CONFIGURATION SPACES FOR WILDLY RAMIFIED COVERS

MICHAEL D. FRIED AND ARIANE MÉZARD

Abstract. Key words: Ramification data, regular ramification data, $\mathcal{R}$-configuration space, families of $\cup_{ij}\mathcal{R}_{ij}$ covers. Locally in the finite topology the $\cup_{ij}\mathcal{R}_{ij}$ configuration space $\mathcal{P}(\cup_{ij}\mathcal{R}_{ij})$ is a natural target for any family of type $\cup_{ij}\mathcal{R}_{ij}$. Further, any family of covers of this type should be the pullback of this target map from a family over a finite cover of the configuration space. A theorem of Garuti applies to show iso-triviality: If a family has a constant map to the configuration space, after finite pullback it is a trivial family.

1. Wild ramification type

Ramification Data: Take k algebraically closed, all extensions separable. Let $k((y))/k((x))$ given by $f(y) = x$ where f is a power series. Write as $\sum_{i=0}^{u}(f_i((y)))^{p^{\nu_i}}$ with $f_i = \sum_{j\geq d_i,(j,p)=1} a_{i,j}y^j$, $d_i > 0$, $a_{i,d_i} \neq 0$ and ν_i strictly decreasing.

1.1. **The operator $L_y^{(\ell)}$.** Let $e_i = d_i p^{\nu_i}$, and $\pi_i(e_i) = p^{\nu_i}$. Let $D_y^{(n)}$ map y^t to $\binom{t}{n}y^{t-n}$. Define $L_y^{(\ell)}$ to be the $\sum_{\nu=0}^{\ell} y^{p^\nu} D_y^{(p^\nu)}$.

Then, $L_y^{(\ell)}$ kills terms with exponents divisible by higher powers of p than p^ℓ but otherwise does not. So, as ℓ increases it is exposing lower degree terms of $f(y)$ with higher powers of p in the exponents. Let $h(-1) = \infty$,

$$h(\ell) = \min_{\psi\in k[[x]],\mathrm{ord}\psi>0} \min(\mathrm{ord}_y(L_y^{(\ell)}(\psi), h(\ell-1,\psi), \ell \in \mathbb{N}.$$

Then $\mathcal{R}(k((y))/k((x)))$ is either the set of $h(\ell)$s that differ from $h(\ell-1)$ or the graph of the pairs $(\pi(e_i), e^i)$ as e_i runs over this set of $h(\ell)$s. The graph may not be convex but its points have strictly decreasing y values.

1.2. **Regular ramification data and tame embeddings.** *Regular ramification data* is the convex hull of this graph. It gives precise data about field extensions immediately useful to applications like Schur covers and Davenport pairs.

Example 1.1. Take $p = 3$. Let $f(y) = x$ be

$$\begin{aligned} f(y) &= y^{729} + (y^4+y^5)^{243} + (y^{110})^9 + (y^{13})^{81} + (y^{370})^3 + y^{10000} \\ &= y^{3^6} + (y^4+y^5)^{3^5} + (y^{110})^{3^2} + (y^{13})^{3^4} + (y^{370})^3 + y^{10000}. \end{aligned}$$

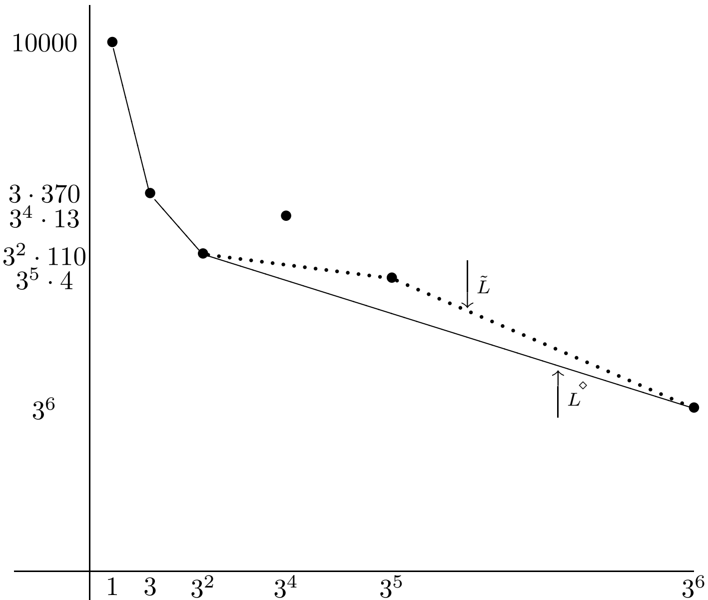
The terms are in order as to their ords.

