been "Yes!," figuring in such topics as the connectedness of the moduli space of curves of genus g (geometry), Davenport's problem (arithmetic) and the genus 0 problem (group theory). One consequence: We then know the definition field of the family components.

Restricting to covers of the sphere by a compact Riemann surface of a *given type*, do all such compose one connected family? Or failing that, do they fall into easily discerned components? The answer has often been "Yes!," figuring in such topics as the connectedness of the moduli space of curves of genus g (geometry), Davenport's problem (arithmetic) and the genus 0 problem (group theory). One consequence: We then know the definition field of the family components.

E.1. Self-contained expositions. The Regular Inverse Galois Problem (§1.2.2 §E.2) has an html definition file: http://www.math.uci.edu/~ mfried → Sect. I.b. → Definitions: Arithmetic of covers and Hurwitz spaces → * R(egular) I(nverse) G(alois) P(roblem): RIGP.html.

Similarly for an overview of **MT**s:
Outline of how Modular Towers generalizes modular curve towers:
http://www.math.uci.edu/~ mfried → Sect. I.a. → Articles: Generalizing modular curve properties to Modular Towers → Item #1 mt-overview.html.

Example conclusion: In Davenport's problem (§E), there are no nontrivial pairs of indecomposable polynomials over $\mathbb{Q}$ with the same value sets modulo all but finitely many primes. There are, however, close calls. For several Nielsen classes, representing a finite number of possible degrees, there are families of polynomial pairs (in particular, genus 0 covers) that do give the same values over a finite extension of $\mathbb{Q}$. These families have each more than one connected component, with none defined over $\mathbb{Q}$. A cover gives a bundle (in this case over $\mathbb{P}^1_z$). Then, each Hurwitz space component attached to a given Nielsen class in Davenport's problem, defines the same family of bundles over $\mathbb{Q}$.

E.2. Conway-Fried-Parker-Völklein. The following is the meaning of the phrase:

The R(egular)I(nverse)G(alois)P(roblem) holds for G:

There is a geometric Galois cover of the sphere with group G, with all its automorphisms defined over $\mathbb{Q}$. Such a regular realization of G corresponds to a rational point on an inner Hurwitz space associated to some Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ for some rational union of conjugacy classes in G (§E.2). Part of the point of this theory is that if a Hurwitz space has no rational points, then there will be no regular realizations corresponding to those conjugacy classes.

§**??** reminds that **MT**s result from a ramification restriction on the RIGP, akin to, but far less restrictive than that used in number fields for the Fontaine-Mazur Conjecture. The Main Conjecture thus says, for each p-perfect finite group, there are p-perfect covers of it for which require increasingly unbounded numbers of conjugacy classes to produce any regular realization of them.

Whatever is $N_{G,p} = |\ker(R^*_{G,p} \to G)|$, then the braid orbits on $\mathrm{Ni}(G, \mathbf{C})$ with $\mathbf{C}$ a collection of p' conjugacy classes realizing giving lift invariants can be as large as $N_{G,p}$. That certainly happens if the conjugacy classes in $\mathbf{C}$ are repeated sufficiently often. The following example appears again in §E.3.1. It shows the .

Note the many uses of H-Mreps as in [**FV91**] or [**?**].

This big topic should be helpful on understanding the major issues in the Conway-Fried-Parker-Völklein Thm. (§E.2) that still stands out as the most definitive result on connectedness of Hurwitz spaces. The [, App.] result is roughly: If you repeat *all* conjugacy classes *sufficiently* many times, then there is one connected component of the Hurwitz space (absolute or inner) of covers of the sphere in a given

Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. We engage expectations from the word *sufficiently*. Also, of necessity, we consider dropping the repetition of all classes. Not only doesn't that include the easy classical results, it defies making classical connections, including the RIGP.

