Limit groups: Mapping class orbits and maximal Frattini quotients of dim. 2 p-Poincaré dual groups

All those Frattini p-extensions

Group theorists consider it unlikely there will ever be a classification of all finite groups. Why?

The unknowable collection of p-groups because of all their non-split extensions. Worse still are nonsplit extensions of p-perfect G ($p||G|$, but no $G \to \mathbb{Z}/p \to 1$) by a p-group.

Yet, algebraic equations in the 20th century faced nonsplit abelian p-group extensions, from moduli of abelian varieties through torsion points.

Goal: Explain a moduli approach to melding *all* p-group extensions into arithmetic geometry, *and* why it is *necessary*.

Universal p-Frattini cover of a p-perfect G

Are there many nonsplit finite group extensions of an arbitrary finite group? Answer: You bet! Consider them all $F_G = \{\varphi : H \to G \to 1\}$. For G of *rank* t, let $\tilde{F}_t$ be the pro-free group of rank t. Fix $\psi : \tilde{F}_t \to G \to 1$.

Take all closed $\{\tilde{H} \le \tilde{F}_t\}$ with $\psi : \tilde{H} \to G \to 1$. Let $\tilde{G}$ be a minimal such $\tilde{H}$. (Existence from Zorn's Lemma; use $\tilde{F}_t$ is complete.) So, $\tilde{G}$ of rank t, is unique up to isomorphism. Further:

- Versality: $\psi : \tilde{G} \to G$ is a Frattini cover that factors through any $\varphi \in F_G$.
- Projectivity: In category of profinite groups.
- Prime Factorization: $\tilde{G}$ is the *fiber product* over G of $\{{}_p\tilde{G}\}_{p||G|}$ (p. 3) [FrJ04, Chap. 22.11].

Universal p-Frattini cover ${}_p\tilde G$ of G

- p-Projectivity: $\psi : {}_p\tilde G \to G$ is maximal subquotient of $\tilde G \to G$ having p-group as kernel.
- Pro-free kernel:

$$\ker_0 \stackrel{\text{def}}{=} \ker_{G,p,0} = \ker({}_p\tilde G \to G)$$

  is pro-free pro-p of finite rank.
- Split case: If $G = P \times^s H$ with P a (normal) p-Sylow of rank t', then ${}_p\tilde G = {}_p\tilde F_{t'} \times^s H$.

Comments on Split case: How to extend H to ${}_p\tilde F_{t'}$ is non-obvious, and hard to find explicitly. Also, Examples # and # 2 of Talk #1 (p. 12) show the "easy" split case doesn't simplify understanding MT components and their cusps.

Characteristic module sequence:

$$\ker_1 = \Phi(\ker_0) \stackrel{\text{def}}{=} \ker_0^p \cdot (\ker_0, \ker_0),$$

$\ker_i = \Phi(\ker_{i-1}), \dots$ and $G_i = {}_p\tilde G / \ker_i$.

Then, $M_i \stackrel{\text{def}}{=} \ker_i / \ker_{i+1}$ is a $\mathbb{Z}/p[G_i]$ module.

Modular Tower for $(G, \mathbf{C}, p)$ with p' set $\mathbf{C}$

Take any p' conjugacy classes $\mathbf{C}$ of G. Form $\mathrm{Ni}(G_k, \mathbf{C})/\langle \mathcal{Q}'', G_k\rangle$ and

$$\mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}} = \mathcal{H}_k^{\mathrm{in,rd}} = \mathcal{H}_k.$$

When $r = 4$: gives a diagram of j-line covers

$$\cdots \to \bar{\mathcal{H}}_{k+1} \to \bar{\mathcal{H}}_k \to \cdots \to \mathbb{P}^1_j. \qquad (1)$$

MT *levels* are moduli spaces: Galois covers with group G in $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ equivalencing

$$(\varphi_i : X \to \mathbb{P}^1_z, \mathrm{Aut}(X/\mathbb{P}^1_z)),\ i = 1, 2,$$

if there is $(\psi : X_1 \to X_2, \alpha \in \mathrm{PGL}_2(\mathbb{C}))$ with $\alpha \circ \varphi_1 = \varphi_2 \circ \psi$ inducing inner automorphism on $G \Leftrightarrow \boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}}$.

