Galois closure groups: Outline proof of the Main Conjecture for $r = 4$; variants of the Regular Inverse Galois Problem; Serre's Open Image Theorem

Cusp and Component trees

A cusp at MT level k maps to its containing component. Use cusp (resp. component) $\mathcal{C}_{G,\mathbf{C},p}$ (resp. $\mathcal{T}_{G,\mathbf{C},p}$) ordered tree language: $\mathcal{C}_{G,\mathbf{C},p} \to \mathcal{T}_{G,\mathbf{C},p}$ (containment of cusps in components). By Groups: A cusp set is a q_2 orbit in a braid orbit on Nielsen classes.

I state things for $r = 4$, but these generalize for arbitrary r. An infinite (geometric) component branch is a projective sequence

$$B' = \{\bar{\mathcal{H}}'_k \subset \bar{\mathcal{H}}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$$

of (geometric) Hurwitz space components. An infinite cusp branch is a projective sequence

$$B = \{\bar{\boldsymbol{p}}_k \in \bar{\mathcal{H}}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$$

of (geometric) points over $j = \infty$. There also exist finite branches, where the last component $\mathcal{H}'_k$ has nothing above it on $\mathcal{H}(G_{k+1}, \mathbf{C})^{\mathrm{in,rd}}$.

When the Main Conjecture Matters

Main Conjecture only applies to infinite component branches. For such a B' there is at least one sequence of cusp sets defined by a projective system $\{{}_k\boldsymbol{g} \in \text{Ni}(G_k, \mathbf{C})^{\text{in}}\}_{k=0}^{\infty}$.

Conjecture from Talk 4, p. 5: A projective sequence on the Component tree is the image of a projective system of g-p' cusps on the cusp tree. For K a completion of a number field, [Ca05a], [DDe04] and [DEm04] consider the case when these are H-M cusps in a projective system of K components. They show the cusp branch consists of a projective system of K points. This should extend to g-p' cusps.

Talk 4 discussed importance of braid orbits of limits groups attached to $(G, \mathbf{C}, p)$: What are the maximal p-Frattini quotients (limit groups) of orientable dimension 2 p-Poincaré dual groups defined by a mapping class group orbit.

Steps in considering limit groups

1. Identify braid orbits of limit groups.

2. Identify braid orbits with limit group ${}_p\tilde{G}$ (so define an infinite component branch B').

3. Decide when B' in # 2 has definition K, $[K : \mathbb{Q}] < \infty$.

4. Decide when levels of B' have K points.

In Main Conjecture, # 4 only matters if # 3 holds.

g-p' Cusp Conjecture: # 2 holds iff a g-p' cusp branch defines the component branch.

When K points force a g-p' Branch

[Fr05c, Lem. 3.1, Princ. 4.8] considers when g-p' cusps force their component branches to be over a number field K. Another approach:

Conjecture 1. If a component branch on a MT has K points at each level, is it the image of a g-p' cusp branch?

Affirmative Case of Conj. 1: The following is from [BFr02,§6.1].

Proposition 2. *If $p = 2$, and there are $\mathbb{R}$ points at each level of a* MT*, then there is an H-M cusp branch defining a component branch with a projective sequence of $\mathbb{R}$ components.*

Comments. What happens (again, only for $p = 2$) is that for level $k \geq 1$, all $\mathbb{R}$ points are on cusp components associated with H-M and *near H-M* cusps. A connected component of $\mathcal{H}_k(\mathbb{R})^{\text{in}}(\mathbb{R})$ associated to an H-M (resp. near H-M) cusp has a (resp. has no) connected component of $\mathcal{H}_{k+1}(\mathbb{R})^{\text{in}}(\mathbb{R})$ above it. □

Frattini Cusp Principles: Revisit Talk 1 p. 11

Talk 1: For modular curves $X_1(p^{k+1})$, cusp of width p^u, $u \ge 1$, has only cusps of width p^{u+1} over it at level $k+1$. Generalizes to MT cusps:

Principle 3 (Frattini Principle 1). *With projective system $\{{}_k\boldsymbol{g} = ({}_kg_1, {}_kg_2, {}_kg_3, {}_kg_4) \in \mathrm{Ni}'_k\}_{k=0}^{\infty}$ of cusp reps, if $p^u | \mathrm{ord}({}_kg_{2\,k}g_3)$, $u \ge 1$, then $p^{u+1} | \mathrm{ord}({}_{k+1}g_{2\,k+1}g_3)$.*

Modular curve $X_1(p^{k+1})$ has a projective system of width one cups above every width 1 cusp. Generalizes to MT cusps:

Principle 4 (Frattini Principle 2). *If ${}_0\boldsymbol{g} \in \mathrm{Ni}(G_0, \mathbf{C})$ represents a g-p' cusp, then above it there is a g-p' cusp branch $\{{}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})\}$.*

Frattini Principle 3, relating cusp and component lifting invariants

With $r = 4$, $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$, denote:

$$\langle g_2, g_3 \rangle = H_{2,3}(\boldsymbol{g}) \text{ and } \langle g_1, g_4 \rangle = H_{1,4}(\boldsymbol{g}).$$

Assume for some $k \geq 0$, ${}_k\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})$. Let $R_{G_k} \to G_k$ be the central extension of G_k with $\ker(R_{G_k} \to G_k)$ the maximal quotient of M_k on which G_k acts trivially. Use similar notation,

$$H_{2,3}({}_k\boldsymbol{g}) = H_{2,3} \text{ and } H_{1,4}({}_k\boldsymbol{g}) = H_{1,4},$$

replacing G_k. Basic lemma says there is a map $\beta_{2,3} : R_{H_{2,3}(\boldsymbol{g})} \to R_G$.
Principle 5 (o-p' cusps). *Assume $p \not| \mathrm{ord}(g_2 g_3)$ but not g-p' cusp (o-p' cusp). Then,* $s_{G,p}(\boldsymbol{g}) =$

$$\beta_{1,4}(s_{H_{1,4},p}((g_4 g_1)^{-1}, g_4, g_1))\beta_{2,3}(s_{H_{2,3},p}(g_2, g_3, (g_2 g_3)^{-1})).$$

o-p' Cusp Conjecture: There is no (infinite) projective system of o-p' cusps.

