Generic absoluteness and universally Baire sets of reals

Trevor Wilson

Miami University, Ohio

July 18, 2016

Definition

- $B \subset \omega^\omega$ is universally Baire (uB) if for every λ there is a λ-absolutely complemented tree T with $p[T] = B$.
- A tree T is λ-absolutely complemented if there is a tree $\tilde{T}$ such that $\Vdash_{\mathrm{Col}(\omega,\lambda)} p[\tilde{T}] = \omega^\omega \setminus p[T]$.

Example

- Σ^1_1 sets are universally Baire. (Schilling)
- If every set has a sharp, then Σ^1_2 sets are universally Baire. (Martin–Solovay)
- More large cardinals imply that more sets of reals are universally Baire.

Definition

A sentence φ is generically absolute if, for every generic extension $V[g]$ of V, we have

$$V \models \varphi \iff V[g] \models \varphi.$$

Example

- Σ^1_2 sentences are generically absolute. (Shoenfield)
- If every set has a sharp, then Σ^1_3 sentences are generically absolute. (Martin–Solovay)
- More large cardinals imply that more sentences are generically absolute.

The continuum hypothesis is Σ^2_1 and is not generically absolute, but we can restrict Σ^2_1 to "nice" sets of reals:

Definition

A sentence is $(\Sigma^2_1)^{\mathrm{uB}}$ if it has the form

$$\exists B \in \mathrm{uB}\,(\mathrm{HC}; \in, B) \models \theta.$$

Theorem

- Σ^1_2 sentences are generically absolute. (Shoenfield)
- If there is a proper class of Woodin cardinals, then $(\Sigma^2_1)^{\mathrm{uB}}$ sentences are generically absolute. (Woodin)

We can force to get a little more generic absoluteness for free, using the compactness theorem for first-order logic.[1]

Definition

A sentence is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ if it has the form

$$\exists x \in \mathbb{R}\, \forall B \in \mathrm{uB}\, (\mathrm{HC}; \in, B) \models \theta[x].$$

Theorem

- Σ^1_3 generic absoluteness is consistent relative to ZFC.
- $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ generic absoluteness is consistent relative to ZFC and a proper class of Woodin cardinals.

Proof on board.

[1] See Hamkins' consistency proof for the *maximality principle*.

Generic absoluteness is related to uB sets:

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. Σ^1_3 generic absoluteness
2. $\Delta^1_2 \subset \mathrm{uB}$.

Theorem (W.)

The following statements are equivalent modulo a proper class of Woodin cardinals:

1. $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ generic absoluteness
2. $(\Delta^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$.

Proof on board.

For higher consistency strength we need real parameters.

Definition

One-step generic absoluteness refers to formulas with real parameters in V.

Corollary

The following statements are equivalent:

1. One-step $\utilde{\boldsymbol{\Sigma}}^1_3$ generic absoluteness
2. $\utilde{\boldsymbol{\Delta}}^1_2 \subset \mathrm{uB}$.

The following statements are equivalent modulo a proper class of Woodin cardinals:

1. One-step $\exists^{\mathbb{R}}(\utilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness
2. $(\utilde{\boldsymbol{\Delta}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$.

Remark

- The compactness theorem does *not* work to show consistency of generic absoluteness with real parameters.
- Forcing to remove a counterexample may add new counterexamples by adding reals.
- At a sufficiently large cardinal, this process reaches a closure point:

Definition

A cardinal κ is Σ_2-reflecting if it is inaccessible and

$$V_\kappa \prec_{\Sigma_2} V.$$

Theorem (Feng–Magidor–Woodin)

The following statements are equiconsistent modulo ZFC*:*

1. *There is a Σ_2-reflecting cardinal*
2. *One-step $\underset{\sim}{\mathbf{\Sigma}}^1_3$ generic absoluteness.*

Proof idea

- If κ is Σ_2-reflecting, then one-step $\underset{\sim}{\mathbf{\Sigma}}^1_3$ generic absoluteness holds in $V^{\mathrm{Col}(\omega,<\kappa)}$.
- If one-step $\underset{\sim}{\mathbf{\Sigma}}^1_3$ generic absoluteness holds, then ω_1^V is Σ_2-reflecting in L.

The forward direction can be adapted:

Theorem (W.)

If κ is Σ_2-reflecting and there is a proper class of Woodin cardinals, then one-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}}$ generic absoluteness holds in $V^{\mathrm{Col}(\omega,<\kappa)}$.

