MATH 281B WINTER 2017 HOMEWORK 2

Due date: Wednesday February 22

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

PD stands for the Axiom of Projective Determinacy which states that every projective set is determined.

1. (1/2 page) Assume $\mathsf{Det}(\boldsymbol{\Delta}^1_{2n})$. Prove that each of the pointclases Π^1_{2n+1}, Σ^1_{2n+2}, $\boldsymbol{\Pi}^1_{2n+1}$ and $\boldsymbol{\Sigma}^1_{2n+2}$ has the uniformization property.

2. (1/3 page) Assume $\mathsf{Det}(\boldsymbol{\Delta}^1_{2n})$. Prove that every nonempty $\Sigma^1_{2n+2}(x)$-set contains a $\Delta^1_{2n+2}(x)$-point. Here we say that, given a pointclass Γ, a point $y \in \mathcal{N}$ is a Γ-point iff $\{s \in \mathsf{SEQ} \mid y \in B_s\} \in \Gamma$.

3. (2/3 page) In the lecture we used Coding Lemma to show that, under AD, there is a nonprincipal countably complete ultrafilter on ω_1. However, using the Coding Lemma is an overkill – much less is needed.

Recall that $\mathcal{D}$ is the set of all Turing degrees. Define a map $h : \mathcal{D} \to \omega_1$ by

$$h(\boldsymbol{d}) = \sup(\{\mathsf{otp}((a)^2_0) \mid a \in \boldsymbol{d} \;\&\; (a)^2_0 \in \mathsf{WO}\}).$$

Show that the projection of the Martin measure under h is a non-principal countably complete ultrafilter on ω_1.

4. (1/2 page) We saw that AD implies the Perfect Set Property PSP and PSP implies that there is no injection from ω_1 into $\mathcal{N}$. There are however different ways of showing the non-existence of such an injection.

Prove that if there is a non-principal countably complete ultrafilter on ω_1 then there is no injection from ω_1 into ${}^{\omega}\{0,1\}$.

Hint. Given an ω_1-sequence of elements of ${}^{\omega}\{0,1\}$, use the ultrafilter to "freeze" all coordinates.

5. (1page) AD postulates that all games where the players play elements of ω are determined. It is however not possible to extend this axiom to games where the players play elements of ω_1, at least not in the naive way.

Given a set $A \subseteq {}^{\omega}\omega_1$, consider the following game:

$$\begin{array}{c|ccccccc} I & a_0 & & a_2 & & a_4 & \cdots & \\ \hline II & & a_1 & & a_3 & & a_5 & \cdots \end{array}$$

where $\alpha_k \in \omega_1$ for all $k \in \omega$. I wins the run $(\alpha_k \mid k \in \omega)$ iff $(\alpha_k \mid k \in \omega) \in A$.

Prove in ZF that there is an undetermined set $A \subseteq {}^{\omega}\omega_1$.

Hint. Try to design a game that produces an injection from ω_1 into $\mathcal{N}$.