MATH 281C SPRING 2017 HOMEWORK 3

Target date: Friday, June 16

Rules: Write as efficiently as possible: Include all relevant points and think carefully what to write and what not. Use common sense to determine what is the appropriate amount of details for a course at this level. Quote any result from the lecture you are referring to; do not reprove the result. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/3 + 1/3 page) Let $\mathbb{P}$ be a poset.

(a) Given $\mathbb{P}$-names $\dot{x}, \dot{y}$, write down a $\mathbb{P}$-name $\dot{z}$ such that $\dot{z}^G = \{\dot{x}^G, \dot{y}^G\}$ whenever G is a filter generic for $\mathbb{P}$ over $\mathbf{V}$.

(b) Given a $\mathbb{P}$-name $\dot{x}$, write down a $\mathbb{P}$-name $\dot{y}$ such that $\dot{y}^G = \bigcup \dot{x}^G$ whenever G is a filter generic for $\mathbb{P}$ over $\mathbf{V}$.

In either case avoid the use of the forcing theorem. If you construct the names the right way then the conclusions in (a) and (b) will hold for all filters G, not just the generic ones.

2. (1/2 page) Let $\mathbb{P}$ be a poset defined as follows.

- **Conditions** in $\mathbb{P}$ are finite strictly increasing sequences $p : n \to \omega_2$ where $n \in \omega$.
- **Ordering** of $\mathbb{P}$ is the extension, i.e. $p \leq q$ iff $p \supseteq q$.

Notice that $\mathbb{P}$ collapses ω_2; this is an easy observation. However, it is not a priori clear what is the cardinality of $\omega_2^{\mathbf{V}}$ in the sense of $\mathbf{V}[G]$ where G is a filter generic for $\mathbb{P}$ over $\mathbf{V}$. It turns out $\omega_2^{\mathbf{V}}$ is countable in $\mathbf{V}[G]$, as follows from the following:

Prove that in $\mathbf{V}[G]$ there is a surjection $f : \omega \to \omega_1^{\mathbf{V}}$.

3. (1/2 page) Prove that if G is generic for $\mathsf{Add}(\aleph_\omega, 1)$ over $\mathbf{V}$ then in $\mathbf{V}[G]$ there is a surjection $f : \omega \to \aleph_\omega^{\mathbf{V}}$.

Here recall that $\mathsf{Add}(\aleph_\omega, 1)$ consists of conditions $p : x \to \{0, 1\}$ were $x \in [\aleph_\omega]^{<\aleph_\omega}$ ordered by reverse inclusion.

4. (1 page) Consider the poset $\mathbb{P}$ defined as follows.

- **Conditions** are bounded closed subsets of ω_1. Thus, each $c \in \mathbb{P}$ has a largest element, which we denote by α_c.
- **Ordering** is by end-extensions, that is, $c \leq d$ iff $d = c \cap (\alpha_d + 1)$.

It is easy to see that $\leq$ is reflexive and transitive.

(a) Prove that $\mathbb{P}$ is ω_1-closed, so ω_1 is not collapsed in generic extensions via $\mathbb{P}$, and the models $\mathbf{V}$ and $\mathbf{V}[G]$ have the same subsets of ω.

(b) If G is a generic filter for $\mathbb{P}$ over $\mathbf{V}$, write $C_G = \bigcup G$. Prove that C_G is a closed unbounded subset of ω_1 in $\mathbf{V}[G]$.

(c) Prove that for every closed unbounded set C in ω_1 such that $C \in \mathbf{V}$ we have $C \not\subseteq C_G$, and if $\omega_1 \setminus C$ is unbounded in ω_1 then $C_G \not\subseteq C$.

(d) Prove that every $a \subseteq \omega$ can be encoded into C_G in way that gives rise to a surjection of ω_1 onto $\mathcal{P}(\omega)$. Thus, $\mathbf{V}[G] \models \mathsf{CH}$.

4. (1/3 page) Let $\mathbb{P}$ be a poset, $B \subseteq \mathbb{P}$ be an open set in $\mathbb{P}$, and $A \subseteq B$ be a maximal antichain contained in B. Prove that if G is a filter generic for $\mathbb{P}$ and $G \cap B \neq \varnothing$ then in fact $G \cap A \neq \varnothing$.

5. (1 page) Given a poset $\mathbb{P}$, define a binary relation $\leq_{\mathbb{P}}^*$ on $\mathbb{P}$ as follows.

$$p \leq_{\mathbb{P}}^* q \iff (\forall p' \in \mathbb{P})(p' \leq_{\mathbb{P}} p \implies p' \,||_{\mathbb{P}}\, q)$$

Notice that $p \leq_{\mathbb{P}} q \implies p \leq_{\mathbb{P}}^* q$, but the converse may fail.