Discrete objects of a MT (1):

- Projective systems of $(q_2 \mapsto)\gamma_\infty$ orbits (*cusps*).
- Projective systems of $\langle \gamma_1, \gamma_\infty\rangle$ orbits.
  The Shift: $q_1q_2q_3 \mapsto \gamma_1$ acts as the shift:

$$(g_1, g_2, g_3, g_4) \mapsto (g_2, g_3, g_4, g_1).$$

MTs and Greatest Hopes for the RIGP

Hope for $[K:\mathbb{Q}] < \infty$: Get all finite quotients of ${}_p\tilde G$ over K_ℓ and even over K, *at one time.* Data for any regular realization of G_k is $(G_k, \mathbf{C}_k)$ (r_k conjugacy classes in G_k).

Simplest if $\mathbf{C}_k$ doesn't change too much.

1. Maybe has r_k bounded.
2. Maybe has r_k fixed at some value r.
3. Maybe has branch points $\boldsymbol{z}^k = \boldsymbol{z}$ (all k).

[FK97, Using Branch Cycle Lemma]: For a number field #1 $\Longrightarrow$ #2, and $\exists$ MT with a K point at each level.
#3 $\Longrightarrow$ $\exists$ projective sequence of realizations: equivalent to K points on a MT. No such projective sequence if $[K:\mathbb{Q}] < \infty$.

Main Conjecture of MTs: If $[K:\mathbb{Q}] < \infty$, $\mathcal{H}(G_k, \mathbf{C})^{\text{in,rd}}(K) = \emptyset$ if k is large.

First step: Decide if $\exists$ a(n infinite) projective sequence of components on a given MT

Geometric Conjecture [Fr04c]: $\exists$ projective sequence of components (PSC) is equivalent to existence of a *g-p' rep.* (at level 0).

Established Facts on g-p' cusps: $[K:\mathbb{Q}] < \infty$.

- p. 8-9 of Alternating Groups Talk: g-p' rep. sufficient for PSC.
- [Fr95, Part III] When $\exists$ *Harbater-Mumford* (H-M) cusps, there is an effective sufficient criterion for PSC over K.
- With condition [Fr95, Part III], there is a K_ℓ point on the MT ([DDe04] and [DEm04]).

Analyzing cusps means representing objects $\mapsto$ cusps (over $\mathbb{R}$ or $W(\mathbb{F}_q)$). Like elliptic curves $\mapsto$ modular curve cusp.

Big improvement over PSC over K to get small dimensional subvarieties on the tower levels over K. First substantial result is [Ca05a] (uses H-M cusps). [Iha86], [IM95] and [Na99] applies to see the Grothendieck-Teichmüller relations in $G_{\mathbb{Q}}$ along MT cusps. Generalizing to g-p' cusps is in cards, but not done yet.

p-Poincaré Duality Setup

Let $\varphi : X \to \mathbb{P}^1_z$, with branch points $\boldsymbol{z}$, a Galois cover in $\text{Ni}(G, \mathbf{C})^{\text{in}}$: represents a braid orbit O.

Use *classical generators* (App. A_1) for $\pi_1(U_{\boldsymbol{z}}, z_0)$. [BFr02, Prop. 4.15] produces a quotient M_φ of $\pi_1(U_{\boldsymbol{z}}, z_0)$ so $\ker(M_\varphi \to G)$ identifies with the pro-p completion of the fundamental group of X. If $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$ corresponds to these choices, denote M_φ by $M_{\boldsymbol{g}}$.

p-Nielsen limit through O is a maximal quotient of $M_{\boldsymbol{g}}$ that is Frattini over G. Equivalence by conjugation braid action fixed on $\boldsymbol{g}$ (automatically includes conjugation by $\ker(M_{\boldsymbol{g}} \to G)$.

Extension Viewpoint: Projective systems

$$\{\boldsymbol{p}_k \in \mathcal{H}(G_k, \mathbf{C})^{\text{in,rd}}\}_{k=0}^{\infty} \text{ over fixed } \boldsymbol{p}_0 \Leftrightarrow$$

extensions of $M_{\boldsymbol{g}} \to G$ to $M_{\boldsymbol{g}} \to {}_p\tilde{G} \to G$.

p-Poincaré duality groups [We05] (extending [Br82] and [Ser91])

Dimension 2 p-Poincaré duality [We05, (5.8)]. Expresses an exact cohomology pairing

$$H^k(M_{\boldsymbol{g}}, U^*) \times H^{2-k}(M_{\boldsymbol{g}}, U) \to \mathbb{Q}_p/\mathbb{Z}_p \stackrel{\text{def}}{=} I_{G,p}$$

where U is any abelian p-power group that is also a $\Gamma = M_{\boldsymbol{g}}$ module, U^* is its dual with respect to $I_{G,p}$ and k is any integer. [Ser91, I.4.5] has the same definition, though that assumes $M_{\boldsymbol{g}}$ is a pro-p-group, while we have the p-perfect group G at its head.

