MATH 281A FALL 2016 HOMEWORK 3

Due date: Friday December 2

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do <u>not</u> reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) A function $g : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is called monotonic iff the following two conditions are satisfied.

(a) $s \subseteq t \implies g(s) \subseteq g(t)$, and

(b) for every $x \in \mathcal{N}$ the lengths $\mathsf{lh}(g(x \restriction n))$ converge to ω as n converges to ω.

Given a monotonic function $g : {}^{<\omega}\omega \to {}^{<\omega}\omega$ we can define a function $g^* : \mathcal{N} \to \mathcal{N}$ by

$$g^*(x) = \bigcup_n g(x \restriction n).$$

Prove the following.

(i) If $g : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is monotonic then $g^* : \mathcal{N} \to \mathcal{N}$ is continuous.

(ii) If $g : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is monotonic then $g^* : \mathcal{N} \to \mathcal{N}$ is Σ^0_1-recursive in a_g where $a_g \in \mathcal{N}$ is a natural coding of g.

(iii) If $f : \mathcal{N} \to \mathcal{N}$ is continuous then there is a monotonic $g : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $f = g^*$.

Remark. Recall that if Γ is a pointclass then a function $f : \mathcal{N} \to \mathcal{N}$ is Γ-recursive iff the relation $P^f \subseteq \mathcal{N} \times \omega$ defined by

$$P^f(x, s) \iff \mathsf{SEQ}(s) \wedge s \subseteq f(x)$$

is in Γ.

Thus, every continuous function $f : \mathcal{N} \to \mathcal{N}$ is Σ^0_1-recursive in some $a \in \mathcal{N}$. Consider the following analogy: The function g in (iii) relates to f in an analogous way as the restriction of f to rationals, should f be considered as a function from reals to reals.

2. (1/3 page) Recall that if $A \subseteq$ is a Π^1_1-set then there is a recursive tree T on $\omega \times \omega$ such that for every $x \in \mathcal{N}$,

$$A(x) \iff <^T_x \in \mathsf{WO}.$$

We proved in the lecture that A is Δ^1_1 iff the function $x \mapsto \mathsf{otp}(<^T_x)$ is bounded below ω_1^{CK}.

Prove that there is a Π^1_1-set (lightface!) $A \subseteq \mathcal{N}$ such that the values $\mathsf{otp}(<^T_x)$ are unbounded in ω_1 (this is the true ω_1!) as x ranges over A.

3. A pointclass Γ has the reduction property iff for any sets $A, B \in \Gamma$ such that $A, B \subseteq \mathcal{N}$ there are sets $A^*, B^* \in \Gamma$ such that

(r1) $A^* \subseteq A$ and $B^* \subseteq B$.
(r2) $A^* \cap B^* = \varnothing$ and $A^* \cup B^* = A \cup B$.

A pointclass Γ has the separation property iff for any sets $A, B \in \Gamma$ such that $A, B \subseteq \mathcal{N}$ there are sets $A', B' \in \Gamma$ such that

(s1) $A \subseteq A'$ and $B \subseteq B'$.
(s2) $A' \cap B' = A \cap B$ and $A' \cup B' = \mathcal{N}$.

Prove the following

(a) **(5 lines)** If $A, B \subseteq \mathcal{N}$ are in Γ and $A \cap B = \varnothing$ then the separation property for A, B is equivalent to the statement that there is a set $D \in \Delta = \Gamma \cap \check{\Gamma}$ such that $A \subseteq D$ and $B \cap D = \varnothing$. (This is the form of separation property we considered in Math 280.)

(b) **(8 lines)** Γ has the reduction property iff $\check{\Gamma}$ has the separation property.

(c) **(1/3 page)** If Γ is normed then Γ has the reduction property. Conclude that Π^1_1 has the reduction property and Σ^1_1 has the separation property. Similarly for relativized and boldface pointclasses. This gives a lightface version of the separation theorem for analytic sets we proved in Math 280.

Recall that Γ is ω-parametrized iff there is a universal set $G \in \Gamma$ such that $G \subseteq \omega \times \mathcal{N}$ and for every $P \in \Gamma$ there is $e \in \omega$ such that $P(x) \iff G(e, x)$ for all $x \in \mathcal{N}$.

(d) **(2/3 page)** Prove that if Γ is ω-parametrized then Γ cannot have both the reduction property and the separation property. In fact, Γ cannot have both the reduction property and the weak form of separation property for disjoint sets discussed in (a) above.

It follows that if n is such that Σ^1_n/Π^1_n is normed then Π^1_n/Σ^1_n is not normed. In particular, Σ^1_1 and Π^1_2 are not normed (and therefore not scaled) pointclasses.

Hint. For (d), fix a universal Γ-set $G \subseteq \omega \times \mathcal{N}$. Let $A, B \subseteq \mathcal{N}$ be defined by

$$A(x) \equiv G(x(0), x) \quad \text{and} \quad B(x) \equiv G(x(1), x).$$

Let A^*, B^* reduce A, B and A', B' separate A^*, B^*. Let $i, j \in \omega$ be such that

$$A'(x) \iff G(i, x) \quad \text{and} \quad B'(x) \iff G(j, x),$$

and let $a \in \mathcal{N}$ be such that $a(0) = j$ and $a(1) = i$. Since $A' \cup B' = \mathcal{N}$ and $A' \cap B' = \varnothing$, the sequence a is an element of exactly one of A' or B'. Use this to get a contradiction.