This applied to show how to find Nielsen classes for which the corresponding inner Hurwitz space has a connected component with definition field $\mathbb{Q}$. They must exist, and some of them must have $\mathbb{Q}$ points, for each centerless group G if the RIGP is correct. Still, the version of the Conway-Fried-Parker-Voelklein result in [**FV92**] required unknown large values of r. It applied to create presentations of $G_{\mathbb{Q}}$, the absolute Galois group of $\mathbb{Q}$, the first, and still, only such proven presentations. The version of CFPV in §E.2 allows us to state connnectedness problems very close to the Liu-Osserman examples that reflect on all aspects of this paper, especially how explicitly lift invariants tie to connectedness results.

E.3. Pure-cycle cases of non-genus zero covers. When $r = 4$, the reduced Hurwitz space of a pure-cycle Nielsen class has a birational embedding in $\mathbb{P}^1_j \times \mathbb{P}^1_j$. It doesn't matter if the covers in the family have genus 0 or not. To see that consider such a cover $\varphi : X \to \mathbb{P}^1_z$. Then, map the four branch points $\varphi_{\boldsymbol{z}}$ to their j invariant $j_{\varphi_{\boldsymbol{z}}}$. Above each branch point z_i is a unique ramified point x_i. So, that gives the j invariant of $\boldsymbol{x}$, which we denote $j_{\varphi_{\boldsymbol{x}}}$. The birational embedding is $\varphi \mapsto (j_{\varphi_{\boldsymbol{z}}}, j_{\varphi_{\boldsymbol{x}}})$. Notice this also holds for modular curves. There is a common reason for both cases, though they do differ.

LEMMA E.1. *Suppose $r = 4$, and $\mathbf{C}$ has the property that each conjugacy class is represented by elements with a disjoint cycle of distinguished length, and also the gcd of all cycle lengths in the conjugacy class is 1. Then, the reduced space embeds in $\mathbb{P}^1_j \times \mathbb{P}^1_j$. This applies to the modular curves $X_0(p)$ because they are the Nielsen class of $(D_p, \mathbf{C}_{2^4})$, and the conjugacy class of multiplication on $\mathbb{Z}/p$ fixes just 0. Why doesn't this work for $(D_{p^{k+1}}, \mathbf{C}_{2^4})$?*

E.3.1. *Start of the* **MT** *for* $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$. Here there is only the prime 2 to consider. This is the "easiest" case of pure-cycle covers of genus exceeding 0. [**Fr06a**, Prop. 6.12] considers this case to show that both level 0 components of the reduced absolute spaces are nonmodular curves, despite — like modular curves — that they embed in $\mathbb{P}^1_j \times \mathbb{P}^1_j$ just as do modular curves.

TABLE 10. **sh**-Incidence Matrix for Ni_0^+

Orbit	${}_{\mathbf{c}}O_{1,1}$	${}_{\mathbf{c}}O_{1,3}$	${}_{\mathbf{c}}O_{3,1}$
${}_{\mathbf{c}}O_{1,1}$	1	1	2
${}_{\mathbf{c}}O_{1,3}$	1	0	1
${}_{\mathbf{c}}O_{3,1}$	2	1	0

Cusp representatives — 1st 3 for Ni^+, 2nd 3 for Ni^- — of the various cusp orbits are in this list using the corresponding subscripts.

- $\boldsymbol{g}_{1,1} = ((1\,2\,3), (1\,3\,2), (1\,3\,4), (1\,4\,3))$
- $\boldsymbol{g}_{1,3} = ((1\,2\,3), (1\,2\,4), (1\,4\,2), (1\,3\,2))$
- $\boldsymbol{g}_{3,1} = ((1\,2\,3), (1\,3\,2), (1\,4\,3), (1\,3\,4))$
- $\boldsymbol{g}_{1,4} = ((1\,2\,3), (1\,2\,4), (1\,2\,3), (1\,2\,4))$
- $\boldsymbol{g}_{3,4} = ((1\,2\,3), (1\,2\,4), (1\,2\,4), (4\,3\,2))$

