MATH 281C SPRING 2017 HOMEWORK 2

Target date: Monday, June 5

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) Recall the Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I} = [\omega]^{<\omega}$ from Math 281A, Homework assignment 1, Problem 2. In part 2(e) you proved that if $(b_n \mid n \in \omega)$ is a strictly decreasing chain in $\mathcal{P}(\omega)/\mathcal{I}$ then $(b_n \mid n \in \omega)$ has a lower bound in $\mathcal{P}(\omega)/\mathcal{I}$.

Prove that $\mathsf{MA}(\kappa)$ implies the following: If $(b_\xi \mid \xi < \kappa)$ is a strictly decreasing chain in $\mathcal{P}(\omega)/\mathcal{I}$ then $(b_\xi \mid \xi < \kappa)$ has a lower bound in $\mathcal{P}(\omega)/\mathcal{I}$.

2. (1/2 page + 1/2 page) Prove that $\mathsf{MA}(\kappa)$ implies the following "generalization" of the Baire Category Theorem.

(1) If X is a compact Hausdorff c.c.c. topological space and $(A_\xi \mid \xi < \kappa)$ is a family of open dense subsets of X then $\bigcap_{\xi<\kappa} A_\xi$ is a dense subset of X.

We say that a topological space is c.c.c. if every family of open pairwise disjoint sets is countable.

Note that if $\kappa = \omega$ then a stronger form of statement (1), where we drop the condition on X being c.c.c., can be proved in ZFC alone.

The converse to statement (1) holds, too. For the moment we address just a restricted verson where the posets in question are of the form $\mathbb{B} \setminus \{0_\mathbb{B}\}$ where $\mathbb{B}$ is a Boolean algebra. Assume (1) holds. Prove:

(2) If $\mathbb{P}$ is a c.c.c. poset of the form $\mathbb{B} \setminus \{0_\mathbb{B}\}$ where $\mathbb{B}$ is a Boolean algebra and $\mathcal{D} = (D_\xi \mid \xi < \kappa)$ is a family of dense subsets of $\mathbb{P}$ then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

3. (2/3 page) Prove that $\mathsf{MA}(\kappa)$ is equivalent with the following superficially weaker statement

$\mathsf{MA}^w(\kappa)$: If $\mathbb{P}$ is a c.c.c. poset of cardinality $\leq \kappa$ and $\mathcal{D} = (D_\xi \mid \xi < \kappa)$ is a family of dense subsets of $\mathbb{P}$ then there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

The next exercise is another step toward the converse of (1) in Problem 2.

4. (1 page) Let X be a topological space. Recall that if $A \subseteq X$ then:

- $\overline{A}$ is the closure of A, that is, the smallest closed set with respect to the inclusion containing A as a subset. Equivalently, $\overline{A}$ is the intersection of all closed sets in X which contain A as a subset.
- A° is the interior of A, that is, the largest open subset of A. Equivalently, A° is the union of all open subsets of A.

Notice that if A is open in X then $A \subseteq \overline{A}^\circ$, and if $A \subseteq B$ are arbitrary subsets of X then $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$.

A set $A \subseteq X$ is called **regular open** iff $A = \overline{A}^\circ$. Prove the following.

(a) If $A \subseteq X$ then $\overline{A}^\circ$ is regular open. An immediate consequence is: The interior of any closed set is regular open.

(b) If $A, B \subseteq X$ are regular open then $\overline{A \cup B}^\circ$ is the smallest regulr open set in X containing both A and B as subsets.

(c) If $A, B \subseteq X$ are regular open then $A \cap B$ is the largest regular open set in X which is a subset of both A and B.

(d) Prove that if X is a topological space then the structure

$$\mathbb{B} = (\mathsf{RO}(X), \wedge, \vee, ', \varnothing, X)$$

is a Boolean algebra such that the associated partial ordering $\leq_{\mathbb{B}}$ is the inclusion, that is, $A \leq_{\mathbb{B}} B$ iff $A \subseteq B$, where

- $\mathsf{RO}(X)$ is the family of all sets which are regular open in X,
- $A \wedge B = A \cap B$ and $A \vee B = \overline{A \cup B}^\circ$, and
- $A' = (X \setminus A)^\circ$.

Warning: The proof of distributivity takes some work.

Then prove that $\mathbb{B}$ is a complete Boolean algebra, where the general join and meet are defined as follows. Given $\mathcal{A} \subseteq \mathsf{RO}(X)$,

$$\bigvee \mathcal{A} = \left(\overline{\bigcup \mathcal{A}}\right)^\circ \quad \text{and} \quad \bigwedge \mathcal{A} = \left(\bigcap \mathcal{A}\right)^\circ \tag{3}$$

Remark. The Boolean algebra $\mathbb{B}$ is called the **regular open algebra of** X and denoted by $\mathbb{RO}(X)$.