MATH 281A FALL 2016 HOMEWORK 2

Due date: Monday, November 14

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. Work in ZF. A Borel code is a pair $c = (T, u)$ where

(a) T is a countable well-founded tree with a single root $r = r_c$; we denote the corresponding tree ordering by $<_T$.
(b) $u : \min(T) \to {}^{<\omega}\omega$ where $\min(T)$ is the set of all minimal elements of T.

Notice that if $c = (T, u)$ is a Borel code and $t \in T$ then $c_t = (T^t, u \restriction \min(T^t))$ is also a Borel code; here $T^t = \{t' \in T \mid t \leq_T t'\}$. Notice also that $r(c_t) = t$.

Given a Borel code $c = (T, u)$, we define an evaluation $B(c)$ of c recursively as follows.

(i) If T consists of only a root, i.e. $T = \{r_c\}$ then $B(c) = B_{u(r_c)}$ where recall that for every $s \in {}^{<\omega}\omega$, the set B_s is the basic open neighborhood in $\mathcal{N}$ determined by the sequence s.
(ii) If r_c has exactly one immediate successor in T, call it t, then

$$B(c) = \mathcal{N} \smallsetminus B(c_t).$$

(iii) If r_c has more than one immediate successors in T then

$$B(c) = \bigcup\{B(c_t) \mid t \text{ is and immediate successor of } r_c \text{ in } T\}.$$

A. Prove the following. For (A3) you may use the Axiom of Choice.

(A1) **(1/3 page)** Every Borel code has an evaluation.
(A2) **(1/3 page)** For every Borel code c, the evaluation $B(c)$ is a Borel subset of $\mathcal{N}$.
(A3) **(1/2 page)** If $B \subseteq \mathcal{N}$ is a Borel set then there is a Borel code c such that $B = B(c)$.

Now make the things more uniform. Given a Borel code $c = (T, u)$, we can code c via some $\tilde{c} \in \mathcal{N}$ as follows.

- $(\tilde{c})_0$ codes a tree ordering $<_{\tilde{c}}$ on ω isomorphic to T.
- $(\tilde{c})_1$ codes the function u; this time as a function from the set of minimal nodes with respect to the ordering $<_{\tilde{c}}$ into SEQ such that $(\tilde{c})_1(k)$ is the code of the sequence $u(t)$ where t is the minimal node in T corresponding to k under the above isomorphism.

From now on, when we talk about Borel Codes we mean elements of $\mathcal{N}$ coding Borel codes in the previous sense.

B. Prove the following

- (B1) **(1/2 page)** (A3) can be proved using merely $\mathsf{AC}_\omega(\mathbb{R})$.
- (B2) **(1 page)** The set BC of all Borel Codes is a Π^1_1-subset of $\mathcal{N}$.

2. We proved in the lecture that there is a universal Σ^0_1-set $G \subseteq \omega \times \omega \times \mathcal{N}$ (lightface!) in the sense that if $A \subseteq \omega \times \mathcal{N}$ is Σ^0_1 then there is some $e \in \omega$ such that

$$A(n, x) \iff G(e, n, x)$$

whenever $n \in \omega$ and $x \in \mathcal{N}$.

- (a) **(1/3 page)** Use this fact to construct a universal Σ^1_1 set $H \subseteq \omega \times \omega$ for all Σ^1_1-subsets of ω.
- (b) **(1/3 page)** Use (a) to construct a Π^1_1-subset of ω which is not Σ^1_1.
- (c) **(1/3 page)** We proved in the lecture that there is a Π^1_1-subset of $\mathcal{N}$ which is not $\boldsymbol{\Sigma}^1_1$. Is there a Π^1_1-subset of ω which is not $\boldsymbol{\Sigma}^1_1$?

3. **(2/3 page)** If $a \in \mathsf{WO}$ we let $\mathsf{otp}(a)$ be the order-type of the well-ordering coded by a. Define

$$\delta^1_1 = \sup\{\mathsf{otp}(a) \mid a \in \mathsf{WO} \ \wedge \ a \text{ is a } \Delta^1_1\text{-subset of } \omega\}.$$

Prove that $\delta^1_1 = \omega_1^{CK}$.

Hint. For the non-trivial part, use Exercise 2.

4. **(2/3 page)** Let $A \subseteq \mathsf{WO}$ be a Σ^1_1-set. Prove that there is a countable ordinal α such that $\mathsf{otp}(a) < \alpha$ for all $a \in A$.

Hint. Use Exercise 2.