MATH 280B WINTER 2016 HOMEWORK 2

Due date: Monday, February 1

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. Given an $\mathcal{L}$ formula $\varphi(u, v_1, \ldots, v_\ell)$, we say that φ defines a partial/total function over an $\mathcal{L}$-structure $\mathcal{M}$ iff the set

$$\{((a_1, \ldots, a_\ell), b) \in M^\ell \times M \mid \mathcal{M} \models \varphi(b, a_1, \ldots, a_\ell)\}$$

is a partial/total function from M^ℓ into M.

(a) **(3 lines)** Write down an $\mathcal{L}$-sentence φ' such that for every $\mathcal{L}$-structure $\mathcal{M}$,

$$\varphi \text{ defines a partial function over } \mathcal{M} \quad \text{iff} \quad \mathcal{M} \models \varphi'.$$

(b) **(2 lines, use the result in (a))** Write down an $\mathcal{L}$-sentence φ'' such that for every $\mathcal{L}$-structure $\mathcal{M}$,

$$\varphi \text{ defines a total function over } \mathcal{M} \quad \text{iff} \quad \mathcal{M} \models \varphi''.$$

(c) **(3 lines)** Let $<^*$ be a well ordering on M and let $\mathcal{L}^*$ be the language obtained by adding a binary relation symbol $\dot{<}$ to $\mathcal{L}$. Also let $\mathcal{M}^*$ be the expansion of $\mathcal{M}$ obtained by adding the interpretation $\dot{<}^{\mathcal{M}^*} = <^*$. Show that $\mathcal{M}^*$ has definable Skolem functions (of course, with respect to the language $\mathcal{L}^*$.)

2. (1/2 page) Let $\mathcal{M}$ be an $\mathcal{L}$-structure which has definable Skolem functions. Let $\mathcal{M}_1, \mathcal{M}_2$ be elementary substructures. Show that $M_1 \cap M_2$ induces an elementary substructure of $\mathcal{M}$. We denote this structure by $\mathcal{M}_1 \cap \mathcal{M}_2$. Generalize this to intersections of arbitrary collections of elementary substructures. Conclude that $\mathcal{M}$ has a smallest elementary substructure, that is, there is some $\mathcal{M}^* \prec \mathcal{M}$ such that $\mathcal{M}^* \prec \mathcal{M}'$ whenever $\mathcal{M}' \prec \mathcal{M}$.

3. (1 page) Let $(I, \leq_I)$ be a linear order and $(\mathcal{M}_i \mid i \in I)$ be a sequence of $\mathcal{L}$-structures satisfying

$$i < j \quad \Longrightarrow \quad \mathcal{M}_i \text{ is a substructure of } \mathcal{M}_j.$$

Such a sequence of $\mathcal{L}$-structures is called a **chain** of structures. If we additionally have

$$i < j \quad \Longrightarrow \quad \mathcal{M}_i \prec \mathcal{M}_j$$

then the sequence is called an **elementary chain**. If λ is an ordinal, the chain $(\mathcal{M}_\alpha \mid \alpha < \lambda)$ of $\mathcal{L}$-structures is **continuous** iff $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$ whenever $\beta < \lambda$ is a limit ordinal.

Let

(a) $M = \bigcup_{i \in I} M_i$,

and define an $\mathcal{M}$-structure with domain M by interpreting $\mathcal{L}$-symbols as follows.

(b) $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for some/all $i \in I$, whenever c is a constant symbol of $\mathcal{L}$.

(c) $f^{\mathcal{M}}(a_1, \ldots, a_\ell) = b$ iff $f^{\mathcal{M}_i}(a_1, \ldots, a_\ell) = b$ for some/all $i \in I$ such that $a_1, \ldots, a_\ell \in M_i$, whenever f is an ℓ-place function symbol of $\mathcal{L}$.

(c) $(a_1, \ldots, a_\ell) \in R^{\mathcal{M}}$ iff $(a_1, \ldots, a_\ell) \in R^{\mathcal{M}_i}$ for some/all $i \in I$ such that $a_1, \ldots, a_\ell \in M_i$, whenever R is an ℓ-place relation symbol of $\mathcal{L}$.

The structure $\mathcal{M}$ is called the **union** of the elementary chain $(\mathcal{M}_i \mid i \in I)$ and often denoted by $\bigcup_{i \in I} \mathcal{M}_i$.

(a) Prove that if $(\mathcal{M}_i \mid i \in I)$ is an elementary chain then for every $i \in I$ we have $\mathcal{M}_i \prec \mathcal{M}$.

(b) Prove that if $(\mathcal{M}_\alpha \mid \alpha < \lambda)$ is a continuous chain of structures such that $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ whenever $\alpha < \lambda$ then this chain is an elementary chain.

4. (1 page) Let $\kappa < \theta$ be uncountable cardinals. Consider the structure $\mathcal{H} = (H_\theta, \in, <^*)$ where $<^*$ is a well-ordering on H_θ. (Recall from Math 280A that H_θ is the set of all sets x that are hereditarily of cardinality $< \theta$, that is, $\mathsf{card}(\mathsf{trcl}(x)) < \theta$.) The corresponding language is $\mathcal{L}^* = \{\in, \dot{<}\}$ with obvious interpretations of symbols.

Given an elementary substructure $\mathcal{M} \prec \mathcal{H}$ we let

$$\delta_M = \sup(M \cap \kappa).$$

Strictly speaking we should write $\delta_{\mathcal{M}}$ instead of δ_M, but it is common to write δ_M, as M uniquely determines $\mathcal{M}$. Obviously $\delta_M \leq \kappa$. Notice also that $\delta_M \subseteq M$ iff $\delta_M = M \cap \kappa$.

(a) Assume κ is regular. Show that for every $a \in H_\theta$ there is some $\mathcal{M} \prec \mathcal{H}$ such that $a \in M$, $\delta_M < \kappa$ and $\delta_M \subseteq M$.

(b) Let $\mathbb{M}$ be the set of all $\mathcal{M}$ as in (a) where κ is still regular. Show that the set

$$D = \{\delta_M \mid \mathcal{M} \in \mathbb{M}\}$$

contains a club.

(c) Show that if $\mathsf{cf}(\kappa) = \omega$ then $\delta_M = \kappa$.

(d) More generally, assume κ is singular. Show that if $\mathcal{M} \prec \mathcal{H}$ is such that $\delta_M \subseteq M$ then $\delta_M = \kappa$.

Hint. For (a) and (b) use the elementary chain construction.

5. (1/2 page) We use the notation from Problem 4. Assume κ is regular uncountable. Prove the following.

(a) If C is a closed unbounded subset of κ and $\mathcal{M} \prec \mathcal{H}$ is such that $C \in M$ and $\delta_M < \kappa$ then $\delta_M \in C$.

(b) If $(C_\alpha \mid \alpha < \kappa)$ is a sequence of closed unbounded subsets of κ and $\mathcal{M} \prec \mathcal{H}$ is such that $(C_\alpha \mid \alpha < \kappa) \in M$ and $\delta_M < \kappa$ then $\delta_M \in \triangle_{\alpha<\kappa} C_\alpha$.

(c) Use (b) of the current problem along with (b) of Problem 4 to give an alternative proof that a diagonal intersection of a κ-sequence of club subsets of κ is in the club filter.

Remark. In (b) recall from Math 280A that $\triangle_{\alpha<\kappa} C_\alpha$ is the diagonal intersection of $(C_\alpha \mid \alpha < \kappa)$. If you prefer, in (b) you may make a simplifying assumption that $\delta_M \subseteq M$.