We say that a poset $\mathbb{P}$ is **separative** iff $\leq_{\mathbb{P}}^* = \leq_{\mathbb{P}}$, which is equivalent to the statement that

$$p \not\leq_{\mathbb{P}} q \implies (\exists p' \in \mathbb{P})(p' \leq_{\mathbb{P}} p \wedge p' \perp_{\mathbb{P}} q) \tag{1}$$

In the following we drop the subscripts $_{\mathbb{P}}$.

Separativity is a generalization of the following property of sets: If $A \not\subseteq B$ then there is some $C \neq \varnothing$ such that $C \subseteq A$ and $C \cap B = \varnothing$; a typical example is $C = A \setminus B$. Many of the posets we come across in this lecture are separative, but there are many natural posets which are not separative.

(a) Let X be a set and $\mathcal{I}$ be an ideal over X (so $\mathcal{I} \subseteq \mathcal{P}(X)$). Define a poset $\mathbb{P}_{\mathcal{I}}$ as follows.
   - **Conditions** are $\mathcal{I}$-positive subsets of X.
   - **Ordering** is inclusion.

   That is, $\mathbb{P} = (\mathcal{P}(X) \setminus \mathcal{I}, \subseteq)$. Assume $\mathcal{I}$ is non-trivial, that is, $\mathcal{I} \neq \{\varnothing\}$. Prove that $\mathbb{P}_{\mathcal{I}}$ is not separative.

(b) Let $\mathbb{P}$ be an arbitrary poset. Prove that $\leq^*$ is reflexive and transitive, so that $(\mathbb{P}, \leq^*)$ is also a poset. Prove that $(\mathbb{P}, \leq^*)$ is separative. This poset is called the **separative quotient of** $\mathbb{P}$.

(c) Given an arbitrary poset $\mathbb{P}$, prove that for any two $p, q \in \mathbb{P}$,

$$p \,||\, q \iff p \,||^* q$$

and conclude that $\mathsf{id}_{\mathbb{P}} : (\mathbb{P}, \leq) \to (\mathbb{P}, \leq^*)$ is a dense embedding. (This embedding is obviously surjective.) Thus, for any poset we can find a forcing equivalent separative poset. However, when doing concrete forcing constructions using a non-separative poset, it is often more convenient to stay with the original non-separative version rather than replacing it with a separative quotient.

(d) Given an arbitrary poset $\mathbb{P}$ and two conditions $p, q \in \mathbb{P}$, prove that

$$p \leq^* q \iff p \Vdash \check{q} \in \dot{G}$$

where $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter, that is,

$$\dot{G} = \{(p, \check{p}) \mid p \in \mathbb{P}\}.$$

Thus, if $\mathbb{P}$ is separative then

$$p \leq q \iff p \Vdash \check{q} \in \dot{G}$$

The term "separative quotient" comes from the quotient construction: Instead of considering the poset $(\mathbb{P}, \leq^*)$, one can define a binary relation $\equiv^*$ on $\mathbb{P}$ by letting $p \equiv^* q$ iff $p \leq^* q \ \wedge \ q \leq^* p$. The relation $\equiv^*$ is an equivalence relation on $\mathbb{P}$ which is a congruence with respect to $\leq^*$. One then can consider the quotient poset $(\mathbb{P}/\equiv^*, \leq^* / \equiv^*)$, which is easily seen to be separative, and prove that the quotient map $k : (\mathbb{P}, \leq) \to (\mathbb{P}/\equiv^*, \leq^* / \equiv^*)$ is a dense embedding. However, the construction in (b) and (c) avoids the use of the power set axiom, which one needs to define the actual quotient, and is in my view simpler and more elegant.

6. (1/2 page) In Problem set 2 you constructed the regular open algebra of a topological space and proved that it is a complete Boolean algebra. Recall that a poset $\mathbb{P}$ can be viewed as a topological space with domain $\mathbb{P}$ where the basic open sets are of the form $p{\downarrow}$ for $p \in \mathbb{P}$. Then $\mathbb{RO}(\mathbb{P})$ denotes the regular open algebra of this topological space. Prove that the map $e : (\mathbb{P}, \leq) \to (\mathbb{RO}(\mathbb{P}) \setminus \{\varnothing\}, \subseteq)$ defined by

$$e(p) = \overline{p{\downarrow}}^{\circ}$$

is a dense embedding of posets. Thus, for any poset $\mathbb{P}$ there is a forcing equivalent complete Boolean algebra.