TABLE 11. **sh**-Incidence Matrix for Ni_0^-

Orbit	${}_{\mathbf{c}}O_{1,4}$	${}_{\mathbf{c}}O_{3,4}$	${}_{\mathbf{c}}O_{3,5}$
${}_{\mathbf{c}}O_{1,4}$	2	1	1
${}_{\mathbf{c}}O_{3,4}$	1	0	0
${}_{\mathbf{c}}O_{3,5}$	1	0	0

- $\boldsymbol{g}_{3,5} = ((1\,2\,3), (1\,2\,4), (1\,4\,3), (2\,3\,1))$

Some comments: $\boldsymbol{g}_{1,1}$ is an H-M rep, and a 2-cusp, while $\boldsymbol{g}_{1,3}$ is the shift of an H-M rep. On the other hand, the cusp orbit of $\boldsymbol{g}_{3,5}$ has length three by Princ. **??**. From Princ. 3.3 we know immediately that the Main Conjecture holds for any H-M cusp branch. Here, however, is a harder question.

QUESTION E.2. FP 3 says there is at least one H-M component branch defining a **MT** for $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$. Does the Main Conjecture hold for every component branch?

Not much of a question if there is only one component branch, or slightly worse there are several component branches, all H-M. Neither of these, however, holds.

References

[BF02] Paul Bailey and Michael D. Fried, *Hurwitz monodromy, spin separation and higher levels of a modular tower*, Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), Proc. Sympos. Pure Math., vol. 70, Amer. Math. Soc., Providence, RI, 2002, pp. 79–220. MR MR1935406 (2005b:14044)

[BiF82] R. Biggers and M. Fried, *Moduli spaces of covers and the Hurwitz monodromy group*, Crelle's Journal **335** (1982), 87–121.

[Ca05] A Cadoret, *Rational points on Hurwitz towers*, preprint as of Jan. 2006 (2006), 1–30.

[CaD08] A. Cadoret and P. Dèbes, *Abelian constraints in Inverse Galois Theory*, preprint (2007).

[D06] P. Dèbes, *An introduction to the Modular Tower Program*, Groupes de Galois arithmétiques et différentiels, Séminaires et Congrès **13** (2006), 127–144.

[DD04] Pierre Dèbes and Bruno Deschamps, *Corps ψ-libres et théorie inverse de Galois infinie*, J. Reine Angew. Math. **574** (2004), 197–218. MR MR2099115 (2005i:12003)

[DE06] Pierre Dèbes and Michel Emsalem, *Harbater-Mumford Components and Towers of Moduli Spaces*, Journal de l'Institut Mathématique de Jussieu, **5/03** (2006), 351–371.

[DFr94] P. Dèbes and M. Fried, *Nonrigid situations in constructive Galois theory*, Pacific Journal **163** (1994), 81–122.

[Fr77] M. D. Fried, *Fields of definition of function fields and Hurwitz families and groups as Galois groups*, Communications in Algebra **5** (1977), 17-82.

[Fr78] ———, *Galois groups and Complex Multiplication*, Trans.A.M.S. **235** (1978), 141-162.

[Fr95a] ———, *Extension of Constants, Rigidity, and the Chowla-Zassenhaus Conjecture*, Finite Fields and their applications, Carlitz volume **1** (1995), 326–359.

[Fr95b] ———, *Introduction to Modular Towers: Generalizing the relation between dihedral groups and modular curves*, Proceedings AMS-NSF Summer Conference, vol. 186, 1995, Cont. Math series, Recent Developments in the Inverse Galois Problem, pp. 111–171.

[Fr02] ———, *Moduli of relatively nilpotent extensions*, Inst. of Math. Science Analysis **1267**, June 2002, Communications in Arithmetic Fundamental Groups, 70–94.

[Fr05] ———, *Relating two genus 0 problems of John Thompson*,Volume for John Thompson's 70th birthday, in Progress in Galois Theory, H. Voelklein and T. Shaska editors 2005 Springer Science, 51–85.

[Fr06a] ———, *The Main Conjecture of Modular Towers and its higher rank generalization*, in *Groupes de Galois arithmetiques et differentiels* (Luminy 2004; eds. D. Bertrand and P. Dèbes), Seminaires et Congres, Vol. **13** (2006), 165–233.