Nub of the (weak) Main Conjecture

Restrict the $(\gamma_0, \gamma_1, \gamma_\infty)$ of Talk 1 to Ni'_k defining level k of B': Gives $(\gamma'_{0,k}, \gamma'_{1,k}, \gamma'_{\infty,k})$ defining the genus $g_{\bar{\mathcal{H}}'_k}$ of $\bar{\mathcal{H}}'_k$: $2(\deg(\bar{\mathcal{H}}'_k/\mathbb{P}^1_j) + g_{\bar{\mathcal{H}}'_k} - 1) =$

$$\text{ind}(\gamma'_{0,k}) + \text{ind}(\gamma'_{1,k}) + \text{ind}(\gamma'_{\infty,k}).$$

There should be no $\mathcal{T}_{G,\mathbf{C},p}$ genus 0 or 1 branches: Two possibilities may negate Main Conjecture.

1. $g_{\bar{\mathcal{H}}'_k} = 0$ for all $0 \le k < \infty$ (B' has genus 0).
2. For k large, $g_{\bar{\mathcal{H}}'_k} = 1$ (B' has genus 1).

Expectation from g-p' and o-p' conjectures: We expect branches from g-p' cusps to give analytic expression for genera as in modular curve tower. Elementary corollary of R-H says, # 2 means, for large k, $\bar{\mathcal{H}}'_{k+1} \to \bar{\mathcal{H}}'_k$ don't ramify. From Princ. 4, this says:

For no value of k does $\bar{\mathcal{H}}'_k$ have a p cusp.

Main Conjecture Nub cont.

To exclude # 1 requires strengthening the negation of # 2. Consider what happens for k large if $2(\deg(\bar{\mathcal{H}}'_k/\mathbb{P}^1_j)-1)=$

$$\text{ind}(\gamma'_{0,k})+\text{ind}(\gamma'_{1,k})+\text{ind}(\gamma'_{\infty,k}): \ \#1 \text{ holds.}$$

Let $\boldsymbol{p}'_k \in \mathcal{H}'_k$ be a p cusp. Assume u_k primes $\boldsymbol{p}'_{k+1} \in \mathcal{H}'_{k+1}$ lie over $\boldsymbol{p}'_k$; $\deg(\mathcal{H}'_{k+1}/\mathcal{H}'_k) \stackrel{\text{def}}{=} \nu_k$. Princ. 3 says each $\boldsymbol{p}'_{k+1}/\boldsymbol{p}'_k$ ramifies to order p and $\nu_k = p \cdot u_k$.

This also holds for any $\boldsymbol{p}'_{k+2} \in \mathcal{H}'_{k+2}$ over $\boldsymbol{p}'_{k+1}$.

1. Then, index contribution of all $\boldsymbol{p}'_{k+2}$s to R-H from $\bar{\mathcal{H}}'_{k+2}$ to $\bar{\mathcal{H}}'_{k+1}$ is $u_k \cdot u_{k+1} \cdot (p-1)$.
2. R-H for $\bar{\mathcal{H}}'_{k+2} \to \bar{\mathcal{H}}'_k$, says $\bar{\mathcal{H}}'_{k+2}$ cannot have genus 0 if $u_k \cdot u_{k+1} \cdot (p-1)$ exceeds $2(p \cdot u_{k+1} - 1)$.

This is so, if u_k is *moderate*: $u_k \geq 4$ for $p = 2$, and $u_k \geq 3$ for $p > 2$.

Example Evidence

Result 6. *Branch B' of $\mathcal{T}_{G,\mathbf{C},p}$ contradicts case # 1 if there is one p cusp at level k having a small number of cusps over it.*

Branch B' contradicts case # 2 if there is one p cusp at a suitably high level k: Princ. 4 $\Longrightarrow$ it forces ramification from $\mathcal{H}'_k$ to $\mathcal{H}'_{k+1}$.

Example 7. Level 0 of $(A_5, \mathbf{C}_{3^4}, p = 2)$ MT has no 2 cusps. Apply Fried-Serre to $\text{Spin}_5 \to A_5$ cover: Level 1 components $\mathcal{H}_{\pm}((G_1(A_5), \mathbf{C}_{3^4})^{\text{in,rd}}$ are full of 2 cusps [BFr02, Cor. 8.3].

Full analysis: $\mathcal{H}_{+}((G_1(A_5), \mathbf{C}_{3^4})^{\text{in,rd}}$, contains all H-M cusps, has genus 12 and degree 16 over the unique component of $\mathcal{H}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$. It also has all real (and so all the $\mathbb{Q}$) points at level 1 [BFr02, Chap. 9]. Further, all except the H-M cusps are 2 cusps.

The other component, $\mathcal{H}_{-}((G_1(A_5), \mathbf{C}_{3^4})^{\text{in,rd}}$ is obstructed, so contributes no more to the weak Main Conjecture.