Here are simple normalizations from a change of variables.

(1.1a) By tame extension of $k((x))$ minimal ord term is a p-power.
(1.1b) $f(y)$ is truncated beyond first p' term: $k((y))/k((x))$ totally wildly ramified.

Thus, $\mathcal{R}(x) = \{729 = 3^6, 972 = 3^5 \cdot 4, 990 = 3^2 \cdot 110, 1110 = 3 \cdot 370, 10000\}$.

Date: December 29, 2007.



$\mathcal{R}(x)$ and $\mathbf{\mathcal{R}}(x)$ for $f(y)$ in Ex. 1.1

$$f(y) = y^{3^6} + (y^4 + y^5)^{3^5} + (y^{110})^{3^2} + (y^{13})^{3^4} + (y^{370})^3 + y^{10000}$$

Example: The minimal integer multiplying all slopes of the regular ramification data to integers bounds the ramification of tame embeddings of $k((y))$ (fixed on $k((x))$) into the algebraic closure of $k((y))/k((x))$. This is the tame index $\mathrm{ind}^{\mathrm{tm}}(y/x)$ of the extension.

1.3. **The configuration space.** Main Theorem of [FM, §2]: Coefficients of semi-canonical polynomials giving all the field extensions of type $\mathcal{R}(y/x) = \mathcal{R}$ form an explicit affine open subset $\mathcal{P}(\mathcal{R})$ of an affine space. The nonzero coefficients are terms with exponents $N(\mathcal{R})$, $\mathbb{N}$ with the following exponents removed:

$$M_1(\mathcal{R}) = \quad \{j < e_0, \text{ or } e_{i-1} < j < e_i \text{ and } \pi(e_{i-1})p \not| j\}; \text{and}$$
$$M_2(\mathcal{R}) = \quad \{j \text{ of form } j(\mu) = \min_{0\le \ell \le t} E_\ell + \mu\pi(E_\ell) \text{ for some } \mu\}.$$

This *configuration space* for ramification data $\mathcal{R}$ has but finitely many points representing each field extension. This space $\mathcal{P}(\mathcal{R})$ is locally versal (in the finite topology) for any family of local extensions of ramification type $\mathcal{R}$.

2. Comparing tame and wild configuration spaces

The fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ over $\mathbb{C}$ has the notion of classical generators (non-intersecting conjugacy classes of loops around branch points satisfying the product-one condition).

2.1. **Tame configuration.** For covers of $U_{\boldsymbol{z}}$ with group G you can use elements in a given set of conjugacy classes satisfying the product-one condition. Given two such covers, you can ask what are the relations between the two covers. Example: Do they fit together in a smooth connected family? Could they be conjugate under the action of $G_{\mathbb{Q}}$? If the sets of conjugacy classes for generators of G are different for the two covers they cannot be connected, and they have have different cyclotomic orbits, they cannot $G_{\mathbb{Q}}$ connected. This is the first and simplest criterion for distinquishing between covers over $\mathbb{C}$.

This is so simple that we can use a very simple configuration space to detect the difference between two covers over $\mathbb{C}$, a space $U_r = \mathbb{P}^r \setminus D_r$ detecting the location of the branch points of a cover in a given Nielsen class. Here are key configuration space properties for the r-branch point tame case, from Grothendieck's Theorem, when the monodromy group G has order prime to the characteristic.

(2.1a) It is a natural target for any family of given type.

(2.1b) Any family of covers of the requisite type is the pullback of this target map from a family over an étale cover of the configuration space.

