Def (a) (P, Σ) is a le-pair (with scope V_θ) iff P is a premouse and Σ is an IS for P defined on V_θ and Σ normalizes well (NW) and has strong hull condensation (SHC)

(b) (P, Σ) is a lbt hod pair iff P is a lpm and ...

Rem If (P, Σ) is an le/lpm pair and $\pi : N \to P$ is elementary then (N, Σ^π) is a le/lpm pair s.t. $\Sigma^\pi(\mathcal{T}) = \Sigma(\pi\mathcal{T})$

(Normalizing well goes from Σ to Σ^π because normalizing commutes with copying.)

Notation $\pi : (P, \Sigma) \longrightarrow (Q, \Lambda)$ iff $\pi : P \to Q$ elem and $\Sigma = \Lambda^\pi$

$(P, \Sigma) \trianglelefteq (Q, \Lambda)$ iff $\exists (\nu, k)$ s.t. $P = Q|(\nu, k)$ and $\Sigma = \Lambda_{(\nu, k)}$

Comparison lemma ~~&~~ next page

Comparison lemma (AD^+). Let (P,Σ), (Q,Λ) be lc/lbt pairs with scope H_C. Then there are countable normal $\mathcal{T}$ on P and $\mathcal{U}$ on Q by Σ and Λ resp. with last models (P_1,Σ_1) and (Q_1,Λ_1), $\Sigma_1 = \Sigma_{\mathcal{T},P_1}$, $\Lambda_1 = \Sigma_{\mathcal{U},Q_1}$ s.t.

(a) $(P_1,\Sigma_1) \trianglelefteq (Q_1,\Lambda_1)$ and P-to-P_1 in $\mathcal{T}$ does not drop

or vice versa:

(b) $(Q_1,\Lambda_1) \trianglelefteq (P_1,\Lambda_1)$ and Q-to-Q_1 in $\mathcal{U}$ does not drop

Follows from Thm 4.3 in the notes:

Thm 4.3 Given δ Woodin, $P \in V_\delta$, Σ is δ-UB with scope V_δ, (P,Σ) an lc/lbt pair, and given $\mathbb{C}$ a maximal lc/lbt-construction of V_δ and s.t. $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ exists then either:

(a) (P,Σ) iterates normally to some $(M^{\mathbb{C}}_{\eta,\ell}, \Omega^{\mathbb{C}}_{\eta,\ell})$ s.t. $\langle \eta,\ell\rangle <_{lex} (\nu,k)$

or

(b) (P,Σ) iterates past $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, perhaps not strictly.

Moreover, there is a $\langle \nu, k \rangle <_{lex} \langle \delta, 0 \rangle$ s.t. (a) holds

To get comparison thm from Thm 4.3 mentioned above:

Given (P, Σ), (Q, Λ) we have that
Code(Σ), Code(Λ) are Suslin-co-Suslin
Let Γ be a good scaled pointclass s.t.
Code(Σ), Code(Λ) $\in \underset{\sim}{\Delta}$.

Let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ-Woodin model
that captures Σ and Λ. ($P \in HC^{N^*}$, have
T_0, T_1 trees on $\omega \times \delta^*$ in N^* s.t.

$$\mathrm{Code}(\Sigma) = \bigcup_{i: N^* \to M^*} p[i(T_0)]$$

$$\mathbb{R} \setminus \mathrm{Code}(\Sigma) = \bigcup_i p[i(T_1)]$$

Apply Thm 4.3 in N^* to (P, Σ) also
to (Q, Λ)

$N^* \models$ "I am uniquely iterable for trees in V_{δ^*}"
so the construction $\mathbb{C}$ converges, as we will prove.

$(P, \Sigma) \rightsquigarrow (M^{\mathbb{C}}_{\nu_0, k_0}, \Omega^{\mathbb{C}}_{\nu_0, k_0})$
$(Q, \Lambda) \rightsquigarrow (M^{\mathbb{C}}_{\nu_1, k_1}, \Omega^{\mathbb{C}}_{\nu_1, k_1})$
So $N^* \models$ conclusion of the comparison for
(P, Σ), (Q, Λ)

Then, by absoluteness, the conclusion is true in our model of AD^+.

