C(onway)F(ried)P(arker)V(oelklein) connectedness results

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Finite group G, collection of distinct (nonidentity) generating conjugacy classes $\mathbf{C}' = \{\mathrm{C}'_1, \dots, \mathrm{C}'_{r'}\}$ of G: *seed classes*.

Basic Topic: Deciphering Hurwitz space components defined by G and conjugacy classes $\mathbf{C}$ subject to:

(*) $\mathbf{C}$ is supported in the seed classes: It has $\mathbf{C}'$-support.

Nielsen Class Interpretation: Find orbits of **sh** and q_1 on

$\mathrm{Ni}(G, \mathbf{C}) = \{(g_1, \dots, g_r) \mid \langle \boldsymbol{g} \rangle = G, \boldsymbol{g} \in G^r \cap \mathbf{C}, g_1 \cdots g_r = 1\}$:

sh : $\boldsymbol{g} \mapsto (g_2, \dots, g_r, g_1), q_1 : \boldsymbol{g} \mapsto (g_1 g_2 g_1^{-1}, g_1, g_3, \dots, g_r)$.

An extra condition appears in the most general results:

(*^{2}) Each seed class has high multiplicity in $\mathbf{C}$: High $\mathbf{C}'$-support.

The cases of consideration

Modular curves: $r' = 1$, $\mathbf{C}'$ is involution class $\mathrm{C}_{\mathrm{inv}}$ in dihedral groups $\{D_{p^{k+1}}\}_{k=0}^{\infty}$, $\mathbf{C} = \mathrm{C}_{\mathrm{inv}^4}$. We compare 4 other cases.

[Cn_1] Moduli space of curves of genus g: Connectedness of Hurwitz spaces defined by $\mathrm{C}_2 =$ 2-cycles in S_n (Clebsch, 1872).

[Cn_2] Hurwitz space components defined by $\mathrm{C}_3 =$ 3-cycles and the parity of a particular linear system (Fried, 1990, [Fr09a]).

[Cn_3] Spaces of genus 0 pure-cycle covers (Liu-Osserman, 2007,[LOs09]).

[Cn_4] Hurwitz spaces for Nielsen classes with all conjugacy classes appearing sufficiently often (CFPV, 1991, [FrV91]).

§I. 3-cycle Hurwitz spaces and Spin Invariants

§II. Lessons from Alternating Group Hurwitz spaces

§III. The **sh**-incidence cusp pairing for $(A_4, \mathbf{C}_{\pm 3^2})$

§I. 3-cycle Hurwitz spaces and Spin Invariants

Topic [Cn$_2$]: Components of Hurwitz spaces defined by the Nielsen class $G = A_n$ and $\mathbf{C} = \mathbf{C}_{3^r}$ (conjugacy classes of) $r \geq n-1$ 3-cycles.

Note: When $n = 4$ (or 3) there are two conjugacy classes of 3-cycles, so $\mathbf{C}_{3^r}$ is ambiguous.

Describing the $(A_n, \mathbf{C}_{3^r})$ Hurwitz space components generalizes Serre's Stiefel-Whitney approach to Spin covers [Ser90].

§I.A. Quick start on Schur Multipliers

Frattini cover $G' \to G$: Group cover with restriction to any proper subgroup of G' not a cover. Get *small* lifting invariant from any *central* Frattini extension $\psi : R \to G$: $\ker(R \to G)$ is a quotient of the Schur multiplier, SM_G, of G.

Def: **C** is $|R/G|'$: For Nielsen class $\mathrm{Ni}(G, \mathbf{C})$, elements of **C** have order prime to $|\ker(R \to G)|$.

When $\ker(\psi)$ is p-part of Schur multiplier, ψ is a p-representation Cover — maximal central p-Frattini extension of G (unique if G is p-perfect).

Special case: R/G is Spin_n/A_n

Spin_n^+ is (unique) nonsplit degree 2 cover of the connected component O_n^+ (of I_n) of orthogonal group. Regard S_n as $< O_n$ (orthogonal group); $A_n < O_n^+$, kernel of the determinant map.

Spin_n: Pullback of A_n to Spin_n^+:

$$\ker(\mathrm{Spin}_n \to A_n) = \{\pm 1\}.$$

Odd order elements of S_n are in A_n. We get much information from this case: **C** is $|\mathrm{Spin}_n/A_n|' = 2'$:

C consists of odd order elements of A_n.

§I.B. R/G Lifting Invariant if **C** is $|R/G|'$

Small Schur-Zassenhaus: Each $g \in$ **C** has a unique same-order lift $\hat{g} \in R$. For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, $\hat{\boldsymbol{g}} \stackrel{\text{def}}{=} (\hat{g}_1, \dots, \hat{g}_r) \in R^r \cap \mathbf{C}$.

