MATH 13 WINTER 2016 HOMEWORK 6

Due: Monday, March 14, 2016 Please turn in at the final exam.

Each group please turn in only one paper. If you prefer to work alone, that is fine: A group can consist of one member.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS: It is crucial that you write your arguments clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.

When preparing the homeworks, please follow the **Rules for homeworks** on the course webpage under **Course information and policies** and also the guidelines under **Grading**. In particular, keep in mind the **Aspects of grading** in the **Grading** section.

1. (6pt; 2pt for each question a,b,c) Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove the following: If both functions f, g are injections then the composition $g \circ f : A \to C$ is also an injection.

(b) Assume f is not an injection and g is an injection. Can the composition $g \circ f : A \to C$ be an injection? (Equivalently, is there a pair of functions $f : A \to B$ and $g : B \to C$ such that f is not an injection, g is an injection and the composition $g \circ f$ is an injection?)

(c) Assume f is an injection and g is not an injection. Can the composition $g \circ f : A \to C$ be an injection? (Equivalently, is there a pair of functions $f : A \to B$ and $g : B \to C$ such that f is an injection, g is not an injection and the composition $g \circ f$ is an injection?)

2. (8pt; 2pt for each question a,b,c,d) If not specified otherwise, A, B, C and D are sets.

(a) Assume $f : A \to C$ and $g : B \to D$ are bijections. Prove that the map $h : A \times B \to C \times D$ defined by

$$h(\langle a, b\rangle) = \langle f(a), g(b)\rangle$$

is a bijection.

(b) Prove that the function $f : [0, 1] \to [0, 9]$ defined by $f(x) = 9x$ is a bijection between intervals $[0, 1]$ and $[0, 9]$.

(c) Prove that the function $g : (0, 1) \to (1, \infty)$ defined by $g(x) = 1/x$ is a bijection between intervals $(0, 1)$ and $(1, \infty)$.

(d) Denote the set of all infinite sequences of integers by S. Prove that the functions $h : S \to S$ defined by

$$h(\langle a_1, a_2, a_3, a_4, \ldots \rangle) = \langle a_2, a_1, a_3, a_4 \ldots \rangle$$

is a bijection.

Remark. So the function h swaps the first two members of the sequence.

3. (6pt; 1pt for each question a(i),a(ii),b(i),b(ii),c(i) and c(ii)) The following exercise focuses on equivalence relations.

(a) The binary relation E on $\mathbb{R}$ is defined as follows:

$$\langle a, b \rangle \in E \quad \Longleftrightarrow \quad a - b \text{ is an integer}$$

(i) Prove that E is an equivalence relation.
(ii) Explain what is the equivalence class $[5]_E$.

(b) Consider a line ℓ in the plane P. The binary relation R on P is defined as follows: If $x, y \in P$ are two points in P then

$$\langle x, y \rangle \in R \quad \Longleftrightarrow \quad \mathsf{dist}(x, \ell) = \mathsf{dist}(y, \ell)$$

where $\mathsf{dist}(x, \ell)$ is the distance of the point x from the line ℓ.

(i) Prove that R is an equivalence relation.
(ii) Consider a point a in the plane P. Explain what is the equivalence class $[a]_R$. Also explain what is $[a]_R$ in the case where a is a point on line ℓ, that is, in the case where $a \in \ell$.

(c) Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ which have the first derivative $f' : \mathbb{R} \to \mathbb{R}$. (This means: the first derivative $f'(a)$ exists for every $a \in \mathbb{R}$.) Let D be a binary relation on F defined by

$$\langle f, g \rangle \in D \quad \Longleftrightarrow \quad f' = g'$$

(i) Prove that D is an equivalence relation.
(ii) Explain what is the equivalence class $[f]_D$. In particular, describe the equivalence class $[c_0]_D$ where $c_0 : \mathbb{R} \to \mathbb{R}$ is the constant function with value 0.