MATH 281A FALL 2016 HOMEWORK 1

Due date: Wednesday, October 26

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

Convention. If A is a set, we write $\mathcal{P}(A)$ to denote both the power set of A and the Boolean algebra $(\mathcal{P}(A), \cap, \cup, {}^c, \varnothing, \omega)$ where ^{c} is the complementation in A. It will be clear from the context which meaning applies. In the cases where there may be misunderstanding, we make the things precise.

1. (1 page) Work in ZFC. Recall that $\mathcal{N}$ denotes the Baire space. Assume that $A \subseteq \mathcal{N}$ is an $\aleph_n$-Suslin set where $n \in \omega \setminus \{0\}$. Prove that A is the union of a family of $\aleph_n$ many Borel sets.

2. (2 1/2 page) Work in ZFC. Let $\mathcal{I} = [\omega]^{<\omega}$, and notice that $\mathcal{I}$ is an ideal on $\mathcal{P}(\omega)$. Consider the Boolean algebra

$$\mathbb{B} = \mathcal{P}(\omega)/\mathcal{I}.$$

For each of the two Boolean algebras $\mathcal{P}(\omega)$ and $\mathbb{B}$ answer the following questions, and give supporting arguments.

(a) **(1/2 page)** Is there a maximal antichain of size ω?
(b) **(1/3 page)** Is every infinite antichain of size ω?
(c) **(2/3 page)** What is the saturation of the respective Boolean algebra?
(d) **(1/2 page)** Is there a strictly descending ω-chain $\vec{b} = (b_n \mid n \in \omega)$ of elements of the algebra such that $\vec{b}$ is bounded from below?
(e) **(1/2 page)** Is every strictly descending ω-chain $(b_n \mid n \in \omega)$ of elements of the algebra bounded from below?

In (c) and (d), the term "bounded from below" means that there is some element b of the Boolean algebra such that $b \leq b_n$ for each $n \in \omega$ and $b \neq 0$.

Important note: Regarding the page limitation required above: Each clause should treat both algebras on the required space. That is, for instance, (a) should use 1/2 page to give answers for both algebras, meaning that one should not use 1/2 page for $\mathcal{P}(\omega)$ and another 1/2 page for $\mathbb{B}$.

3. (2/3 page) Work in ZFC. Let $\mathbb{B}$ be a Boolean algebra and

$$S(\mathbb{B}) = \text{the set of all ultrafilters on } \mathbb{B}.$$

For each $b \in \mathbb{B}$ let

$$P_b = \{U \in S(\mathbb{B}) \mid b \in U\}$$

and

$$B^* = \{P_b \mid b \in \mathbb{B}\}$$

Prove that B^* induces a Boolean subalgebra of $\mathcal{P}(S(\mathbb{B}))$, in other words, there is a Boolean subalgebra of $\mathcal{P}(S(\mathbb{B}))$ whose elements are precisely all elements of B^*. Explain what this means. Denote this subalgebra by $\mathbb{B}^*$. Let $h : \mathbb{B} \to \mathbb{B}^*$ be the map defined by

$$h(b) = P_b.$$

Prove that h is an isomorphism between $\mathbb{B}$ and $\mathbb{B}^*$.