Proof on board.

Question

What is the consistency strength of a proper class of Woodin cardinals and one-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}}$ generic absoluteness?
Can we get *any* nontrivial lower bound?

Definition

Two-step generic absoluteness says that one-step generic absoluteness holds in every generic extension
(real parameters from generic extensions are allowed.)

Corollary

The following statements are equivalent:

1. Two-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness
2. $\undertilde{\mathbf{\Delta}}^1_2 \subset \mathrm{uB}$ in every generic extension.

The following statements are equivalent modulo a proper class of Woodin cardinals:

1. Two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness
2. $(\undertilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ in every generic extension.

For $\utilde{\boldsymbol{\Sigma}}^1_3$, there is an equivalence with large cardinals:

Theorem (Feng–Magidor-Woodin)

The following statements are equivalent:

1. Two-step $\utilde{\boldsymbol{\Sigma}}^1_3$ generic absoluteness

1′. $\utilde{\boldsymbol{\Delta}}^1_2 \subset \mathrm{uB}$ in every generic extension

2. $\Sigma^1_2 \subset \mathrm{uB}$

2′. $\utilde{\boldsymbol{\Sigma}}^1_2 \subset \mathrm{uB}$ in every generic extension

3. Every set has a sharp.

Proof idea

- Given sharps, use the Martin–Solovay tree.
- To get sharps, use Jensen's covering lemma.

For $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$, only some of these results carry over:

Theorem (W.)

Consider the statements:

1. Two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness

1′. $(\underset{\sim}{\mathbf{\Delta}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ in every generic extension

2. $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$

2′. $(\underset{\sim}{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ in every generic extension.

Then modulo a proper class of Woodin cardinals we have:

- $1 \iff 1'$ (noted already)
- $2 \iff 2'$ (proof on board)
- $2, 2' \implies 1, 1'$ (obvious).

Remark

Unlike for Σ^1_3, generic absoluteness for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}}$ is not known to follow from any large cardinal.
However, it can be *forced* from large cardinals:

Theorem (Woodin)

Assume there is a proper class of Woodin cardinals and a strong cardinal κ. Then $V^{\mathrm{Col}(\omega,2^{2^\kappa})}$ satisfies

1. Two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness
2. $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$.

Remark

2^{2^κ} bounds the number of measures on κ.

Theorem (W.)

Assume there is a proper class of Woodin cardinals and a strong cardinal κ. Then $V^{\mathrm{Col}(\omega,\kappa^+)}$ satisfies

1. Two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness
2. $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$.

Remark

κ^+ bounds the number of subsets of V_κ in $L(j(T), V_\kappa)$ where

- $j : V \to M$ witnesses some amount of strongness of κ
- T is a tree for Σ^2_1 in the derived model of V at κ.

The consistency strength of two-step generic absoluteness:

Theorem (Sargsyan, W., Woodin)

The following statements are equiconsistent modulo a proper class of Woodin cardinals:

1. Two-step $\exists^{\mathbb{R}}(\undertilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness

1′. $(\undertilde{\boldsymbol{\Delta}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ in every generic extension

2. $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$

2′. $(\undertilde{\boldsymbol{\Sigma}}^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ in every generic extension.

3. There is a strong cardinal.

It remains to show $\mathrm{Con}(1) \implies \mathrm{Con}(3)$ modulo a proper class of Woodin cardinals.

First note an analogous result in the projective hierarchy:

Theorem (Hauser, Woodin)

The following statements are equiconsistent:

- Two-step $\utilde{\mathbf{\Sigma}}^1_4$ generic absoluteness
- There is a strong cardinal.

Proof idea

- If κ is strong, then forcing to collapse 2^{2^κ} (or just κ^+) gives two-step $\utilde{\mathbf{\Sigma}}^1_4$ generic absoluteness.
- If two-step $\utilde{\mathbf{\Sigma}}^1_4$ generic absoluteness holds, there is a strong cardinal in the core model K.

Recall we want to show $\text{Con}(1) \implies \text{Con}(3)$:

1. There is a p.c. of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\text{uB}}$ generic absoluteness holds
2. There is a p.c. of Woodin cardinals and $(\Sigma^2_1)^{\text{uB}} \subset \text{uB}$
3. There is a p.c. of Woodin cardinals and a strong cardinal.

