MATH 281B WINTER 2017 HOMEWORK 3

Due date: Monday, March 6

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

All structures considered in this assignment are structures for LST or some extension of LST. When we say "structure M" we mean structure $(M, \in)$, so $\in$ is a relation present in all structures we are considering. When we say "set M", in many contexts we mean the "structure M".

1. Work in ZF. Let $(M_\alpha \mid \alpha \in \mathbf{On})$ be a cumulative hierarchy and $M = \bigcup_{\alpha \in \mathbf{On}} M_\alpha$. By Tarski's theorem on undefinability of truth, there is no class sequence of closed proper classes $(C_\varphi \mid \varphi \in \mathsf{FML})$ such that for every $\varphi(v_0, \dots, v_{k-1}) \in \mathsf{FML}$ and every $\alpha \in C_\varphi$,

$$M_\alpha \models \varphi(\vec{a}) \iff M \models \varphi(\vec{a}) \tag{1}$$

whenever $\varphi \in \mathsf{FML}$, $\alpha \in \mathbf{On}$ and $\vec{a} = (a_0, \dots, a_{k-1})$ is such that $a_0, \dots, a_{k-1} \in M_\alpha$. Formulate this rigorously!

However:

(a) Prove that there is a class sequence $(C_\varphi \mid \varphi \in \mathsf{FML} \ \wedge \ \varphi \text{ is atomic})$ such that (1) holds for every atomic $\varphi(v_0, \dots, v_{k-1}) \in \mathsf{FML}$ and every α and $\vec{a}$ as above.

(b) Prove that for every $n \in \omega$, the result from (a) can be extended to all $\varphi \in \mathsf{FML}$ which are Σ_n.

(c) Prove that there is an ordinal $\alpha \in \mathbf{On}$ such that $V_\alpha \prec_{1000} \mathbf{V}$.

2. Work in ZF. Recall that if α is an ordinal of uncountable cofinality and $(C_n \mid n \in \omega)$ is a sequence of closed unbounded subsets of α then $\bigcap_{n \in \omega} C_n$ is a closed unbounded subset of α. The point here is that one does not need any instance of the Axiom of Choice to prove this.

Prove that if there exists an ordinal α such that $V_\alpha \models \mathsf{ZF}$ then the smallest such α has countable cofinality.

If $\sigma : M \to M'$ where M, M' are transitive and $n \in \omega + 1$, we say that σ is Σ_n-preserving iff for any Σ_n-formula $\varphi(\vec{v})$ and any tuple $\vec{a} = (a_1, \dots, a_\ell)$ of elements in M of the same length as $\vec{v}$,

$$M \models \varphi(\vec{a}) \iff M' \models \varphi(\sigma(\vec{a})) \tag{2}$$

where $\sigma(\vec{a}) = (\sigma(a_1), \dots, \sigma(a_\ell))$. By this terminology, σ is Σ_ω-preserving iff σ is fully elementary.

3. Work in ZF. If M, M' are transitive sets and $\sigma : M \to M'$ is a Σ_0-preserving map then σ is called cofinal iff $M' = \bigcup \operatorname{rng}(\sigma)$. (Think what this says!)

Let $A \subseteq M$ be such that (M, A) is an amenable structure; this means that $x \cap A \in M$ whenever $x \in M$. Assume further that M has the following closure property: If $x_1, \dots, x_\ell \in M$ are finitely many elements then there is a transitive $y \in M$ sucht that $x_1, \dots, x_\ell \in y$.

Prove that there is a unique set $A' \subseteq M'$ such that $\sigma : (M, A) \to (M', A')$ is Σ_0-preserving, namely $A' = \bigcup\{\sigma(A \cap x) \mid x \in M\}$, and the structure (M', A') is amenable. (Here the preservation is understood for the extended language which has a predicate symbol which is interpreted as A in (M, A) and A' in (M', A').)

A Q-formula is a formula of the form $(\forall u)(\exists v \supseteq u)\varphi$ where φ is a Σ_1-formula and u has no free occurrence in φ. A map $\sigma : M \to M'$, where M, M' are transitive, is Q-preserving iff for every Q-formula $\varphi(\vec{w})$ and every tuple $\vec{a}$ in M of the same length as $\vec{w}$,

$$M \models \varphi(\vec{a}) \iff M' \models \varphi(\sigma(\vec{a}))$$

We will often use the following fragment of ZF which we denote by BS and call the basic set theory:

$$\begin{aligned} \mathsf{BS} = {} & \text{Existence} + \text{Extensionality} + \text{Foundation} + \text{Pairing} + \text{Union} + \\ & \text{Infinity} + \Sigma_0\text{-separation} + \text{Existence of Cartesian products} \end{aligned}$$

Notice that if M is nonempty transitive then M automatically satisfies the Existence, Extensionality and the Foundation axiom. Notice also that if we replace the axiom postulating the existence of Cartesian products by Σ_0-collection then we obtain a theory stronger than BS, as the existence of Cartesian products will then follow

4. Work in ZF. Assume $\sigma : M \to M'$ is Σ_0-preserving and cofinal. Prove:

(a) σ is Σ_1-preserving.
(b) σ is Q-preserving.
(c) If $M \models \mathsf{BS}$ and satisfies Δ_0-collection then σ is Σ_2-preserving.
(d) More generally, if $M \models \mathsf{BS}$, satisfies Π_n-collection and Π_n-separation then σ is Σ_{n+2}-preserving. Thus, if $M \models \mathsf{ZF}$ then σ is fully elementary and $M' \models \mathsf{ZF}$.