[Fr08a] ———, On my home page http://math.uci.edu/˜mfried, section: Ib. Definitions and discussions from Major Article Themes: → * Book: Intro. and Chp.s on Riemann's Existence Thm

[Fr08b] ———, On my home page http://math.uci.edu/˜mfried, section: Ib. Definitions and discussions from Major Article Themes: → * Definitions: Arithmetic of covers and Hurwitz spaces.

[Fr11] ———, *Alternating groups and moduli space lifting Invariants*, Arxiv #0611591v4. Israel J. Math. **179** (2010) 57–125 (DOI 10.1007/s11856-010-0073-2).

[Fr11b] ———, *Frattini towers and the shift-incidence cusp pairing*, in preparation.

[FK97] M. Fried and Y. Kopeliovic, *Applying Modular Towers to the inverse Galois problem*, Geometric Galois Actions II Dessins d'Enfants, Mapping Class Groups and Moduli, vol. 243, Cambridge U. Press, 1997, London Math. Soc. Lecture Notes, pp. 172–197.

[FV91] Michael D. Fried and Helmut Völklein, *The inverse Galois problem and rational points on moduli spaces*, Math. Ann. **290** (1991), no. 4, 771–800. MR MR1119950 (93a:12004)

[FV92] M. Fried and H. Völklein, *The embedding problem over an Hilbertian-PAC field*, Annals of Math **135** (1992), 469–481.

[GAP00] The GAP group, especially A. Hulpke, **GAP** — *Groups, Algorithms and Programming*, Ver. 4.2; 2000 (http://www.gap-system.org).

[Ha84] D. Harbater, *Mock covers and Galois extensions*, J. Algebra **91** (1984), 281-293.

[Is94] I.M. Isaacs, *Algebra, a Graduate Course*, Brooks/Cole Publishing, 1994.

[LOs06] F. Liu and B. Osserman, *The Irreducibility of Certain Pure-cycle Hurwitz Spaces*, to appear in AJM.

[MM99] G. Malle and B.H. Matzat, *Inverse Galois Theory*, Springer 1999, Monographs in Mathematics, ISBN 3-540-62890-8.

[Mu72] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Comp. Math. **24** (1972), 129–174.

[Se68] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, 1st ed., McGill University Lecture Notes, Benjamin, New York • Amsterdam, 1968, written in collaboration with Willem Kuyk and John Labute; 2nd corrected ed. by A. K. Peters, Wellesley, MA, 1998.

[Ser90] Jean-Pierre Serre, *Relèvements dans $\tilde{A}_n$*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 8, 477–482. MR MR1076476 (91m:20010)

[Ser92] J.-P. Serre, *Topics in Galois theory*, no. ISBN #0-86720-210-6, Bartlett and Jones Publishers, notes taken by H. Darmon, 1992.

[Ser96] J.-P. Serre, *Galois Cohomology*, 4th Edition, Springer: Translated from the French by Patrick Ion; original from Springer LN5 (1964), 1996.

[Sh71] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Pub. of Math. Soc. of Japan **11**, Princeton U. Press, 1971.

[Vo96] H. Völklein, *Groups as Galois Groups*, an Introduction, Cambridge Studies in Mathematics 1996 **53**, ISBN 0-521-56280-5.

[WS] Modular Tower web site: **http://www.math.uci.edu/˜mfried** → §I.a, 5th line down [Articles: Generalizing modular curve properties to Modular Towers] and → §I.b, 3rd listing down: Definitions: Modular Towers and Profinite Geometry.

[Wei05] ———, *Maximal ℓ-frattini quotients of ℓ-poincare duality groups of dimension 2*, volume for O. H. Kegel on his 70th birthday, Arkiv der Mathematik–Basel, 2005.

[Wm73] A. Williamson, *On primitive permutation groups containing a cycle*, Math. Zeit. **130** (1973), 159–162.

UC Irvine, emeritus

E-mail address: mfried@math.uci.edu