Finite group theory and Connectedness of moduli spaces of Riemann Surface covers

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http://math.uci.edu/~mfried → §1.a → # Generalizing modular curve properties to Modular Towers

→ #1 mt-overview.html

Let $\varphi : X \to \mathbb{P}^1_z$ be a function on a Riemann surface X, with φ Galois having group G. Then, φ defines these quantities:

- Unordered branch points $\boldsymbol{z} = \{z_1, \dots, z_r\} \in U_r$ (undered distinct points on $\mathbb{P}^1_z$);
- Conjugacy classes **C** $= \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G; and
- A *Poincaré extension* of groups:

$$\psi_\varphi : M_\varphi \to G \text{ with } \ker_\psi \stackrel{\text{def}}{=} \ker(M_\varphi \to G) = \pi_1(X).$$

Using Classical Generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$

Denote $(\mathcal{P}_1, \ldots, \mathcal{P}_r)$ (see App. A) $\mapsto$ an isotopy class of r generators $\bar{\boldsymbol{g}} = (\bar{g}_1, \ldots, \bar{g}_r)$.

Refer to their images in M_φ also as $\bar{\boldsymbol{g}}$, and their images in G by $(g_1, \ldots, g_r) = \boldsymbol{g}$.

Then, $\boldsymbol{g}$ is in the Nielsen class of $(G, \mathbf{C})$:

$$\mathrm{Ni}(G, \mathbf{C}) \stackrel{\mathrm{def}}{=} \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \Pi(\boldsymbol{g}) \stackrel{\mathrm{def}}{=} g_1 \cdots g_r = 1\}.$$

Notion: Given classical generators $\bar{\boldsymbol{g}}$ of M_φ, rename it $M_{\bar{\boldsymbol{g}}}$: ψ_φ becomes $\psi_{\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$, by $\bar{g}_i \mapsto g_i$.

Part I: E(xtension) P(roblem) (Item #1 below)

Given $(\boldsymbol{g}, \mathbf{C})$, and a prime p:

1. When does $\psi_{\boldsymbol{g}}$ extend to all $H \to G \to 1$ with p-group kernel? Abelianized version (App. C): To all H with $\ker(H \to G)$ abelian.

2. How does this depend on $\boldsymbol{g}$?

3. What equivalence relation on extensions gives a reasonable description of all cases?

4. Why should this concern mathematics?

Non-obvious Reductions

- Complete $M_{\bar{\boldsymbol{g}}}$ so $\ker_{\psi_\varphi} =$ pro-p completion of $\pi_1(X)$.
- Restrict in #1 (p. 3) to p-Frattini covers of G.
- Any $g \in$ **C** must have order prime to p.
- G is p-perfect (no $\mathbb{Z}/p$ quotient; or #1 impossible).

Equivalent: When are all p-Frattini covers $H \to G \to 1$ achieved by unramified extensions $Y_H \to X$?

Deformation equivalence of extensions

If φ were a cyclic cover of $\mathbb{P}^1$, we could write it by hand. It isn't. Further, why deal one cover at-a-time? Consider all covers with $(G, \mathbf{C})$ as their data: In the Nielsen class.

Deformation Conclusion: Can always start by fixing branch points $\boldsymbol{z}^0$. Any cover (with branch points $\boldsymbol{z} \in U_r$; r unordered points in $\mathbb{P}^1_z$) deforms to a cover with branch points $\boldsymbol{z}^0$. Then, $M_{\bar{\boldsymbol{g}}}$ and any of its extension properties deform with it.

One cover defines a family: $\varphi : X \to \mathbb{P}^1_z \implies$

1. Permutation representation of $\pi_1(U_r, \boldsymbol{z}^0) \stackrel{\text{def}}{=} H_r$ *Hurwitz monodromy* on orbit Ni'_φ — independent of classical generators — of $[\varphi] \in \text{Ni}(G, \mathbf{C})$.
2. An unramified connected cover $\mathcal{H}(G, \mathbf{C})_\varphi \to U_r$: Hurwitz space component containing φ.