Basic Idea: $M_{\boldsymbol{g}}$ has a finite index subgroup satisfying Poincaré duality: pro-p completion of $\pi_1(X)$.

Weak Orientability: When U is a $\mathbb{Z}/p[G]$ module, the pairing rt. side has trivial $M_{\boldsymbol{g}}$ action.

Test for going from MT Level k to Level $k+1$

[FrK97] Lift principle: If $G_2' \to G_1' \to G$ are p-Frattini covers, with $\ker(G_2' \to G_1') = M$ an irreducible non-trivial G module, then $\boldsymbol{g}_1 \in \text{Ni}(G_1', \mathbf{C})$ lifts to $\boldsymbol{g}_2 \in \text{Ni}(G_2', \mathbf{C})$.

Test for going from braid orbit $O_k \le \text{Ni}(G_k, \mathbf{C})$ to $O_{k+1} \le \text{Ni}(G_{k+1}, \mathbf{C})$.

Let $R_k \to G_k$ be maximal among central, exponent p Frattini extensions of G_k. Then, $\ker(R_k \to G_k) = \text{Sc}_k$ is the maximal exponent p quotient of G_ks Schur multiplier.

Theorem 1 (W Test A). *$s_{R_k/G_k}(O) = 0$ is iff test for $\exists\, O_{k+1}$ (use $s_{R/G}$ lift inv., p. 3, Talk 2).*

Proof for [W Test A] [Fr05c, Cor. 4.12]

Proof. Let $\boldsymbol{g}_k \in O_k \Leftrightarrow \psi : M_{\boldsymbol{g}_0} \to G_k$. Need: If fiber of $\text{Ni}(G_{k+1}, \mathbf{C}) \to \text{Ni}(G_k, \mathbf{C})$ over $\boldsymbol{g}_k$ is empty, then $s_{R/G_k}(\boldsymbol{g}) \neq 0$ for $R_k \leq R \leq G_k$ with $\ker(R \to G_k) = \mathbb{Z}/p$. [Fr95, Prop. 2.7] says $H^2(G_k, M_k) = \mathbb{Z}/p$: It is 1-dimensional.

Obstruction to lifting ψ to G_{k+1} is inflation of a generator of $H^2(G_k, M_k)$ to $H^2(M_{\boldsymbol{g}}, M_k)$. p-Poincaré duality says this is

$$H_0(M_{\boldsymbol{g}}, D \otimes M_k) \simeq D \otimes_{\mathbb{Z}/p[M_{\boldsymbol{g}}]} M_k,$$

with $D = \mathbb{Z}/p$ the duality module for $\mathbb{Z}/p[M_{\boldsymbol{g}}]$ (on which it acts trivially). So, $D \otimes_{\mathbb{Z}/p[M_{\boldsymbol{g}}]} M_k$ is the maximal quotient of M_k on which $M_{\boldsymbol{g}}$ (so G_k) acts trivially. □

Limit Group Test and return to Modular Curves

Theorem 2 (F-K-W Test B). *For $G^* \to G$ a limit group, there is a unique p-Frattini extension $G^{**} \to G^*$ with* $\ker(G^{**} \to G^*)$ *an irreducible module, and that module must be* $\mathbf{1}_G$.

Examples p. 11-12, Talk 1:
Example 1: All modular curves.

Projectively complete

$$F_3 = \langle \boldsymbol{\sigma} = \sigma_1, \dots, \sigma_4 \quad \text{mod } \sigma_1\sigma_2\sigma_3\sigma_4 = 1 \rangle.$$

Denote result by $\widehat{F}_{\boldsymbol{\sigma}}$.

