MATH 13 WINTER 2016 HOMEWORK 5

Due: Wednesday March 9 Please turn in at the lecture.

Each group please turn in only one paper. If you prefer to work alone, that is fine: A group can consist of one member.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your arguments clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.

When preparing the homeworks, please follow the **Rules for homeworks** on the course webpage under **Course information and policies** and also the guidelines under **Grading**. In particular, keep in mind the **Aspects of grading** in the **Grading** section.

1. (4pt) Given are sets A, B, C and a set U such that $A, B, C \subseteq U$. The complements of sets A, B, C are computed with respect to U, so for instance $A^c = U \setminus A$. In each case decide whether the given statement is true for all such sets A, B, C, U or not, and then prove (if true) or disprove it (if false).

(a) **(1pt) (1pt)** $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.
(b) **(1pt)** $(A \setminus B) \setminus C = A \setminus (B \cup C)$.
(c) **(2pt)** $A \times B^c = (A \times B)^c$.

2. (4pt) Given are the following a binary relations.

(i) R is a binary relation on $\mathbb{N}$, so $R \subseteq \mathbb{N} \times \mathbb{N}$ and is defined as follows.

$$(a, b) \iff a \equiv b \mod 7$$

(ii) S is a binary relation on $\mathbb{R}$, so $S \subseteq \mathbb{R} \times \mathbb{R}$ and is defined as follows.

$$(x, y) \in S \iff \text{there is an integer } n \text{ such that } n \leq x, y < n + 1.$$

(iii) T is a binary relation on $\mathbb{Z} \times \mathbb{Z}$, so $T \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ and is defined as follows.

$$((a, b), (c, d)) \in T \iff \max(a, b) = \max(c, d)$$

Here $\max(a, b)$ is the larger number of a, b if one of them is smaller than the other, and $\max(a, b) = a = b$ if $a = b$.

(iv) V is a binary relation on $\mathbb{N} \times \mathbb{N}$, so $V \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ and is defined as follows.

$$((a, b), (c, d)) \in V \iff \gcd(a, b) = \gcd(c, d).$$

Answer the following questions. In either case prove your answer.

(a) **(0.5pt for each relation)** Is $R^{-1} = R$?
(b) **(0.5pt for each relation)** Assume $(a, b) \in R$ and $(b, c) \in R$. Is $(a, c) \in R$?

Formulate the corresponding questions for relations S, T and V, answer them, and prove your answer.

3. (9pt) Given are the following functions. Recall that $\mathbb{N}$ consists of all **positive** integers.

(i) $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$.
(ii) $g : \mathbb{R} \to \mathbb{Z}$ defined by

$$g(x) = \text{the smallest integer } n \text{ such that } x < n$$

(iii) $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $h(a, b) = 2^a \cdot 3^b$.
(iv) $u : \mathbb{Z} \to \{0, 1, 2, 3, 4\}$ defined by

$$u(a) = \text{the remainder when } a^3 \text{ is divided by } 5$$

(v) $w : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by $w(A) = A^c$.

Answer the following questions; in either case provide a justifying proof.

(a) **(0.5pt each)** Is the function f, g, h, u, w injective?
(b) **(1pt for u, 0.5pt for g, h, w)** Is the function g, h, u, w surjective?
(c) **(1pt each)** Compute $g[[0, \infty)]$, $f^{-1}[\{0\}]$, $g^{-1}[\{0, 1\}]$ and $u^{-1}[\{3\}]$.

4. (3pt) Construct bijections as directed, and in each case prove that the function you construct is a bijection.

(a) **(1pt)** Construct a bijection from A to B where

$$\begin{aligned} A &= \{a \in \mathbb{Z} \mid a \equiv 1 \mod 3\} \\ B &= \{b \in \mathbb{Z} \mid b \equiv 2 \mod 3\} \end{aligned}$$

(b) **(1pt)** Let A be a set. Construct a bijection from A to $\{0\} \times A$.
(c) **(1pt)** Let A, B be sets. Construct a bijection from $A \times B$ to $B \times A$.