Equivalences of covers and Nielsen classes.

[Abs.] $\varphi' : X' \to \mathbb{P}^1_z \sim \varphi \Leftrightarrow \boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in N_{S_n, \mathbf{C}}(G)$.

[Inn.] φ Galois with $\mu : \text{Aut}(X/\mathbb{P}^1_z) \xrightarrow{\text{isom}} G \sim (\varphi', u') \Leftrightarrow$ $\boldsymbol{g} = h\boldsymbol{g}'h^{-1}, h \in G$.

Braid equivalence of extensions

H_r has two generators given by their action on $\bar{\boldsymbol{g}}$:

- Shift: $\mathbf{sh} : \bar{\boldsymbol{g}} \mapsto (\bar{g}_2, \dots, \bar{g}_r, \bar{g}_1)$; and
- 2ndTwist: $q_2 : \bar{\boldsymbol{g}} \mapsto (\bar{g}_1, \bar{g}_2\bar{g}_3\bar{g}_2^{-1}, \bar{g}_2, \bar{g}_4, \dots)$: $q_{i+2} \stackrel{\text{def}}{=} \mathbf{sh}^i q_2 \mathbf{sh}^{-i}$.

Braid Comments: H_r is automorphism group of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$ preserving classical generators. It acts compatibly on these:

- Inner Nielsen Classes: $\mathrm{Ni}(G, \mathbf{C})/G \stackrel{\text{def}}{=} \mathrm{Ni}^{\mathrm{in}}$
- Absolute Nielsen classes: $\mathrm{Ni}(G, \mathbf{C})/N_{S_n}(G) \stackrel{\text{def}}{=} \mathrm{Ni}^{\mathrm{abs}}$ (given $G \le S_n$ a permutation representation)
- Poincaré extensions: $\psi_{\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$, *preserving* extension properties of $\psi_{\boldsymbol{g}}$

Part II: Introduction to/Existence of Modular Towers

GOAL 1: Given $(G, \mathbf{C}, p)$, understand projective systems of H_r orbits acting on $\{\mathrm{Ni}(H, \mathbf{C})^{\mathrm{in}}\}_{H \to G}$ (lift $\mathbf{C}$ uniquely to p' conjugacy classes in H): Running over p-Frattini covers $H \to G$.

Reduction: Take $G_1 \to G = G_0$ to be the maximal p-Frattini cover of G with elementary p group kernel. Let $G_{k+1} = G_1(G_k)$. In GOAL 1 need only the case H runs over the G_k s.

Def: M(odular) T(ower): A projective system $\{O_k = H_r(\boldsymbol{g}_k)\}_{k=0}^{\infty}$ of H_r orbits on $\{\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$.

Inductive Existence of a MT: [Lum, Cor. 4.19], [We]

Let $\mu_k : R_k \to G_k$ be the universal exponent p central extension of G_k:

- $G_{k+1} \to G_k$ factors through μ_k.
- $\ker(R_k \to G_k) =$ max. elementary p-quotient of G_ks Schur multiplier.

Proposition 1 (App. C– Abel. Vers.). *If p-perfect G has no p-center, then neither does G_k, $k \geq 1$.*

$H_r(\boldsymbol{g}_k) \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$ is in the image of $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{in}} \Leftrightarrow$ $\boldsymbol{g}_k$ is in the image of $\mathrm{Ni}(R_k, \mathbf{C}) \Leftrightarrow H_r$ orbit of $M_{\boldsymbol{g}} \to G$ extends through all G_k.

Part III: Cusps and the p Cusp Problem

Cusp on an H_r orbit $O \subset \mathrm{Ni}(G, \mathbf{C})$:

- $r \geq 5$: An orbit of $\mathrm{Cu}_r = \langle q_2 \rangle$
- $r = 4$: An orbit of $\mathrm{Cu}_4 = \langle q_2, \mathbf{sh}^2, q_1 q_3^{-1} \rangle$.