Proposition 3. *Denote the quotient of $\widehat{F}_{\boldsymbol{\sigma}}$ by*

$$\sigma_i^2 = 1, \; i = 1, 2, 3, 4 \; (\textit{so } \sigma_1\sigma_2 = \sigma_4\sigma_3)$$

by $\hat{D}_{\boldsymbol{\sigma}}$. Then, $\prod_{p\neq 2} \mathbb{Z}_p^2 \times^s J_2 \equiv \hat{D}_{\boldsymbol{\sigma}}$ and $\mathbb{Z}_p^2 \times^s J_2$ is the unique $\mathbf{C}_{2^4}$ p-Nielsen class limit.

Argument for Prop. 3 (more in [Fr05c, §6.1])

Goal: show $\hat{D}_{\boldsymbol{\sigma}}$ is $\tilde{\mathbb{Z}}^2 \times^s J_2$ and $\sigma_1\sigma_2$ and $\sigma_1\sigma_3$ are independent generators of $\tilde{\mathbb{Z}}^2$. Then, σ_1 acts on $\tilde{\mathbb{Z}}^2$ by multiplication by -1.

First: $\sigma_1(\sigma_1\sigma_2)\sigma_1 = \sigma_2\sigma_1$ shows σ_1 conjugates $\sigma_1\sigma_2$ to its inverse. Also,

$$(\sigma_1\sigma_2)(\sigma_1\sigma_3)=(\sigma_1\sigma_3)\sigma_3(\sigma_2\sigma_1)\sigma_3=(\sigma_1\sigma_3)(\sigma_1\sigma_2)$$

shows the said generators commute. The maximal possible quotient is $\mathbb{Z}_p^2 \times^s \{\pm 1\}$.

Second: $G = V \times^s J_2$, V a nontrivial quotient of $\mathbb{Z}^2$, gives nonempty Nielsen classes. Use a cofinal family of Vs, $(\mathbb{Z}/p^{k+1})^2$, $p \neq 2$. Two proofs, one using elliptic curves and the other pure Nielsen class, appear in [Fr05a,§6.1.3]. That shows $\mathbb{Z}_p^2 \times^s \{\pm 1\}$ is a limit group. Uniqueness comes from Talk 1.

Heisenberg analysis of modular curve Nielsen classes [Fr05c, App. A.2]

Loewy layers (App. A_4) show Prop. 3 is an example of [F-K-W Test B].

First: $(\mathbb{Z}\times\mathbb{Z})\times^s\mathbb{Z}/2$ is an orientable p-Poincaré duality group if p is odd: Finite-index subgroup $\mathbb{Z}\times\mathbb{Z}$ is fundamental group of the torus. Denote the matrix $\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$ by $M(x,y,z)$. Heisenberg group with entries in ring R:

$$\mathbb{H}_{R,3} = \{M(x,y,z)\}_{x,y,z\in R}.$$

Consider $\mathbf{1}^-$ and $\mathbb{Z}_p^-$: $g\in S_n \mapsto$ mult. by $\mathrm{Det}(g)$.

Proposition 4 ([Fr05c, App. B2]).

$$\mathbb{H}_{\mathbb{Z}/p,3} \to (\mathbb{Z}/p)^2 \text{ by } M(x,y,z)\mapsto (x,y)$$

is Frattini. The p-Frattini module $M_0(G_0)$ of $G_0 = (\mathbb{Z}/p)^2\times^s\mathbb{Z}/2$ has $\mathbf{1}_{G_0}\oplus\mathbf{1}^-_{G_0}\oplus\mathbf{1}^-_{G_0}$ at its head. Extension defined by $\mathbf{1}_{G_0}$ gives Heisenberg group, obstructing MT *at level 1. Also gives infinite limit group*

$$(\mathbb{Z}_p)^2\times^s\mathbb{Z}/2 = (\mathbb{Z}_p^-)^2\times^s\mathbb{Z}/2.$$

$(G, \mathbf{C})$ having many Limit Groups

Talk 5 does Example 2, p. 12 of Talk 1, with $H = \mathbb{Z}/3 = \langle \alpha \rangle$, considering limit groups of $\text{Ni}((\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3, \mathbf{C}_{\pm 3^2}) = \text{Ni}_p$ $(p \neq 3)$. Note: $(\mathbb{Z}/2)^2 \times^s \mathbb{Z}/3 = A_4$.