2.2. **Tame versus wild isotrivial result.** The success of Hurwitz spaces on the Inverse Galois Problem and in vastly generalizing modular curve towers is based on conditions precisely identifying one étale cover that works for all families. When p divides $|G|$, we retain (a), but we must relax (b) to be an étale cover only of the range (not of the whole configuraton space).

A consequence of even this weak form is that a constant map to the target space implies the family is iso-trivial. The Fried-Mezard paper produces a definition of type and a configuration space that works like (a) except you must use the finite topology. From this one gets the following iso-triviality result that takes account of all the local wildly ramified extensions.

Given a family $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ of type $\cup_{ij} \mathcal{R}_{ij}$, refer to a finite cover $\mu : \mathcal{P}' \to \mathcal{P}$ providing the map Ψ' to the configuration space as a *configuration cover*.

Proposition 2.1 (Isotrivial Proposition)**.** *Suppose after pullback to a configuration cover $\mu : \mathcal{P}' \to \mathcal{P}$, the map Ψ' is constant. Then, after pullback to a further finite cover $\mathcal{P}'' \to \mathcal{P}'$ of $\mathcal{P}'$, the family is isomorphic to $\mathcal{P}'' \times X_0 \to \mathcal{P}'' \times \mathbb{P}^1_z$ by a map that is the identity in the first factor.*

2.3. **Galois closure and Garuti's Theorem.** With Hurwitz spaces the topic of Galois closure is a significant issue, especially displaying in a cover the difference between having not having automorphisms. The key here is that all covers in a tame r-branch point family (of not necessarily Galois covers) have the same geometric Galois group (given by the family version of the Galois closure construction). This is not true for wildly ramified covers. Even over $\mathbb{C}$, classical geometers hardly recognized the Galois closure, using primarily simple-branched covers.

Suppose $\Phi : \mathcal{T} \to \mathcal{P} \times \mathbb{P}^1_z$ be a family of degree n covers as above over a field k. Form the Galois closure of Φ: Normalize a component $\hat{\mathcal{T}}$ of the n-fold fiber product $\mathcal{T}^{(n)} \stackrel{\text{def}}{=} X^{(n)}_{\varphi} \stackrel{\text{def}}{=} X^{(n)}_{\mathcal{P} \times \mathbb{P}^1_z}$ of the fat diagonal of Φ. Next let $\hat{\mathcal{P}}$ be the integral closure of $\mathcal{P}$ in the function field of $\hat{\mathcal{T}}$. This gives a family $\hat{\Phi} : \hat{\mathcal{T}} \to \hat{\mathcal{P}} \times \mathbb{P}^1_z$. For tame covers the fibers are Galois and irreducible. For wildly ramified covers, not necessarily so.

The proof of Thm. 2.1 uses the following result of Garuti [Ga].

Theorem 2.2. *Let Y_k be a smooth projective curve over an algebraically closed field k of positive characteristic p. Let $f_k : X_k \to Y_k$ be a finite Galois covering of Y_k, with Galois group G. Let R be a characteristic zero, complete discrete valuation ring, of residue field k, and fix a smooth lifting Y of Y_k to R. Then after a finite extension R'/R, there exists a generically étale covering of proper normal R'-curves $f' : Y' \to Y' = Y \times_R R'$ of Galois group G such that the special fibre*

X'_k can only have cusps as singular points and admits a G-equivariant morphism of normalisation $X_k \to X'_k$ which is an isomorphism outside the ramified points.

References

[FM] M. D. Fried and Ariane Mézard, *Configuration spaces of wildly ramified covers*, MSRI Fall 1999 on Arithmetic fundamental groups, PSPUM series of AMS, to appear soon.

[Ga] M. Garuti, *Prolongement de revêtements galoisiens en géométrie rigide. [Extension of Galois coverings in rigid geometry]* Compositio Math. **104** (1996), no. 3, 305–331. MR 98m:14023

[Gr] A. Grothendieck, Revêtements étales et groupe fonda mental (SGA I). Lecture notes in mathematics **224**, Springer-Verlag (1971).

UC Irvine, Irvine, CA 92697, USA
E-mail address: mfried@math.uci.edu