Essential data is from conjugacy class of $\mathrm{Cu}_r \implies$ can substitute q_i for q_2.

MiddleProduct: $(g_1, g_2, g_3, g_4) \mapsto \mathrm{ord}(g_2 g_3) \stackrel{\mathrm{def}}{=} (\boldsymbol{g})\mathbf{mpr}$.

p cusp: represented by $\boldsymbol{g} \in O$ for which $p^{\mu_p(\boldsymbol{g})} || (\boldsymbol{g})\mathbf{mpr}$, $\mu_p(\boldsymbol{g}) > 0$ (p-mult. of $\boldsymbol{g}$).

Other cusp types for $r = 4$ (App. B for $r > 4$)

- g(roup)-p': $U_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle$ and $U_{2,3}(\boldsymbol{g}) = \langle g_2, g_3\rangle$ are p' groups

- o(nly)-p': Not a p cusp, but $U_{1,4}(\boldsymbol{g})$ or $U_{2,3}(\boldsymbol{g})$ not p'.

GOAL 2: Given a MT, $\{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, classify when there is a p cusp on O_k for $k >> 0$.

Proposition 2 (g-p' MT). *If O_0 has a g-p' cusp, then a $\mathrm{MT}, \mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, lies over it.*

Part IV: MT Geometric correspondence

$\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty} \Leftrightarrow \{\mathcal{H}'_k\}_{k=0}^{\infty}$ where:

- $\mathcal{H}'_k$s are (normal) absolutely irreducible algebraic varieties (dim$=r-3$);

- Cusps at level k correspond to divisors on the normal compactification $\bar{\mathcal{H}}'_k$.

- ${}_0\boldsymbol{g} \in O_0$ a p cusp $\implies \mu_p({}_k\boldsymbol{g}) = k + \mu_p({}_0\boldsymbol{g})$. Gives order of p dividing ramification index of the divisor.

- $r = 4$: $\mathcal{H}'_k$, upper-half plane quotient, j-line cover, ramific. order dividing 3 (resp. 2) over 0 (resp. 1).

Inner (resp. absolute) Reduced spaces [BFr02, §2]

Reduced equiv.: $\varphi : X \to \mathbb{P}^1_z \sim \beta \circ \varphi$, $\beta \in \mathrm{PGL}_2(\mathbb{C})$.

j-invariant: $\boldsymbol{z} \in U_4 \mapsto j_{\boldsymbol{z}} \in U_\infty \stackrel{\mathrm{def}}{=} \mathbb{P}^1_j \setminus \{\infty\}$ of $\boldsymbol{z}$. Normalize so $j = 0$ and 1 are *elliptic points*: $j_{\boldsymbol{z}}$ with more than a Klein 4-group stabilizer in $\mathrm{PGL}_2(\mathbb{C})$.

Let $\mathcal{Q}'' = \langle (q_1 q_2 q_3)^2 = \mathbf{sh}^2, q_1 q_3^{-1} \rangle \le H_4$. Reduced classes of covers with j-invariant $j' \in U_\infty \Leftrightarrow$ elements of reduced Nielsen classes: $\mathrm{Ni}(G, \mathbf{C})^*/\mathcal{Q}''$ (where $* = \mathrm{in}$ or abs).

H_4 on reduced Nielsen classes factors through the *mapping class group*: $\bar{M}_4 \stackrel{\mathrm{def}}{=} H_4/\mathcal{Q}'' \equiv \mathrm{PSL}_2(\mathbb{Z})$.

p Cusps and Main MT Conjecture

Main Conj.: K a number field, then $\mathcal{H}'_k(K) = \emptyset$ for $k >> 0$. For $r = 4$, let g'_k be the genus of $\bar{\mathcal{H}}'_k$.