$s_{R/G}$: $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}) \mapsto$

$$s_{R/G}(\boldsymbol{g}) = \Pi(\hat{\boldsymbol{g}}) \stackrel{\text{def}}{=} \hat{g}_1 \cdots \hat{g}_r \in \ker(R \to G).$$

If $G \le A_n$, $\varphi_n : \mathrm{Spin}_n \to A_n$, and **C** has only odd order elements, lifting $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ to $\mathbf{C} \cap (\psi_n^{-1}(G))^r$ still makes sense. If $\psi_n^{-1}(G) \to G$ splits, then

$s_{\psi_n^{-1}(G)/G}(\boldsymbol{g})$ is always trivial.

§I.C. A Formula for the Spin-Lift Invariant

For odd order $g \in A_n$, let $w(g)$ be the number disjoint cycles of length l in g with $\frac{l^2-1}{8} \equiv 1 \mod 2$.

Theorem 1 (Fried-Serre). *If* $\varphi : X \to \mathbb{P}^1$ *is in Nielsen class* $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$*, then* $\deg(\varphi) = n$*, X has genus 0, and* $s(\varphi) = (-1)^{n-1}$.

Generally, for any genus 0 Nielsen class of odd order elements, and representing $\boldsymbol{g} = (g_1, \dots, g_r)$*,* $s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g})$ *is constant, equal to* $(-1)^{\sum_{i=1}^{r} w(g_i)}$.

Serre asked me in 1989 about 3-cycle case for genus 0 in 1989. [Ser90] proved general case. Short proof [Fr09a, Cor. 2.3] shows how general case follows from 3-cycle case.

Meaning of Spin lifting invariant

Assume $\varphi : X \to \mathbb{P}^1_z$ has odd order branching; $\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z$ (its geometric Galois closure of) with group A_n. Then, $s_{\mathrm{Spin}_n/A_n}(\varphi) = 1 \implies \exists \mu : Y \to \hat{X}$ unramified: $\varphi \circ \mu$ is Galois with group Spin_n.

Exercise: Genus 0 assumption doesn't apply to

$$\boldsymbol{g}_1 = ((1\,2\,3)^{(3)}, (1\,4\,5)^{(3)}), \text{ or to}$$

$$\boldsymbol{g}_2 = ((1\,2\,3)^{(3)}, (1\,3\,4), (1\,4\,5), (1\,5\,3)),$$

but you can easily compute $s(\boldsymbol{g}_1) = 1, s(\boldsymbol{g}_2) = -1$.

§I.D. Main Theorem: H_r orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$, $r \geq n-1 \geq 3$

There are either 1 or 2 components, each determined by its *Spin (lift) invariant* value [Fr09a].

Example: Let $G = A_4$ and $\mathbf{C}_{\pm 3^2}$ two repetitions of the two conjugacy classes $\mathrm{C}_{\pm 3}$ (with respective representatives $(1\,2\,3)$ and $(3\,2\,1)$) of 3-cycles in A_4. Then, $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$ contains

$$\boldsymbol{g}_{4,+} = ((1\,3\,4), (1\,4\,3), (1\,2\,3), (1\,3\,2)) \text{ and}$$

$$\boldsymbol{g}_{4,-} = ((1\,2\,3), (1\,3\,4), (1\,2\,4), (1\,2\,4)).$$

$s(\boldsymbol{g}_{4,+}) = +1$ and $s(\boldsymbol{g}_{4,-}) = -1$: Can't braid $\boldsymbol{g}_{4,+}$ to $\boldsymbol{g}_{4,-}$!

General Braid Principle: Lift invariants are constant on covers representing points on each component.

I.E. Constellation of spaces $\mathcal{H}(A_n, \mathbf{C}_{3^r})^*$, *=abs/in.

Label each component $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^*$ (resp. $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^*$) at locus for (n, r) with a symbol $\oplus$ (resp. $\ominus$) for lift invariant $+1$ (resp. -1). Next page explains the diagram.

$\xrightarrow{g \geq 1}$	$\ominus\oplus$	$\ominus\oplus$	. . .	$\ominus\oplus$	$\ominus\oplus$	$\xleftarrow{1 \leq g}$
$\xrightarrow{g=0}$	$\ominus$	$\oplus$	. . .	$\ominus$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	. . .	n even	n odd	$4 \leq n$

Theorem 2. *Generalizing Fried-Serre: For* $r = n-1$, $n \geq 5$, $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{in}}$ *has exactly one connected component and* $\Psi^{\mathrm{in}}_{\mathrm{abs}} : \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{in}} \to \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$ *has degree 2.*

Row of tag $\xrightarrow{g \geq 1}$ illustrates that Nielsen class + lift invariant determines Hurwitz space component.

