MATH 281C SPRING 2017 HOMEWORK 1

Target date: Wednesday, May 24

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Assume $A \subseteq \mathbf{On}$ is such that $A \subseteq \omega_1^{\mathbf{L}[A]}$. Prove that $\mathbf{L}[A] \models \mathsf{CH}$.

2. (a) **(10 lines)** Construct a set $A \subseteq \omega_2^{\mathbf{V}}$ such that $\omega_1^{\mathbf{L}[A]} = \omega_1^{\mathbf{V}}$ and $\omega_2^{\mathbf{L}[A]} = \omega_2^{\mathbf{V}}$. (Notice that if M is a proper class inner model and $\omega_2^M = \omega_2^{\mathbf{V}}$ then $\omega_1^M = \omega_1^{\mathbf{V}}$, although this observation is not needed for the task.)

(b) **(10 lines)** Assume CH fails in $\mathbf{V}$. Construct a set A satisfying the requirements in (a) such that additionaly we have $\mathbf{L}[A] \models \neg\mathsf{CH}$.

3. (1/2 page) This exercise is not limited to $\mathbf{L}$; the result hold in any model of ZFC. Fix a large regular θ and a well-ordering $<_\theta$ of H_θ. Work in the language $\mathcal{L} = \mathsf{LST} \cup \{\dot{<}\}$ and consider the $\mathcal{L}$-structure $(H_\theta, \in, <_\theta)$ where $\dot{<}$ is interpreted as $<_\theta$. In the following we will write "H_θ" for both this structure and for the set H_θ alone; what we mean will be clear from the context. It should be stressed that unless specified otherwise, all structures used in this exercise are $\mathcal{L}$-structures. In particular, all elementary substructures of H_θ are calculated with respect to $\mathcal{L}$, i.e. they are elementary also with respect to the well-ordering $<_\theta$.

Let $\nu < \omega_2$. Assume X, Y are countable elementary substructures of H_θ such that

- $\nu \in X, Y$, and
- $\omega_1 \cap X = \omega_1 \cap Y$.

(a) Prove that $\nu \cap X = \nu \cap Y$.

In particular, then $\mathsf{otp}(\nu \cap X) = \mathsf{otp}(\nu \cap Y)$.

For each $\nu < \omega_2$ and each elementary chain $\vec{X} = (X_\xi \mid \xi < \omega_1)$ such that each X_ξ is a countable elementary substructure of H_θ, let $\alpha_\xi^{\vec{X}} = \omega_1 \cap X_\xi$ whenever $\xi < \omega_1$ and $C_{\vec{X}} = \{\alpha_\xi^{\vec{X}} \mid \xi < \omega_1\}$. Notice that $C_{\vec{X}}$ is a closed unbounded subset of ω_1. Define a function $h_{\vec{X}} : C_{\vec{X}} \to \omega_1$ by

$$h_{\vec{X}}(\alpha_\xi^{\vec{X}}) = \mathsf{otp}(\nu \cap X_\xi)$$

Prove that if $\vec{X}, \vec{Y}$ are two elementary chains as above then there is a closed unbounded set $C \subseteq C_{\vec{X}} \cap C_{\vec{Y}}$ such that for every $\alpha \in C$:

(b) There is some $\xi < \omega_1$ such that $\alpha_\xi^{\vec{X}} = \alpha = \alpha_\xi^{\vec{Y}}$, and

(c) $h_{\vec{X}}(\alpha) = h_{\vec{Y}}(\alpha)$.

Thus, the function $h_{\vec{X}}$ "almost" does not depend on $\vec{X}$: For $\vec{X}, \vec{Y}$ functions $h^{\vec{X}}, h^{\vec{Y}}$ disagree only on a non-stationary set, i.e. a set contained in the complement of a closed unbounded set.