Proposition 3. *If $g'_0 > 0$, (resp. $= 0$) and, for some k, $\mathcal{H}'_k$ has a p cusp (resp. three p cusps), then Main Conj. holds for $\mathcal{O}$.*

$\bar{M}_4 = \langle \gamma_0, \gamma_1, \gamma_\infty \rangle, q_1q_2 \mapsto \gamma_0$ (order 3),
shift $= q_1q_2q_3 \mapsto \gamma_1$ (order 2),
$q_2 \mapsto \gamma_\infty$ ($j = \infty$ monodromy generator),
satisfying the product-one relation: $\gamma_0\gamma_1\gamma_\infty = 1$.

Riemann-Hurwitz on components

Interpret R-H: Denote $(\gamma_0, \gamma_1, \gamma_\infty)$ acting on an H_r orbit $O' \le \mathrm{Ni}(G, \mathbf{C})^{*,\mathrm{rd}}$ by $(\gamma'_0, \gamma'_1, \gamma'_\infty)$, *branch cycles* for $\bar{\mathcal{H}}' \longrightarrow \mathbb{P}^1_j$ corresponding to O'.

- Points over 0 (resp. 1) $\Leftrightarrow$ orbits of γ_0 (resp. γ_1).
- The index contribution $\mathrm{ind}(\gamma_\infty)$ from a cusp with rep. $\boldsymbol{g} \in \mathrm{Ni}^{*,\mathrm{rd}}_{G,\mathbf{C}}$ is $|(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''| - 1$.

Part V: Point of connectedness results: Locate where are the p (and other types) of cusps.

Constellations of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ [AGLI, §1]

$\xrightarrow{g\geq 1}$	$\ominus \oplus$	$\ominus \oplus$	. . .	$\ominus \oplus$	$\ominus \oplus$	$\xleftarrow{1\leq g}$
$\xrightarrow{g=0}$	$\ominus$	$\oplus$	. . .	$\ominus$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	. . .	n even	n odd	$4 \leq n$

Theorem 4 (tag $\stackrel{g=0}{\longrightarrow}$, $r = n-1$, $n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm in}$ *has one component. Further,* $\Psi^{\rm in}_{\rm abs} : \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm in} \to \mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^{\rm abs}$ is deg. 2.

Theorem 5 (tag $\stackrel{g\geq 1}{\longrightarrow}$, $r \geq n \geq 5$). $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm in}$ *has two components,* $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\rm in}$ *(symbol* $\oplus$*) and* $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\rm in}$ *(symbol* $\ominus$*). Further*

$$\Psi^{\rm in,\pm}_{\rm abs} : \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\rm in} \to \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\rm abs} \text{ has degree 2.}$$

For $n = 4$, two 3-cycle classes C_{+3}, C_{-3} in A_4, $\mathbf{C} = \mathbf{C}_{+3^{s_1}\cdot -3^{s_2}}$: $\mathrm{Ni}(G, \mathrm{C}_{\pm 3^{s_1,s_2}})$ nonempty iff

$$s_1 - s_2 \equiv 0 \bmod 3 \text{ and } s_1 + s_2 = r.$$

Frattini covers

Frattini cover $G' \to G$ is a group cover (surjection) with restriction to a proper subgroup not a cover. Get a lifting invariant from a *central* Frattini cover.

Central Frattini from A_n: Spin_n^+ the nonsplit degree 2 cover of the connected component O_n^+ of the orthogonal group. Regard $S_n \subset O_n$; $A_n \subset O_n^+$. Denote pullback of A_n to Spin_n^+ by Spin_n. Identify $\ker(\mathrm{Spin}_n \to A_n)$ with $\{\pm 1\}$.

F-S Small lifting invariants ([LUM,§1], [Ser90a])

Odd order $g \in A_n$ has a unique odd order lift, $\hat{g} \in \mathrm{Spin}_n$. Let $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C})$ with $\mathbf{C}$ odd-order. *Small* lifting invariant:

$$s(\boldsymbol{g}) = s_{\mathrm{Spin}_n}(\boldsymbol{g}) = \hat{g}_1 \cdots \hat{g}_r \in \{\pm 1\}.$$

For g odd-order, let $w(g)$ by the number of cycles in g with lengths (ℓ) with $\frac{\ell^2-1}{8} \equiv 1 \mod 2$.