Theorem 3. *For* $r \geq n \geq 5$, $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$ *has two connected components, symbols* $\oplus$ $\ominus$. *Denote images in* $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ *by* $\mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$*:*

$$\Psi^{\mathrm{in},\pm}_{\mathrm{abs}} : \mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}} \to \mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}} \text{ (degree 2)}.$$

Each of $\mathcal{H}_{+}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$ and $\mathcal{H}_{-}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$ has definition field $\mathbb{Q}$, and a dense subset of $\mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ give $(A_n, S_n, \mathbf{C}_{3^r})$ geometric/arithmetic monodromy realizations.

§I.F. Braiding 3-cycle Nielsen classes to Normal Form

Aim braiding general $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ to $(g, g^{-1}, g_3, \dots, g_r)$. Proof inducts on lengths of *d(isappearing) s(equence)s*: A *ds* for 1 in $\boldsymbol{g}$ is a chain of 3-cycles with this effect:

$$1 \mapsto i_1 \mapsto i_2 \mapsto \cdots \mapsto i_{k-1} \mapsto 1.$$

Can always braid cycles $i_1, \dots, i_k$ to 1st k positions of $\boldsymbol{g}$. Coalescing Lemma of [Fr09a] says: If $n \ge 4$ can braid to

1. $(g_1, g_1^{-1}, \dots)$; or 2. $((1\,2\,3), (1\,3\,4), (1\,4\,2), \dots)$; or

3. $((1\,2\,3)^{(3)}, \dots)$(Strong Coelescing says, don't need #3.)

If $n \ge 5$, can braid to where #1 holds. Special cases: Braid 6-tuples with $n \le 7$ to a normal form.

Example: Braid $\boldsymbol{g}_1$ on p. 8 to a *H(arbater)-M(umford)* rep.:

$$(g_1, g_1^{-1}, \dots, g_s, g_s^{-1}).$$

§II. Lessons from Alternating Group Hurwitz spaces

[AL$_1$] Each $\mathcal{H}_+(A_n, \mathbf{C}_{3^r}) \Leftrightarrow$ braid orbit of a special rep. $\boldsymbol{g}$ (g(roup)-$2'$ rep; §II.E), sometimes $\mathrm{H\text{-}M}$, but for r odd, definitely not.

[AL$_2$] For $R \to G$ a p-representation cover (p. 3) and $\mathbf{C}$ is $|R/G|'$, at least as many braid orbits as

$$S_{R/G,\mathbf{C}} = |\{s_{R/G}(\boldsymbol{g}) \in R/G\}|_{\boldsymbol{g}\in \mathrm{Ni}(G,\mathbf{C})}.$$

[AL$_3$] $S_{R/G,\mathbf{C}} = |R/G|$ if $\mathbf{C}$ has high $\mathbf{C}'$ support.

[AL$_4$] $\exists r_0$ so that if $r \ge r_0$, $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ braids to $(\boldsymbol{g}^*, U_{\text{H-M},n})$: $\boldsymbol{g}^* \in \mathrm{Ni}(A_{n^*}, \mathbf{C}_{3^{r^*}})$, $r^* \le r_0$, $U_{\text{H-M},n}$ a *generating* $\mathrm{H\text{-}M}$ rep.

[AL$_5$] All $U_{\text{H-M},n}$ s are braid equivalent: Thm. 2 (p. 10). [AL$_3$] plus [AL$_4$] means high $\mathbf{C}'$ support $\implies$ exactly two braid orbits.

§II.A. Dropping the $|R/G|'$ condition

Clue from oldest connectedness result, [Cn$_1$] (p. 1): There is one braid orbit on $\mathrm{Ni}(S_n, \mathbf{C}_{2^r})$, $r \ge (n-1)$: nonempty iff r is even ([BiF86, App.], same proof in [Vo95, Lem. 10.15]).

Nontrivial Schur-multiplier: SM_{S_n}, $n \ge 4$, is $(\mathbb{Z}/2)^2$. If [AL$_3$] held, we might guess there are 4 Hurwitz space components (at least for r large), not 1 as given by Clebcsh.

Explanation for one component: No way to specify unique lift of 2-cycle $g \in S_n$ to representation cover $R_n \to S_n$.

For lift $\hat{g}$ of g and $\hat{h}$ of any $h \in S_n$, though $g^{-1}hgh^{-1} = 1$, $\hat{g}^{-1}\hat{h}\hat{g}\hat{h}^{-1} \in \ker(R_n \to S_n)$ may not be 1 (App. B).

