MATH 280B WINTER 2016 HOMEWORK 4

Due date: Wednesday, March 2

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (1page) We sketched an argument, using the back-and-forth construction, showing that the theory DLO of dense linear orderings without endpoints is $\aleph_0$-categorical, that is, has precisely one model up to isomorphism. Recall hat the language of DLO consists of a single binary relation symbol $\dot{<}$.

For $n \in \mathbb{N}^+$ let

$$\mathcal{L}_n = \{\dot{<}\} \cup \{c_i \mid i \in \mathbb{N}\} \cup \{D_i \mid i < n\}$$

where c_i are new constant symbols and D_i are unary relation symbols, and

$$\begin{aligned} T_n = \mathsf{DLO} \cup \{c_i \dot{<} c_j \mid i < j\} \quad & \cup \quad \{\text{“Each } D_i \text{ is a dense subset of the model”}\} \\ & \cup \quad \{\text{“The family } \{D_i \mid i < n\} \text{ partitions the model”}\} \\ & \cup \quad \{c_i \in D_0 \mid i \in \mathbb{N}\} \end{aligned}$$

Show that T_n is an $\mathcal{L}_n$-theory. Then show that T_n has precisely $(n+2)$ countable models up to isomorphism.

2. (2/3 page) Let $\mathcal{U}$ be an ultrafilter over I and $(\mathcal{M}_i \mid i \in I)$ be an indexed system of $\mathcal{L}$-structures. Let $\mathcal{M}$ be the ultraproduct of $(\mathcal{M}_i \mid i \in I)$ by $\mathcal{U}$.

Show by induction on the complexity of terms that if $t(v_1, \ldots, v_\ell)$ is a term and $a_1, \ldots, a_\ell \in \prod_{i \in I} M_i$ then

$$t^{\mathcal{M}}([a_1], \ldots, [a_\ell]) = [i \mapsto t^{\mathcal{M}_i}(a_1(i), \ldots, a_\ell(i))].$$

3. (10 lines) Let $\mathcal{U}$ be a principal ultrafilter over I and $(\mathcal{M}_i \mid i \in I)$ be an indexed system of $\mathcal{L}$-structures. Let $\mathcal{M}$ be the ultraproduct of $(\mathcal{M}_i \mid i \in I)$ by $\mathcal{U}$. Describe how $\mathcal{M}$ compares to the structures $\mathcal{M}_i$.

4. (2/3 page) Let $\mathcal{L} = \{\dot{R}\}$ be the language with a single binary relation symbol R. Let $\mathcal{U}$ be an ultrafilter over I. We say that $\mathcal{U}$ is **ω-complete** iff

for every countable family $\{A_\ell \mid \ell \in \mathbb{N}\} \subseteq \mathcal{U}$ we have $\bigcap_{\ell \in \mathbb{N}} A_\ell \in \mathcal{U}$.

Let $(\mathcal{M}_i \mid i \in I)$ be an indexed system of $\mathcal{L}$-structures and let $\mathcal{M}$ be the ultraproduct of $(\mathcal{M}_i \mid i \in I)$ by $\mathcal{U}$. Assume that for every $i \in I$, the relation $R^{\mathcal{M}_i}$ is well-founded and for each $i \in \mathbb{N}$ the relation $R^{\mathcal{M}_i}$ has an infinite chain

$$a_0^i \, R^{\mathcal{M}_i} \, a_1^i \, R^{\mathcal{M}_i} \ldots a_n^i \, R^{\mathcal{M}_i} \, a_{n+1}^i \, R^{\mathcal{M}_i} \ldots \tag{1}$$

Prove:

$$R^{\mathcal{M}} \text{ is well-founded} \quad \text{iff} \quad \mathcal{U} \text{ is } \omega\text{-complete}.$$

If you want more challenge, assume instead of (1) that for each $i \in \mathbb{N}$ there is a finite chain

$$a_0^i \; R^{\mathcal{M}_i} \; a_1^i \; R^{\mathcal{M}_i} \; \dots \; R^{\mathcal{M}_i} \; a_{n(i)}^i \tag{2}$$

such that $\lim_{i\to\infty} n(i) = \infty$.

5.(2/3 page) Prove the compactness theorem using ultraproducts. Given a language $\mathcal{L}$ let Σ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let

$$I = \text{the set of all finite subsets of } \Sigma$$

and to each $i \in I$ assign some $\mathcal{L}$-structure $\mathcal{M}_i$ such that

$$\mathcal{M}_i \models \sigma \quad \text{whenever} \quad \sigma \in i.$$

For each $i \in I$ let

$$A_i = \{j \in I \mid i \subseteq j\}.$$

Show that

$$\{A_i \mid i \in I\}$$

is a centered system. Argue that there is an ultrafilter $\mathcal{U}$ over I such that

$$\{A_i \mid i \in I\} \subseteq \mathcal{U}$$

and look at the ultraproduct of $(\mathcal{M}_i \mid i \in I)$ by $\mathcal{U}$.

6. (2/3 page) Let $\mathcal{L}$ be a countable language, $(\mathcal{M}_i \mid i \in \mathbb{N})$ be an indexed system of $\mathcal{L}$-structures, and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $\mathcal{M}$ be the ultraproduct of $(\mathcal{M}_i \mid i \in \mathbb{N})$. Prove that $\mathcal{M}$ is $\aleph_1$-saturated. (We are not assuming anything about the structures $\mathcal{M}_i$!)