Theorem 6 (F-S). *On any braid orbit, $s(\boldsymbol{g})$ is constant (explains Const. diag. comps). If genus 0 Nielsen class, then $s(\boldsymbol{g}) = (-1)^{\sum_{i=1}^r w(g_i)}$.*

Pure-cycle components

- $g \in S_n$ is *pure-cycle* if exactly one cycle has length > 1.
- Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is *pure-cycle* if all conjugacy classes are pure-cycle (a d-cycle).
- If $d_1, \ldots, d_r$ are the pure-cycle lengths, denote the Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})^*$ (* an equivalence).

Assume $G \le S_n$ transitive and $\mathbf{C}^{S_n} \stackrel{\mathrm{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$ image of $\mathbf{C}$ in S_n, with d_is all odd. Necessary condition $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ is nonempty: Genus

$$\mathbf{g} = \mathbf{g}_{d_1 \cdots d_r} \stackrel{\mathrm{def}}{=} \frac{\sum_{i=1}^{r} d-1}{2} - (n-1) \text{ is non-negative.}$$

Liu-Osserman genus 0 result [LOs06]

Theorem 7. *If $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}_{d_1 \cdots d_r})$ has genus 0, then $G = A_n$, and H_r is transitive on it.*

Compactify the reduced inner space:

$$\bar{\mathcal{H}}(A_n, \mathbf{C}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n, d_1 \cdots d_4}.$$

Consider $\{\bar{\mathcal{H}}(G_k(A_n), \mathbf{C}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4})^{\mathrm{in,rd}} \stackrel{\mathrm{def}}{=} \bar{\mathcal{H}}_{n, d_1 \cdots d_4, k}\}_{k=0}^{\infty}$ with $G_k(A_n) \to A_n$ the universal exponent 2^k 2-group extension of A_n.

Remaining Goal: Give an idea of why the Main Conjecture holds for the Liu-Osserman examples.

Cusp rep listings, $r = 4$, $\mathrm{Ni}^{\text{abs or in}}_{(\frac{n+1}{2})^4}$, all d_i s equal

Define $x_{i,j} = (i\, i{+}1 \cdots j)$. g-$2'$ cusps here are shifts of HM reps. $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Mod conjugation by A_n they are

$$\begin{aligned} \mathrm{HM}_1 &\stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}) \\ \mathrm{HM}_2 = (\mathrm{HM}_1)q_1 &\stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}) \end{aligned}$$

Proposition 8. *For* $n \equiv 5 \mod 8$, HM_1 *and* HM_2 *are are not inner equivalent* $\implies$ *one braid orbit on* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

For $n \equiv 1 \mod 8$, *if* $h \in S_n \setminus A_n$, *there is no braid between* $\boldsymbol{g}$ *and* $h\boldsymbol{g}h^{-1}$ $\implies$ *two braid orbits on* $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

$\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ Table of Cusps reps.(row starts $\mathrm{ord}(g_2g_3)$): HM_1 and **sh** applied to $\mathrm{Cu}_4(\mathrm{HM}_1) = \{\mathrm{HM}_{1,t} = (x_{\frac{n+1}{2},1}, x_{1+t,\frac{n+1}{2}+t}, x_{\frac{n+1}{2}+t,n+t}, x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}$.

1: $(\mathrm{HM}_{1,0})$**sh** $= (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

3: $(\mathrm{HM}_{1,1})$**sh** $= (x_{2,\frac{n+3}{2}}, (\frac{n+3}{2}\ \dots\ n\,1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

5: $(\mathrm{HM}_{1,2})$**sh** $= (x_{3,\frac{n+5}{2}}, (\frac{n+5}{2}\ \dots\ n\,1\,2), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

$\dots$

n: $(\mathrm{HM}_{1,\frac{n-1}{2}})$**sh** $= (x_{\frac{n+1}{2}\,n}, (n\,1\ \dots\ \frac{n-1}{2}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

n: $(\mathrm{HM}_{1,\frac{n+1}{2}})$**sh** $= ((\frac{n+3}{2}\ \dots\ n\,1), x_{1,\frac{n+1}{2}}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