§II.B. Combine 2-cycle and 3-cycle cases for CFPV

Pick a representative $g \in \mathrm{C}'$ for each C' in $\mathbf{C}'$. Pick any lift $\hat{g}$ of it to R. Let $M_{\mathbf{C}'}$ be the subgroup of $\ker(R \to G)$ generated by $\{\hat{h}\hat{g}\hat{h}^{-1}\hat{g}^{-1}\}$, $h \in G$ and $1 \le i \le r$, with $hgh^{-1}g^{-1} = 1$.

General version of [AL_3]: Define $S_{R/G,\mathbf{C}}$ to be $|R/G|/|M_{\mathbf{C}'}|$.

Without sufficiently high $\mathbf{C}'$ support these may happen.

Reason 1: Since $R/M_{\mathbf{C}'} \to G$ is Frattini, replacing the entries of $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ by the lifted entries $\hat{\boldsymbol{g}} \in (R/M_{\mathbf{C}'})^r$ assigned above, gives $\langle \hat{\boldsymbol{g}} \rangle = R/M_{\mathbf{C}'}$. Still, $\{s_{(R/M_{\mathbf{C}'})/G}(\boldsymbol{g})\}_{\boldsymbol{g} \in \mathrm{Ni}}$ might not achieve all of $|R/G|/|M_{\mathbf{C}'}|$.

Reason 2: With $m \in (R/G)/M_{\mathbf{C}} \stackrel{\text{def}}{=} \mathrm{SM}^*_{G,\mathbf{C}'}$, there may be more than one braid orbit on $\{\boldsymbol{g} \in \mathrm{Ni} \mid s_{(R/M_{\mathbf{C}'})/G}(\boldsymbol{g}) = m\}$.

§II.C. Geometric Branch-Generation (CFPV)

For Nielsen classes $\mathrm{Ni}(G, \mathbf{C})$, with $\mathbf{C}$ of high $\mathbf{C}'$ support, there are exactly $SM^*_{G,\mathbf{C}'}|$ braid orbits (so, geometric components of corresponding Hurwitz spaces).

Common 2-cycle and 3-cycle results: Each 2-cycle (resp. 3-cycle) Nielsen element braids to a $\mathrm{H\text{-}M}$ rep. (resp. a tuple list – mildly dependent on n – juxtaposed with an $\mathrm{H\text{-}M}$ rep.).

Akin to this, Conway and Parker used something similar: $U_G = (\dots, g^{\mathrm{ord}(g)}, \dots)_{g \in G \setminus \{1\}}$. As in [AL$_5$] (p. 12), up to braid equivalence, the order of juxtaposition is irrelevant. [AL$_4$] implies a semigroup equivalence on two arrays $\boldsymbol{g}, \boldsymbol{g}^*$ running over all allowable $\mathbf{C}$: $(U^{(j)}_{\mathrm{H\text{-}M},n}, \boldsymbol{g})$ braids to $(U^{(j^*)}_{\mathrm{H\text{-}M},n}, \boldsymbol{g}^*)$.

Conway-Parker observation

For $U_{\text{H-M},n}$ (or for U_G) the semigroup equivalence gives a group structure on the union of arrays running over all classes supported in $\mathbf{C}'$. Following their rough outline, and correcting some points, [FrV91, App.] does the case when $\mathbf{C}'$ includes all non-trivial conjugacy classes of G.

The length of U_G and mysteries behind $\mathrm{SM}^*_{G,\mathbf{C}'}$ make it difficult to explicitly compute components. So, [FrV91, Lem. 4] showed, if you replace G with a cover, and use all conjugacy classes, then $|\mathrm{SM}^*_{G,\mathbf{C}'}| = 1$ if $\mathbf{C}'$ contains all nontrivial classes.

§II.D. Arithmetic Branch-Generation [FrV91, Prop. 1]

Assume G is centerless and $\mathbf{C}'$ is a distinct *rational union* of (nontrivial) classes in G. An infinite set $I_{G,\mathbf{C}'}$ indexes distinct absolutely irreducible $\mathbb{Q}$ varieties $\Theta_{G,\mathbf{C}',\mathbb{Q}} = \{\mathcal{H}_i\}_{i \in I_{G,\mathbf{C}'}}$.

- There is a finite-one map $i \in I_{G,\mathbf{C}'} \mapsto {}_i\mathbf{C}$ (r_i unordered conjugacy classes of G supported in $\mathbf{C}'$); and
- the R(egular) I(nverse) G(alois) P(roblem) holds for G with conjugacy classes $\mathbf{C}$ supported in $\mathbf{C}'$ $\Leftrightarrow$

$$\exists\, i \in I_{G,\mathbf{C}'} \text{ with } \mathcal{H}_i \text{ having a } \mathbb{Q} \text{ point.}$$

The conjugacy class collections run over rational unions of classes with high $\mathbf{C}'$ support. Then, the component with lifting invariant 1 will certainly have definition field $\mathbb{Q}$.

