MATH 281B WINTER 2017 HOMEWORK 4

Target date: Friday, March 17

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. Work in $\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$. Recall the notion of Borel code from HW2 sheet, Fall 2016. View a Borel code c as an element of $\mathcal{N}$. Also, recall that $B(c)$ is the Borel set coded by c; this can be also called "the evaluation of the Borel code c".

Assume M is a transitive model of ZF^-. If $c \in M$ then $B^M(c)$ is the evaluation of c as calculated in M. The relativization R^M then has the obvious meaning.

(a) Prove that the relation $R(x,c) \equiv x \in B(c)$ is Δ^1_1. Thus, $B^M(c) = B(c) \cap M$ whenever $c \in M$.
(b) Prove that $B(c) = \varnothing$ iff $B(c) \cap M = \varnothing$ whenever $c \in M$.
(c) Prove that if $c, c' \in M$ then $B(c) = B(c')$ iff $B^M(c) = B^M(c')$.

2. Work in $\mathsf{ZF} + \mathsf{DC}$. Let M be a proper class model of ZF^-. Prove that if there is a transitive model $P \in \mathbf{V}$ such that $P \models \mathsf{ZF}$ then there is a transitive model $Q \in M$ such that $Q \models \mathsf{ZF}$.

3. Work in $\mathsf{ZF} + AD$. Let $R \subseteq \mathcal{N} \times \mathcal{N}$ be a binary relation defined by

$$R(x,y) \iff y \notin \mathsf{OD}_{\{x\}}$$

Thus, R is lightface definable in $\mathbf{V}$. Prove that:

(a) For every $x \in \mathcal{N}$ the section $R_x = \{y \in \mathcal{N} \mid R(x,y)\}$ is nonempty.
(b) R does not have a uniformization that is ordinal definable from an element of $\mathcal{N}$.

4. Work in ZF. Assume there is no injection $f : \omega_1 \to \mathcal{N}$ that is ordinal definable from an element of $\mathcal{N}$. Prove that $\omega_1^{\mathbf{V}}$ is a limit cardinal in HOD.

Thus, if ω_1^V is regular then it is weakly inaccessible in HOD. That is, the nonexistence of an ω_1-sequence of distinct reals which is ordinal definable from a real implies the consistency of large cardinals.