MATH 280C SPRING 2016 HOMEWORK 3

Due date: Monday June 13

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (1 page) Given a well-founded relation $R \subseteq X \times X$ we let

$$\mathsf{rank}(R) = \sup(\{\mathsf{rank}_R(x) + 1 \mid x \in X\})$$

Given a well-founded tree T on a set A then the well-founded relation in question is the reverse inclusion; in this we write $\mathsf{rank}(T)$ in place of $\mathsf{rank}(\supseteq\restriction (T \times T))$. Notice that $\mathsf{rank}(T) = \mathsf{rank}_T(\varnothing) + 1$.

(a) Give an example of a tree on ω whose rank is $\omega + 1$.

(b) Prove by induction on $\alpha < \omega_1$ that for every $\alpha < \omega_1$ there is a tree T on ω such that $\mathsf{rank}(T) = \alpha + 1$.

2. (3/2 page) Given are trees S, T on sets A, B respectively. Consider a map $\sigma : S \to T$ with the following properties:

(i) $s \subseteq s' \implies \sigma(s) \subseteq \sigma(s')$

(ii) For every $x \in [S]$ the union $\bigcup_{n\in\omega} \sigma(x \restriction n)$ is infinite, in other words, this union is an element of $[T]$.

Here recall that the body of a tree U consists of all branches through U, that is, infinite sequences x such that $x \restriction n \in U$ for all $n \in \omega$.

(a) Let $\sigma : S \to T$ satisfy (i) and (ii). Prove that the map $\tilde{\sigma} : [S] \to [T]$ defined by

$$\tilde{\sigma}(x) = \bigcup_{n\in\omega} \sigma(x \restriction n)$$

is continuous.

(b) Prove that if $\tau : [S] \to [T]$ is a continuous map then there is a map $\sigma : S \to T$ satisfying (i) and (ii) such that $\tau = \tilde{\sigma}$.

3. (3/2 page) One can define the notion of κ-Suslin set for arbitrary Polish space X. Given a map $s \mapsto C_s$ defined on $\kappa^{<\omega}$ such that $C_s \subseteq X$ for every $s \in \kappa^{<\omega}$, define $\mathcal{A}(\vec{C}_s)$ by

$$\mathcal{A}(\vec{C}_s) = \bigcup_{f\in\kappa^\omega} \bigcap_{n\in\omega} C_{f\restriction n}$$

(a) Prove that if $s \mapsto C_s$ is a map as above where each C_s is a closed subset of X then there is a closed set $C \subseteq X \times \kappa^\omega$ such that $p[C] = \mathcal{A}(\vec{C}_s)$.

(b) Prove that if $C \subseteq X \times \kappa^\omega$ is a closed set then there is a map $s \mapsto C_s$ as above such that each C_s is a closed subset of X and $p[C] = \mathcal{A}(\vec{C}_s)$.

4. (2/3 page) Recall that a function $f : X \to Y$ is Borel measurable if and only if $f^{-1}[A]$ is a Borel subset of X whenever A is an open subset of Y. Given a basis $\mathcal{B}$ of topology of Y, a function f is Borel measurable iff $f^{-1}[B]$ is a Borel subset of X whenever $B \in \mathcal{B}$.

(a) Let $f : \mathcal{N} \to \mathcal{N}$ be a function such that f, viewed as a subset of $\mathcal{N} \times \mathcal{N}$, is $\mathbf{\Sigma}^1_1$. Prove that f is Borel measurable.

(b) Let $f : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ be a Borel measurable function which implicitly defines a function $g : \mathcal{N} \to \mathcal{N}$. That is, for every $x \in \mathcal{N}$ there is exactly one $y \in \mathcal{N}$ such that $f(x, y) = 0$ where we write $0 = (0, 0, 0, \dots)$. The function g is then defined by

$$g(x) = \text{the unique } y \in \mathcal{N} \text{ such that } f(x, y) = 0$$

Prove that g is Borel measurable. Conclude that if $h : \mathcal{N} \to \mathcal{N}$ is a Borel measurable bijection then so is h^{-1}; such a map h is called a Borel isomorphism.

Hint. For (a) use Suslin's separation theorem. For (b) use (a).

5. (1 page) Recall that the Perfect Set Property asserts that every set $A \subseteq \mathcal{N}$ is either countable or else contains an perfect subset. Work in ZF.

(a) Assume there is a well-ordering of $\mathcal{N}$. Prove that there are disjoint sets A, B both of size $2^{\aleph_0}$ such that every perfect subset of $\mathcal{N}$ has nonempty intersection with both A, B. Conclude that A is a set of size continuum without the Perfect Set Property. That is, if there is a well-ordering of $\mathcal{N}$ then the Perfect Set Property fails.

(b) Assume Perfect Set Property holds. Prove that there does not exist any injection $f : \omega_1 \to \mathcal{N}$.

Hint. For (a) first prove that there is an enumeration $(C_\alpha \mid \alpha < 2^{\aleph_0})$ of all perfect sets and then diagonalize. For (b) use (a).