MATH 281B WINTER 2017 HOMEWORK 1

Due date: Monday February 6

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

In this collection if exercises Γ is a pointclass with the following properties.

(a) Γ contains all Δ^0_0-recursive sets, is closed under $\wedge$ and $\vee$ and number quantifiers, and has the substitution property.
(b) Γ is ω-parametrized.
(c) Γ is normed.

Recall that Γ has the substitution property iff Γ is closed under substitution of functions which are Γ-recursive on their domains. Also, a function $f : \mathcal{N} \to \mathcal{N}$ is Γ-recursive on its domain iff there is a set $P \subseteq \mathcal{N} \times \mathsf{Seq}$ such that $P \in \Gamma$ and $P^f \subseteq P$, where recall that

$$P^f(x, s) \iff s \subseteq f(x).$$

It is easy to see that f is Γ-recursive iff there is a set $Q \subseteq \mathcal{N} \times \omega$ such that $Q \in \Gamma$ and for all $x \in \mathrm{dom}(f)$ and all $m, n \in \omega$,

$$Q(x, m, n) \iff f(x)(m) = n. \tag{1}$$

The formulation in (1) will be useful in the exercises below.

Given an $a \in \mathcal{N}$, a point $x \in \mathcal{N}$ is Γ-recursive in a, briefly x is a $\Gamma(a)$-point or $x \in \Gamma(a)$, iff $\{s \in \mathsf{Seq} \mid s \subseteq x\} \in \Gamma(a)$. Alternatively, following (1), x is a $\Gamma(a)$-point iff $\{(m, n) \in \omega \times \omega \mid x(m) = n\} \in \Gamma(a)$.

Finally recall that $\Delta = \Gamma \cap \check{\Gamma}$.

The following set of exercises gives a comprehensive answer to the uniformization problem for subsets of $\mathcal{N}^k$ under enough determinacy.

0. (1/2 page) Assume $\mathsf{Det}((\Delta)_{2n})$. Let $a \in \mathcal{N}$. Prove that every $\Pi^1_{2n+1}(a)$-subset of $\mathcal{N} \times \mathcal{N}$ can be uniformized by a $\Pi^1_{2n+1}(a)$-subset of $\mathcal{N} \times \mathcal{N}$ and every $\Sigma^1_{2n+2}(a)$-subset of $\mathcal{N} \times \mathcal{N}$ can be uniformized by a $\Sigma^1_{2n+2}(a)$-subset of $\mathcal{N} \times \mathcal{N}$.

Hint. Look at the periodicity theorems.

1. (2/3 page) Prove that there is a partial function $d : \omega \times \mathcal{N}$ which is Γ-recursive on its domain such that for every $x, y \in \mathcal{N}$,

$$y \text{ is a } \Delta(x)\text{-point} \iff (\exists i \in \omega) d(i, x) = y. \tag{2}$$

Note that the right hand side in (2) implicitly requires that $(i, x) \in \mathrm{dom}(d)$.

Hint. Let $G \subseteq \omega \times \mathcal{N} \times \omega \times \omega$ be a universal Γ-set. Look at the Γ-uniformization G^* of G such that for all $(i, x, m) \in \omega \times \mathcal{N} \times \omega$,

$$(\exists n)G(i, x, m, n) \implies (\exists n)G^*(i, x, m, n).$$

2. (3 lines) Let f be a partial function that is $\Gamma(a)$-recursive on its domain. Prove that $\mathrm{dom}(f) \in \Gamma(a)$.

3. (1/2 page) (Restricted quantification) Let $A \subseteq \mathcal{N} \times \mathcal{N} \times \mathcal{N}$ be such that $A \in \Gamma$. Consider the set $B \subseteq \mathcal{N} \times \mathcal{N}$ defined by

$$B(x, y) \iff (\exists z \in \Delta(y))A(x, y, z).$$

Prove that $B \in \Gamma$.

Hint. Use the previous exercises.

In the following two exercises note that if $n = 0$ then no assumption on determinacy is needed.

4. (1/2 page) Assume $\mathsf{Det}(\boldsymbol{\Delta}^1_{2n})$. Given an $x \in \mathcal{N}$, prove that there is a nonempty $\Pi^1_{2n}(a)$-subset of $\mathcal{N}$ which has no $\Delta^1_{2n+1}(a)$ element.

Hint. Look at set $A \subseteq \mathcal{N}$ which is $\Sigma^1_{2n+1}(a)$ but not $\Pi^1_{2n+1}(a)$. Use the previous exercises.

5. (1/3 page) Assume $\mathsf{Det}(\boldsymbol{\Delta}^1_{2n})$. Given an $a \in \mathcal{N}$, prove that there is a $\Pi^1_{2n}(a)$-subset of $\mathcal{N} \times \mathcal{N}$ which cannot be uniformized by any $\boldsymbol{\Sigma}^1_{2n+1}$-set.