$\dots$

5: $(\mathrm{HM}_{1,n-2})$**sh** $= ((n{-}1\,n\,1\ \dots\ \frac{n-3}{2}), x_{\frac{n-3}{2},n-2}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

3: $(\mathrm{HM}_{1,n-1})$**sh** $= ((n\,1\ \dots\ \frac{n-1}{2}), x_{\frac{n-1}{2},n-1}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$

sh-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$

Pairing on Cu_4 orbits: $(O, O') \mapsto |O \cap (O')\mathbf{sh}|$. $O_{5,5;2}$ (resp. $O_{1,2}$) indicates 2nd **mpr** 5, width 5 (resp. only **mpr** 1, width 2) orbit. **sh**-incidence gives $\bar{\mathcal{H}}(A_5, \mathbf{C}_{34})^{\mathrm{in,rd}}$ has genus 0: $2(18+\mathbf{g}-1) = 2\cdot 18/3 + 18/2 + (1+2\cdot 2+2\cdot 4) \implies \mathbf{g} = 0$.

Orbit	$O_{5,5;1}$	$O_{5,5;2}$	$O_{3,3;1}$	$O_{3,3;2}$	$O_{1,2}$
$O_{5,5;1}$	0	2	1	1	1
$O_{5,5;2}$	2	0	1	1	1
$O_{3,3;1}$	1	1	0	1	0
$O_{3,3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Complete orbit for $\bar{M}_4 = \langle \mathbf{sh}, \gamma_\infty \rangle$ on $\mathrm{Ni}_{34}^{\mathrm{in,rd}}$ in 2-steps: Apply $(\mathbf{sh} \circ \mathrm{Cu}_4)^2$ to H-M rep.

Cusp Tree Conclusions in Liu-Osserman cases

Apply F-S lift inv. to $(g_2, g_3, (g_2g_3)^{-1})$ for Ni_{34}: Level 0 o-$2'$ cusps $O_{5,5,\bullet}$ and $O_{3,3,\bullet}$ have only 2 cusps above them: $(A_5, \mathbf{C}_{3^4}, p = 2)$ cusp tree has only g-$2'$ or 2 cusp branches.

Theorem 9. *If a cusp branch is both H-M and p, then* MT *cusp tree contains a spire: a modular curve cusp tree* $\Longrightarrow$ *Main Conjecture holds. At level 1, holds for (L-O)* $n = 5$*, but not for* $n = 9$*.*

Fried-Serre Lifting Invariant formula

Level 0 Dilemma: $\mathrm{ord}(g_2g_3) = \ell$ in each line of **Table** p. 26 is odd: no 2 cusps at level 0.

Help by Level 1: The exact condition for each cusp at level 1 above the cusp in **Table** p. 26 to be a 2 cusp is that $\frac{\ell^2-1}{8} \equiv 1 \mod 2$.

Main Conjecture?: $n = 5$, $\ell \in \{1,3,3,5,5\}$, so four (> 3) 2 cusps ($\frac{3^2-1}{2} \equiv \frac{5^2-1}{2} \equiv 1 \mod 2$).

$n = 9$, $\ell \in \{1,3,5,7,9\}$: two 2 cusps ($\Leftrightarrow \ell = 3,5$) at level 1 for certain, but above cusps with middle products 7 and 9, not clear there is a 2 cusp. Need more info on level 1 cusps.

Connections between 3 arithmetic problems MT/RIGP/STC using A_5

[STMT] Strong Tors. Conj. $\Longrightarrow$ Main MT Conj. and ($\sim\Leftrightarrow$).

Ram_{r_0}: Choose any r_0. For $k \geq 0$, use covers in $\mathrm{Ni}(G_k, \mathbf{C}_k)$ with at most r_0 classes in $\mathbf{C}_k$.

Question 10 (RIGP(A_5,p=2,r_0) Quest.). Is there r_0, so the RIGP holds for all G_k s from covers in Ram_{r_0}?