Using the result

The centerless condition assures *fine moduli* holds for the Hurwitz space components. So, a $\mathbb{Q}$ point corresponds to an actual regular realization of G. For finite field results:

- So, long as the prime of reduction does not divide $|G|$ then reduction is good, and Hurwitz space components modulo that prime correspond to characteristic zero components.
- Each Hurwitz space component is defined over a computable cyclotomic field depending on the Nielsen class and the lifting invariant value. From class field theory for cyclotomic fields you can compute the definition fields of reductions.

§II.E. Identifying components using g-p' reps.

Suppose you don't know the p part of the Schur multiplier. Yet, can find a braid orbit O with p-lifting invariant 1, if $\boldsymbol{g} \in O$ is a g(roup)-p' rep. A 1st order *g-p' rep.* partitions as $(\boldsymbol{h}_1 = \{g_1, \dots, g_{i_1}\}, \dots, \boldsymbol{h}_t = \{g_{i_{t-1}+1}, \dots, g_{i_t}\}), \ i_t = r$ satisfying:

1. $\langle \boldsymbol{h}_i \rangle = G_i$ is a p' group; and
2. $\langle \Pi(\boldsymbol{h}_i), i = 1, \dots, t \rangle$ is also a p' group.

Higher order (inductive definition) of g-p' cusp is in App. A.

g-p' reps. are transparent to Schur multipliers [Fr06,Princ. 3.6]

Proposition 4. *With $G^* \to G$ a p-Frattini cover, above a g-p' rep. $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ is a g-p' rep. $\tilde{\boldsymbol{g}} \in \mathrm{Ni}(G^*, \mathbf{C})$.*

Proof: Use notation for $\boldsymbol{g}$ a g-p' rep. By Schur-Zassenhaus, each $G_i = \langle \boldsymbol{h}_i \rangle$ lifts to $\tilde{G}_i \le G^*$, uniquely up to conjugacy by $\ker_{G^*} = \ker(G^* \to G)$. Let $\tilde{\boldsymbol{h}}_i$ be the corresponding lift of $\boldsymbol{h}_i$, with $\Pi(\tilde{\boldsymbol{h}}_i) = \tilde{m}_i$, $i = 1, \dots, t$.

Now we choose $\tilde{\boldsymbol{h}}_i$ to satisfy product-one: $\prod_{i=1}^t \tilde{m}_i = 1$.

Schur-Zassenhaus: $H = \langle m_1, \dots, m_t \rangle$ lifts to $\tilde{H} \le G^*$. With $\tilde{m}'_i$ the corresponding lift of m_i in H,

$$\prod_{i=1}^t m_i = 1 \implies \prod_{i=1}^t \tilde{m}'_i = 1.$$

The p' m_j s lift uniquely up to conjugacy. With $\tilde{m}'_i = u_i \tilde{m}_i u_i^{-1}$, $u_i \in \ker_0$, $(u_1 \tilde{\boldsymbol{h}}_1 u_t^{-1}, \dots, u_t \tilde{\boldsymbol{h}}_t u_t^{-1})$ lifts $\boldsymbol{g}$ to $\mathrm{Ni}(G^*, \mathbf{C})$.

III. The **sh**-incidence cusp pairing for $(A_4, \mathbf{C}_{\pm 3^2})$

Goal: There are two components $\bar{\mathcal{H}}_{\pm}$ (p. 10). We want their branch cycle description $(\gamma_0^{\pm}, \gamma_1^{\pm}, \gamma_\infty^{\pm})$ as j-line covers.

First: Nielsen class $\mathrm{Ni}(A_3, C_{\pm 3^2})^{\mathrm{in}} = \mathrm{Ni}(A_3, C_{\pm 3^2})^{\mathrm{ab}}$. Elements $\Leftrightarrow$ 6 arrangements of conjugacy classes. Outer automorphism of A_n ($n = 3$ or 4; conjugate by $(1\,2) \in S_n$) sends a conjugacy class arrangement to its complement.
List of the arrangements, and complements:

$$\begin{array}{lll} [1]\, + - + - & [2]\, + + - - & [3]\, + - - + \\ [4]\, - + - + & [5]\, - - + + & [6]\, - + + -. \end{array}$$

Cusps in $(A_4, \mathbf{C}_{\pm 3^2})$ over [1]