Remark

- $\text{Con}(1) \implies \text{Con}(2)$ is due to Sargsyan and me. It will be discussed below.
- $\text{Con}(2) \implies \text{Con}(3)$ is due to Sargsyan. (Similar to Steel's proof of Woodin's theorem that "there is a limit of Woodin cardinals λ and a $<\lambda$-strong cardinal" is consistent relative to $\text{AD}^+ + \theta_0 < \Theta$.)

Assume (1): there is a proper class of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness holds.

- Fix a singular limit λ of Woodin cardinals.
- Take a set $A \subset \lambda$ coding V_λ.
- Define $\mathrm{Lp}^{\mathrm{uB}}(A)$ as the union of all sound mice over A, projecting to A, with uB iteration strategies.
- By $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness between $V^{\mathrm{Col}(\omega,\lambda)}$ and $V^{\mathrm{Col}(\omega,\lambda^+)}$, the height of this mouse satisfies

$$o(\mathrm{Lp}^{\mathrm{uB}}(A)) < \lambda^+.$$

(Failure of covering by mice.)

Remark

There are two versions of $\mathrm{Lp}^{\mathrm{uB}}(A)$ for uncountable sets A. We can pass to a generic extension to make them equivalent:

Lemma (Sargsyan–W.)

If there is a proper class of Woodin cardinals then after forcing to collapse some cardinal to ω, for any sound premouse $\mathcal{M}$ built over any set of ordinals A and projecting to A, the following statements are equivalent:

1. Every countable $\overline{\mathcal{M}}$ elementarily embedding into $\mathcal{M}$ has a univerally Baire iteration strategy.
2. $\mathcal{M}$ has a universally Baire iteration strategy after forcing to collapse it to ω.

- Take a hull $X \prec H(\lambda^+)$ such that $|X| < \lambda$ and $X^\omega \subset X$.
- Let $\pi_X : M_X \cong X$ be the uncollapse map and $\pi_X(\bar{A}) = A$.
- Because $o(\mathrm{Lp}^{\mathrm{uB}}(A)) < \lambda^+$ we may take X cofinal in $o(\mathrm{Lp}^{\mathrm{uB}}(A))$, so by a standard argument X is mouse-full:

$$\mathrm{Lp}^{\mathrm{uB}}(\bar{A}) \subset M_X.$$

- We may assume $D(V, \lambda)$ satisfies mouse capturing, which means mouse-fullness is equivalent to OD-fullness:

$$\mathrm{OD}^{D(V,\lambda)} \cap \mathcal{P}(\bar{A}) \subset M_X,$$

where $D(V, \lambda)$ is the derived model of V at λ. (Otherwise by Sargsyan there is a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular," which is stronger than our desired conclusion.)

Lemma (W.)

If $X \prec H(\lambda^+)$ as above is OD-full, then $V^{\mathrm{Col}(\omega,|X|)}$ satisfies $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}_\lambda$, the pointclass of λ-universally Baire sets.

Proof idea

- This is similar to obtaining $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$ by collapsing 2^{2^κ} (or κ^+) where κ is strong.
- Instead of a strongness embedding, we use an ultrapower by the extender from the uncollapse map π_X.
- Fullness is used to apply this ultrapower to certain sets.

Finally, pressing down on λ gives a generic extension with $(\Sigma^2_1)^{\mathrm{uB}} \subset \mathrm{uB}$, which was (2).

Question

From a proper class of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness, can we *directly* construct a fullness-preserving iteration strategy for a $(\Sigma^2_1)^{\mathrm{uB}}$-suitable premouse, without first constructing trees for $(\Pi^2_1)^{\mathrm{uB}}$?

Remark

This would give a more descriptive-inner-model-theoretic construction of an inner model with a proper class of Woodin cardinals and a strong cardinal.

Question

If there is a proper class of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}}$ generic absoluteness holds, must there be an inner model M with a proper class of Woodin cardinals and a strong cardinal κ where $(\kappa^+)^M < \omega_1^V$?

Question

If two-step $\utilde{\mathbf{\Sigma}}^1_4$ generic absoluteness holds, must there be an inner model M with a strong cardinal κ where $(\kappa^+)^M < \omega_1^V$?

Remark

In both cases, generic absoluteness is obtained by collapsing the successor of a strong cardinal to ω, but the reversal gives *no upper bound* on the strong cardinal in the inner model.