Theorem 11. *If the answer is "Yes!," then there are $2'$ conjugacy classes* $\mathbf{C}$ *(no more than r_0) in G, and a projective system $\{\mathcal{H}'_k \subset \mathcal{H}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}$ (a Modular Tower component branch over $\mathbb{Q}$) each having a $\mathbb{Q}$ point ([D06] [FrK97]).*

All $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ satisfy Main Conjecture for $p = 2$

All odd $1, \dots, n$ on left side of Table, p. 22, so number of 2 cusps at level 1 goes up with with n. For $n = 17$ get ≥ 2 more: $\frac{11^2-1}{8} \equiv \frac{13^2-1}{8} \equiv 1 \mod 2$.

MTs encodes a huge portion of the RIGP into questions about towers of varieties (MTs). A simple *ramification assumption* on regular realizations forces K points at all tower levels on some MT. Generalization of Mazur-Merel implies this should be impossible: STC $\implies$ MC (Cadoret, [STMT]). So, if STC holds, the ramification assumption must be wrong.

Question 12. When is there an umbrella result for both the Fried + L-O cases?

Appendix A: Using Lifting Invariant on p. 19

List of 3-tuples $(g_2, g_3, (g_2g_3)^{-1})$, with parameter $1 \le k \le \frac{n-1}{2}$:

- $\mathrm{ord}(g_2g_3) = 2k+1$; and $\langle g_2, g_3 \rangle$ is isomorphic to $A_{k+\frac{n+1}{2}}$.

[LUM, Fratt. Princ. 3]: Since 2 part of the Schur multiplier of A_n is just $\mathbb{Z}/2$, all cusps at level 1 above an o-$2'$ cusp are 2-cusps if and only if $s_{\mathrm{Spin}_n/A_n}(g_2, g_3, (g_2g_3)^{-1}) = -1$. Apply F-S formula (p. 9): In each case $(g_2, g_3, (g_2g_3)^{-1})$ has genus 0. So lifting invariant satisfies: $k \implies (-1)^{\frac{(2k+1)^2-1}{2}}$. Example: $n = 9$, $k = 1 \implies -1$, $2 \implies -1$, $3 \implies +1$, $4 \implies +1$.

Appendix B: Why I took all the d_i s equal

Basic Conjecture: A MT whose levels are uniformly defined over one number field is defined by a g-p' cusp branch [LUM, Conj. 1.5] (evidence in [LUM, §4.4].

Group theory: Odd pure-cycles generate an alternating (or cyclic) group $\Longrightarrow$ a g-$2'$ cusp must be an H-M rep. $\Longrightarrow$ d_i s equal in pairs. So, dealing with $\{\mathcal{H}_{n,d_1^2\cdot d_2^2,k}\}_{k=0}^{\infty}$.

Case of $\{\bar{\mathcal{H}}'_{n,d_1^2\cdot d_2^2,k}\}_{k=0}^{\infty}$ where $d_1 \neq d_2$. Fact: Genus of $\bar{\mathcal{H}}_{n,d_1^2\cdot d_2^2,0}$ exceeds 0. One possibility: All $\mathcal{H}_{n,d_1^2\cdot d_2^2,k}$ s are the same space. Producing a single 2-cusp, however, at level 1 excludes this: so, the same argument works.

Abbreviated References: [LUM] has much more

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[Def-Lst]Select from the list in www.math.uci.edu/conffiles_rims/deflist-mt/full-deflist-mt.html of present MT-related definitions. 09/05/06 examples: Branch-Cycle-Lem CFPV-Thm Cusp-Comp-Tree FS-Lift-Inv Hurwitz-Spaces Main-MT-Conj Modular-Towers Nielsen-Classes RIGP Strong-Tors-Conj mt-rigp-stc p-Poincare-Dual sh-Inc-Mat. A similar URL, www.math.uci.edu/conffiles_rims/deflist-mt/full-paplist-mt.html, is a repository for not just mine, but also of the growing list of those joining the MT project.

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