$\mathcal{Q}'' = \langle q_1 q_3^{-1}, \mathbf{sh}^2 \rangle$ equates elements in this list with their complements. So, inner reduced classes and absolute (not reduced) classes are the same. Conclude:
$\mathcal{H}(A_3, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}} \to \mathbb{P}^1_j$ has degree three with branch cycles

$$(\gamma_0^*, \gamma_1^*, \gamma_\infty^*) = ((1\,3\,2), (2\,3), (1\,2)).$$

Map $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2}) \to \mathrm{Ni}(A_3, \mathbf{C}_{\pm 3^2})$: If $\boldsymbol{g} \mapsto [1]$ (with no loss $g_1 = (1\,2\,3)$), then either
$\boldsymbol{g} = \boldsymbol{g}_{1,1} = ((1\,2\,3), (1\,3\,2), (1\,3\,4), (1\,4\,3))$ or $g_1 g_2$ has order 2. Listing the 4 order 2 elements gives 5 elements in $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ lying over [1].

III.A. Effect of q_2 (middle twist)

q_2^2 on $\boldsymbol{g}_{1,1}$ conjugates middle two by $(1\,4)(2\,3)$ to give

$$\boldsymbol{g}_{1,2} = ((1\,2\,3),(4\,2\,3),(4\,2\,1),(1\,4\,3)):\text{ length 4 cusp.}$$

q_2^2 fixes $\boldsymbol{g}_{1,3} = ((1\,2\,3),(1\,2\,4),(1\,4\,2),(1\,3\,2))$: length 2 cusp.

Similarly, q_2^2 on $\boldsymbol{g}_{1,4} = ((1\,2\,3),(1\,2\,4),(1\,2\,3),(1\,2\,4))$ conjugates middle pair by $(1\,3)(2\,4) \implies$

$\boldsymbol{g}_{1,5} = ((1\,2\,3),(1\,2\,4),(2\,4\,3),(1\,4\,3))$ gives a length 4 cusp.

III.B. Effect of γ_∞ on $\boldsymbol{g} \in (A_4, \mathbf{C}_{\pm 3^2})$ over [3]

The H-M rep. $\boldsymbol{g}_{3,1} = ((1\,2\,3), (1\,3\,2), (1\,4\,3), (1\,3\,4))$ maps to [3] in A_3. Applying γ_∞ gives

$$\boldsymbol{g}_{3,2} = ((1\,2\,3), (1\,2\,4), (1\,3\,2), (1\,3\,4)),$$

the same as conjugating on the middle two by $(2\,4\,3)$. The result is a length 3 γ_∞ orbit.

On Nielsen elements over [3], γ_∞ has one length 3 orbit and two of length one. See by listing 2nd and 3rd positions $(g_1 = (1\,2\,3))$. Label as

$$1' = ((1\,3\,2),(1\,4\,3)), 2' = ((1\,2\,4),(1\,3\,2)),$$
$$3' = ((1\,2\,4),(2\,3\,4)), 4' = ((1\,2\,4),(1\,2\,4)),$$
$$5' = ((1\,2\,4),(1\,4\,3)).$$

Effect of γ_∞: It fixes $4'$ and maps $5'$ to

$((1\,2\,3),(2\,3\,4),(1\,2\,4),(3\,1\,2))$ (conjugate by $(1\,2\,3)$ to $5'$).

These computations establish the orbit lengths:

$$(g_{1,1})\gamma_\infty = ((1\,2\,3),(1\,4\,2),(1\,3\,2),(1\,4\,3)) = (3')\textbf{sh},$$
$$(g_{1,3})\gamma_\infty = ((1\,2\,3),(1\,4\,2),(1\,2\,4),(1\,3\,2)) = (1')\textbf{sh}.$$

There are cusps of width 2,3 and 4 (orbit of $1' \to 2' \to 3'$ cycle): Ni_0^+ are Nielsen reps. in this $\bar{M}_4$ orbit.

III.C. **sh**-incidence Matrix on $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$

sh-incidence matrix of Ni_0^+ comes from knowing $\boldsymbol{g}_{1,1}, \boldsymbol{g}_{1,2}, \boldsymbol{g}_{1,3}$ over [1] are permuted as a set by **sh**. They map by γ_∞ respectively to $\boldsymbol{g}_{2,1}, \boldsymbol{g}_{2,2}, \boldsymbol{g}_{2,3}$ over [2]. Under γ_∞ these map resp. to $\boldsymbol{g}_{1,2}, \boldsymbol{g}_{1,1}, \boldsymbol{g}_{1,3}$, while $\boldsymbol{g}_{3,1}, \boldsymbol{g}_{3,2}, \boldsymbol{g}_{3,3}$ cycle among each other. So, three γ_∞ orbits, $O_{1,1}$, $O_{1,3}$ and $O_{3,1}$ on Ni_0^+ named for the subscripts of a representing element.