1. ${}_p\tilde{F}_2 \times^s \mathbb{Z}/3$ is a limit group because Ni_p contains an H-M rep.
2. There are two braid orbits $O_{0,1}$ (H-M) and $O_{0,2}$ ($\text{Spin}_4 \to A_4(O_{0,2}) = -1$) on Ni_2.
3. Ni_2 has many limit groups, all so far, fitting into a pattern.
4. Six braid orbits on $\text{Ni}(G_1(\mathbb{Z}/p)^2 \times^s \mathbb{Z}/3), \mathbf{C}_{\pm 3^2})$:
    - $O_{g=0,1}$ and $O_{g=0,2}$ $\Leftrightarrow$ genus 0, complex conjugate curves $\bar{\mathcal{H}}_{g=0,i} \to \mathbb{P}^1_j$, $i = 1, 2$;
    - $O_{H-M,1}$ and $O_{H-M,2}$ (H-M orbits) $\Leftrightarrow$ genus 1 curves;
    - $O_{g=3,1}$ and $O_{g=3,2}$ $\Leftrightarrow$ genus 3 curves over $\mathbb{Q}$ covering $\mathbb{P}^1_j$.

Talk 5 has #4: $\text{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$ braid orbits, applied to $\text{Ni}(G_1(A_5), \mathbf{C}_{3^4})$, level 1 MTcomponents for $A_5, p = 2$, any conjugacy classes.

App. A_4: Loewy layers of modular curves

Jacobson radical of $\mathbb{Z}/p[G] J_{G,p} = J$: Intersection of maximal left (or right) ideals of $\mathbb{Z}/p[G]$.

Basic Lemma: $M/J_{G,p}M$, the *first* Loewy layer of G module M, is maximal semi-simple G quotient of M. For *Loewy layers* continue series inductively: $J \cdot M$ replaces M.

Knowing M from its Loewy layers requires info on nonsplit subquotients M' of M of this form:

$$0 \to S_1 \to M' \to S_2 \to 0$$

(S_1, S_2 irreducible in the $\ell + 1$st, ℓth layer).
Let $F_u(G) = \{g \in G \mid g - 1 \in J^u\}$: $F_1(G) = G$.
Input for $H_G(t)$ is dimensions $n_1, n_2, \ldots, n_u, \ldots$ of graded pieces of Jenning's Lie algebra:

- uth graded piece is F_u/F_{u+1}; and
- commutators and pth powers from Fs with lower subscripts generate F_u.

For G a p-group, and $M = \mathbb{Z}/p[G]$, J is the augmentation ideal: $\ker(\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g)$.

Easiest non-trivial case of Loewy layers

Jenning's Thm. [Ben91, Thm. 3.14.6] gives Loewy layer dimensions as a Hilbert polynomial $H_G(t)$ (variable t). Only p-group irreducible is $\mathbf{1}_G$. Arrows from levels ℓ to $\ell-1$ give all. Conclude: For $G=(\mathbb{Z}/p)^n$, $n_1=n$, F_u/F_{u+1} is trivial for $u\ge 2$: general case

$$\prod_{u\ge 1}(\frac{1-t^{pu}}{1-t^u})^{n_u} \implies H_{(\mathbb{Z}/p)^n}(t)=(\frac{1-t^p}{1-t})^n.$$

Lemma 5. *So:* $H_{(\mathbb{Z}/p)^2}(t)=(1+t+\ldots+t^{p-1})^2$*; respective Loewy layers of* $\mathbb{Z}/p[(\mathbb{Z}/p)^2]$ *have the dimensions* $1,2,\ldots,p,p-1,\ldots,1$*. With* $(\mathbb{Z}/p)^2=\langle x_1,x_2\rangle$*, symbols* $x_1^\alpha x_2^{\ell-\alpha}$ $0\le\alpha,\ell-\alpha<p$ *generate* $\mathbf{1}$*s at Loewy layer* ℓ*. Arrows from* $\mathbf{1}\leftrightarrow x_1^\alpha x_2^{\ell-\alpha}$ *go to* $\mathbf{1}$*s associated to* $x_1^\alpha x_2^{\ell-1-\alpha}$ *and to* $x_1^{\alpha-1}x_2^{\ell-\alpha}$ *under above constraints.*

Proof. Loewy arrows come from subquotient $R=\mathbb{Z}/p[G]$ module extensions of $\mathbf{1}$ by $\mathbf{1}$. Use the Poincaré-Birkoff-Witt basis for the universal enveloping algebra of R [Ben91, p. 88]. □