<u>REM</u> The comparison process involves hitting the same $\mathcal{E}$ on both sides on the P-side and Q-side.

<u>Theorem</u> (AD^+). Let (P, Σ) be a lbr hod pair. Then (P, Σ) is solid and universal. It follows that core(P) exists.
(Here we need that the $\Omega^{\mathbb{C}}_{\nu,k}$ are ~~mice~~ "mice" under unique iterability.)

<u>Corollary</u> $\mathbb{C}^{N^*}$ does not break down for $(N^*, \Sigma^*, \delta^*)$ as above.

<u>REM</u> Condensation works, too
E.g. if (P, Σ) is a lbr hod pair,
$P|\xi \models ZFC$ for $\xi < \hat{o}(P)$, $Q = \mathrm{Hull}^P_\omega(\phi)$,
$\pi : Q \to P|\xi$ elementary
Then $(Q, \Sigma^\pi) \trianglelefteq (P, \Sigma)$, $Q \triangleleft P|\omega_1^P$.

Proof involves iterating a phalanx with a companying $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$. The technique also gives

Theorem (AD^+) If (P,Σ) is a lbr hod pair and $P \models ZFC$ with scope HC then

$P \models$ UBH for nice trees (plus two trees) using extenders from $\dot{E}^P$

(known for (P,Σ) a le-pair)

Corollary For (P,Σ) as above:

$P \models \lambda$ a limit of Woodins

and g on $Col(\omega,<\lambda)$ is P-generic

$$\mathbb{R}^* = \bigcup_{\eta<\lambda} \mathbb{R}^{P[g\cap Col(\omega,<\eta)]}$$

$$Hom^* = \{ p[T] \cap \mathbb{R}^* \mid \exists \alpha < \lambda \quad P[g \cap Col(\omega,\alpha)] \models T \text{ is } \lambda\text{-a.c.} \}$$

then for each $\langle \nu,k\rangle < \langle \lambda, 0\rangle$ there is a term $\tau \in P$ s.t. for all such g, $\mathbb{R}^*$

$$\tau_g = \Sigma_{P|\langle \nu,k\rangle} \cap HC$$

also

$\Vdash \tau \in Hom^*$

also these τ_g's are Wadge cofinal in Hom^*.

Proof Sargsyan's thesis, "Generic interpretability"

Def (AD^+) Hod pair capturing (HPC) is the statement

$$\forall A \subseteq \mathbb{R}\ \exists \text{ lbr hod pair } (P, \Sigma) \text{ s.t. } A \leq_W \mathrm{Code}(\Sigma)$$

for A Suslin co-Suslin.

~~Theorem Assume~~

Corollary (of the proof of Wadge cofinality)

$$L(\mathbb{R}^*, \mathrm{HOM}^*) \models \mathrm{HPC}$$

for $\mathbb{R}^*$, Hom^* as above.

Theorem ($AD_{\mathbb{R}}$ + HPC) $\mathrm{HOD}|\Theta$ is an lpm.

Rem By a result of Steel-Trang can weaken $AD_{\mathbb{R}}$ to AD^+, but need a stronger version of HPC.

Local HOD computation:

Given AD^+ + HPC; for every Γ s.t. $L(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Gamma$ we have

$$L(\Gamma, \mathbb{R}) \models \mathrm{HPC}.$$

Proof Connect Suslin cardinals to $o(M_\infty(P, \Sigma))$ for (P, Σ) a lbr hod pair.

<u>Using this connection:</u> Let (M, Σ) be a ~~ctbl~~ lbr mod pair,

$M \models \delta$ a WLW $+ (\exists \lambda > \delta)(\lambda$ a limit of Woodins$)$

(This may be a very strong requirement)

Let

$$D = L(\mathbb{R}^*, \mathrm{Hom}^*) \overset{\text{def}}{=} D(M, < \lambda)$$

Then there is a pointclass Γ s.t.

$L(\Gamma, \mathbb{R}) \models$ LSA + for every $s : \omega \to \Theta$, every OD(s) set of reals is Suslin co-Suslin

<u>REM</u> This is essentially due to Sargsyan.