The data above shows

$|O_{1,1} \cap (O_{3,1})\mathbf{sh}| = 2$, $|O_{1,3} \cap (O_{3,1})\mathbf{sh}| = 1$. Compute: $(\boldsymbol{g}_{1,3})\mathbf{sh}$ is $g_{1,1}$ so $|O_{1,1} \cap (O_{1,3})\mathbf{sh}| = 1$.

sh-incidence Matrix for Ni_0^+

Remaining entries from symmetry of **sh**-incidence matrix and sum in a row or col. is total in set labeling that row or col.

Orbit	$O_{1,1}$	$O_{1,3}$	$O_{3,1}$
$O_{1,1}$	1	1	2
$O_{1,3}$	1	0	1
$O_{3,1}$	2	1	0

Similarly, the **sh**-incidence matrix of Ni_0^- comes from the following data. Elements $\boldsymbol{g}_{1,4}, \boldsymbol{g}_{1,5}$ over [1] map by γ_∞ respectively to $\boldsymbol{g}_{2,4}, \boldsymbol{g}_{2,5}$ over [2], and these map respectively to $\boldsymbol{g}_{1,5}, \boldsymbol{g}_{1,4}$, while γ_∞ fixes both $\boldsymbol{g}_{3,4}, \boldsymbol{g}_{3,5}$.

sh-incidence Matrix for Ni_0^-

So, there are three γ_∞ orbits, $O_{1,4}$, $O_{3,4}$ and $O_{3,5}$ on Ni_0^-.

Orbit	$O_{1,4}$	$O_{3,4}$	$O_{3,5}$
$O_{1,4}$	2	1	1
$O_{3,4}$	1	0	0
$O_{3,5}$	1	0	0

Lemma 5. *In general,* **sh***-incidence matrix is same as matrix from replacing* **sh** $= \gamma_1$ *by* γ_0. *Only possible elements fixed by either lie in* γ_∞ *orbits* O *with* $|O \cap (O)\mathbf{sh} \neq 0|$.

On Ni_0^+ *(resp.* Ni_0^-*),* γ_1 *fixes 1 (resp. no) element(s), while* γ_0 *fixes none.*

III.D. Wohlfahrt shows neither component is a modular curve

Label components of $\bar{\mathcal{H}}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{abs,rd}}$: $\bar{\mathcal{H}}_{\pm}^{\mathrm{abs,rd}}$.

Goal: Show $\bar{\mathcal{H}}_{\pm}^{\mathrm{abs,rd}}$ are not modular curves. By Wohlfahrt, if the degree nine cover of $\mathbb{P}_j^1$ is modular, the group of the cover is a quotient of $\mathrm{PSL}_2(12)$. If the degree 6 cover is modular, the group is a quotient of $\mathrm{PSL}_2(4)$.

As $\mathrm{PSL}_2(\mathbb{Z}/4)$ has the λ-line as a quotient, with 2,2,2 as the cusp lengths, these cusp lengths are wrong for second orbit to correspond to λ-line. Similarly, for the other cover, as $\mathrm{PSL}_2(\mathbb{Z}/12)$ has both $\mathrm{PSL}_2(\mathbb{Z}/4)$ and $\mathrm{PSL}_2(\mathbb{Z}/3)$ as a quotient, so cusp lengths are wrong.

App. A: Higher Order g-p' representatives

Formulated by Darren Semmen: Some rooted planar tree, has elements of G labeling its vertices, and these hold.

1. The root has label 1.
2. Leaves of the tree have labels $g_1, \ldots, g_r$ in clockwise order.
3. Vertex labels one level up and adjacent to vertex x generate a p'-group with product (clockwise order) the label of x.