4. Work in $\mathbf{L}$.

(a) **(5 lines)** Let X be a countable elementary substructure of J_{ω_2}, let J_β be its transitive collapse, and $\alpha = X \cap \omega_1$. Let J_γ be the smallest elementary substructure of J_{ω_1} such that $\alpha < \gamma$. Prove that $\beta < \gamma$.

(b) **(5 lines)** Define a function $f : \omega_1 \to \omega_1$ by

$$f(\alpha) = \text{the least } \beta > \alpha \text{ such that } \alpha \text{ is not a cardinal in } J_{\beta+1}.$$

Notice that f is a well-defined total function on ω_1. Prove that for every $\nu < \omega_2$ and every elementary chain $\vec{X}$ as in Problem 3 there is a closed unbounded set $C \subseteq \omega_1$ such that

$$h_{\vec{X}}(\alpha) < f(\alpha) \quad \text{whenever } \alpha \in C.$$

What would be the well-ordering $<_\theta$ in this case?

(c) **(5 lines)** Prove that in fact there is a closed unbounded set $C \subseteq \omega_1$ such that

$$g_{\vec{X}}(\alpha) < f(\alpha) \quad \text{whenever } \alpha \in C$$

where $g : C_{\vec{X}} \to \omega_1$ is defined by

$$g(\alpha_\xi) = \mathsf{otp}(\omega_2 \cap X_\xi)$$

5. (1/2 page) In this exercise, h_α is the uniform Σ_1-Skolem function for J_α. Recall also that $<^*$ is the canonical ordering on $[\mathbf{On}]^{<\omega}$ where $s <^* t$ iff $s \subseteq t$ or s, t are incompatible and $s <_{\text{lex}} t$ when s, t are viewed as descending sequences of ordinals.

Assume $p \in [\omega\alpha]^{<\omega}$ is such that:

(a) There is a Σ_1-formula $\varphi(v, w)$ such that $a_{p,\varphi} \notin J_\alpha$ where

$$a_{p,\varphi} = \{i \in \omega \mid J_\alpha \models \varphi(i, p)\},$$

and

(b) p is the $<^*$-least element of $[\omega\alpha]^{<\omega}$ satisfying (a), that is, if $q \in [\omega\alpha]^{<\omega}$ is such that $q <^* p$ then $a_{q,\varphi} \in J_\alpha$ for every Σ_1-formula $\varphi(v, w)$.

Prove that $J_\alpha = \{h_\alpha(i, s \cup p) \mid i \in \omega \;\&\; s \in [\omega]^{<\omega}\}$, that is, J_α is the canonical Σ_1-Skolem hull of $[\omega\alpha]^{<\omega} \cup \{p\}$.

6. (5 lines) We proved in the lecture that $\mathsf{PSP}(\mathbf{\Pi}^1_1)$, the perfect set property for all boldface coanalytic sets, implies that $\omega_1^{\mathbf{V}}$ is strongly inaccessible in $\mathbf{L}$. Thus, the implication $\mathsf{Con}(\mathsf{ZFC}) \implies \mathsf{Con}(\mathsf{ZFC} + \mathsf{PSP}(\mathbf{\Pi}^1_1))$ is not provable, at least if we believe that the theory $\mathsf{Con}(\mathsf{ZFC} +$ "There is an inaccessible cardinal" is consistent.

On the other hand, the implication $\mathsf{Con}(\mathsf{ZFC}) \implies \mathsf{Con}(\mathsf{ZFC} + \mathsf{PSP}(\Pi^1_1)$ is provable (and similarly for relativized pointclasses $\Pi^1_1(a)$ for any fixed $a \in \mathcal{N}$). This follows from this exercise and the fact that we can arrange $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{V}}$ relative to $\mathsf{Con}(\mathsf{ZFC})$ (we will see this latter fact later). Use appropriate theorems from the lecture to prove the following.

Assume $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{V}}$. Then $\mathsf{PSP}(\Pi^1_1)$, and in fact $\mathsf{PSP}(\Sigma^1_2)$ holds.