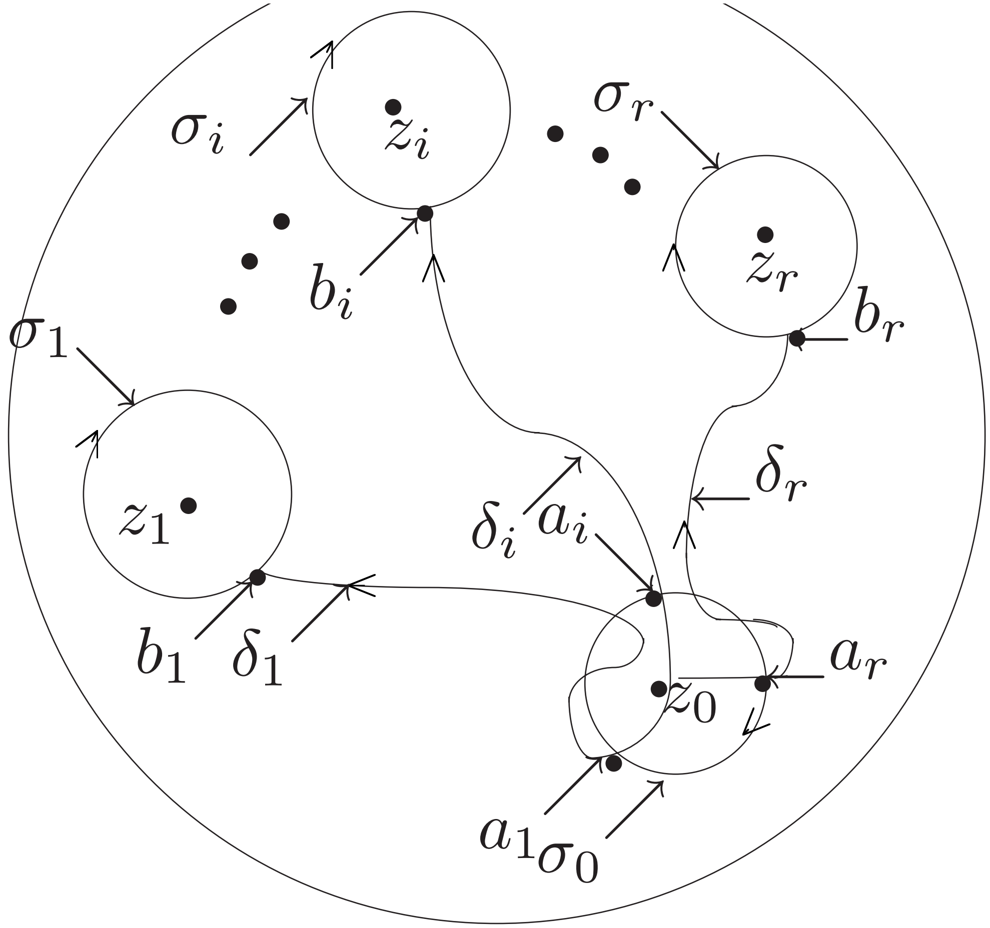
Includes more than first order g-p' reps., but harder to detect.

App. B: Comments on Schur multiplier of S_n

Here is a representation cover (not unique, S_n is not 2-perfect – it has $Z/2$ as a quotient by determinant): As a fiber product over $\mathbb{Z}/2$, $R_n = \hat{S}_n \times_{\mathbb{Z}/2} \mathbb{Z}/4$ with $\hat{S}_n$ the pullback of S_n to Spin^+ (p. 4; see [Se92, §9.1.3] for more detail).

For $g \in S_n$ a 2-cycle (resp. product of two disjoint 2-cycles), lifts to $\hat{S}_n$ have order 2 (resp. order 4). Corresponding statement for lift to $S_n \times_{\mathbb{Z}/2} \mathbb{Z}/4$ is order 4 (resp. order 2).

Problem: For a lift $\hat{g}$ of g to R_n, show conjugacy class $\mathrm{C}_{\hat{g}}$ of $\hat{g}$ in R_n has 4 elements over $\hat{g}$. Why does this say the lift invariant value $s_{R_n/S_n}(\boldsymbol{g})$ of $\boldsymbol{g} \in \mathrm{Ni}(S_n, \mathbf{C}_{2^r})$ must be 1 (p. 6)?



App. D: Classical Generators (next 2 pages): By running over all possible classical generators, a cover defines a Nielsen class (p. 1).

Paths/Pieces in the figure

Ordered closed paths $\delta_i\sigma_i\delta_i^{-1} = \bar{\sigma}_i$, $i = 1, \ldots, r$, are *classical generators* of $\pi_1(U_{\boldsymbol{z}}, z_0)$. Homomorphism $\pi_1(U_{\boldsymbol{z}}, z_0) \to G$ defines Nielsen class element by $\bar{\sigma}_i \mapsto g_i$, $i = 1, \ldots, r$.

Discs, $i = 1, \ldots, r$: D_i with center z_i; all disjoint, each excludes z_0; b_i be on the boundary of D_i.

Clockwise orientation: Boundary of D_i is a path σ_i with initial and end point b_i; δ_i a simple *simplicial* path: initial point z_0 and end point b_i. Assume δ_i meets none of $\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_r$, and it meets σ_i only at its endpoint.

Meeting Boundary of D_0: Defining the Product-One Condition

D_0 intersections: D_0 with center z_0; disjoint from each $D_1, \ldots, D_r$. Consider a_i, first intersection of δ_i and boundary σ_0 of D_0.

Crucial ordering: Conditions on $\delta_1, \ldots, \delta_r$:

- pairwise nonintersecting, except at z_0; and
- $a_1, \ldots, a_r$ are in order clockwise around σ_0.

Since paths are simplicial, last condition is independent of D_0, for D_0 sufficiently small.

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