Math 281C Spring 2017

Completion of the Proof of the Forcing Theorem

Recall the statements $(*)$ and $(**)$ from the lecture.

$(*)$ The following are equivalent.

(i) $p \Vdash_{\mathbb{P}} \varphi(x_1, \ldots, x_\ell)$.

(ii) $M[G] \models \varphi(x_1^G, \ldots, x_\ell^G)$ whenever G is a filter generic for $\mathbb{P}$ over M such that $p \in G$.

and

$(**)_G$ If $M[G] \models \varphi(x_1^G, \ldots, x_\ell^G)$ then there is some $p \in G$ such that $p \Vdash_{\mathbb{P}} \varphi(x_1, \ldots, x_\ell)$.

We want to prove $(**)$ for the formula "$v_1 = v_2$" by recursion on $\in$. We thus assume that we are given sets x and y, and both $(*)$ and $(**)$ hold for the statement "$x' = y'$" whenever $\mathsf{rank}(x') \leq \mathsf{rank}(x)$ and $\mathsf{rank}(y') \leq \mathsf{rank}(y)$.

Claim. $(**)_G$ holds for the statement "$x = y$", that is, given a filter G generic for $\mathbb{P}$ over M,

$$x^G = y^G \implies (\exists p \in G)(p \Vdash x = y),$$

Proof. Let $\Phi(p, x, y)$ be the formula

$$(\forall (q, z) \in x)(\forall p' \leq p)[p' \leq q \implies (\exists \bar{p} \leq p')(\exists (q', z') \in y)(\bar{p} \leq q' \ \wedge \ \bar{p} \Vdash z = z')]$$

Standard arguments similar to those done in lecture show that the following are equivalent.

- $M \models \Phi(p, x, y)$
- If H is a filter generic for $\mathbb{P}$ over M then $x^H \subseteq y^H$.

So $M \models \Phi(p, x, y) \wedge \Phi(p, y, x)$ iff $p \Vdash_{\mathbb{P}}^M x = y$. So one can also define the forcing relation "$p \Vdash x = y$" this way, which is a bit more polished and easier to negate.

Define a set $D \subseteq \mathbb{P}$ as follows.

$$D = \{p \in \mathbb{P} \mid \text{(a)} \vee \text{(b)} \vee \text{(c)}\}$$

where (a),(b) and (c) are as follows.

(a) $\Phi(p, x, y) \wedge \Phi(p, y, x)$

(b) $\neg\Phi(\tilde{p}, x, y)$ holds for densely many $\tilde{p} \leq p$

(c) $\neg\Phi(\tilde{p}, y, x)$ holds for densely many $\tilde{p} \leq p$.

The set D is dense in $\mathbb{P}$. To see this, consider an arbitrary $p_0 \in \mathbb{P}$. If there is some $p \leq p_0$ such that $\Phi(p, x, y) \wedge \Phi(p, y, x)$ then we are done. From now on assume that $\neg\Phi(p, x, y) \vee \neg\Phi(p, y, x)$ holds for every $p \leq p_0$. If $\neg\Phi(p, x, y)$ holds for densely many $p \leq p_0$ then $p_0 \in D$ and we are done. In the remaining case there is some $p \leq p_0$ such that for all $\tilde{p} \leq p$ we have $\Phi(\tilde{p}, x, y)$. But then $\neg\Phi(\tilde{p}, y, x)$ must hold for every $\tilde{p} \leq p$, so $p \in D$. This completes the proof of density of D.

We are now ready to prove $(**)_G$. Since G is generic, there is some $p \in G \cap D$. We show that options (b) and (c) are not possible, which will complete the proof. By symmetricity, we may consider option (b) only. So assume p is such that (b) holds, that is, $\neg\Phi(\tilde{p}, x, y)$ holds for densely many $\tilde{p} \leq p$. By calculating the negation and looking more closely at the meaning of the statement, we conclude that for densely many $p' \leq p$ the following holds.

$$(\exists (q,z) \in x)[p' \leq q \ \wedge \ (\forall (q',z') \in y)(\forall \bar{p} \leq p')(\bar{p} \leq q' \implies \bar{p} \not\Vdash z = z')] \tag{1}$$

As G is generic and $p \in G$, there is some $p' \leq p$ as in (1) in G. Fix such a p' and pick some $(q,z) \in x$ witnessing this. Then $q \in G$ as $p' \leq q$, and $z^G \in x^G$. Now we are assuming $x^G = y^G$, so $z^G \in y^G$, which means that there is some $(q',z') \in y$ such that $q' \in G$ and $z^G = (z')^G$. But then, using $(*)$ for the statement "$z = z'$", we can find some $\bar{p} \in G$ such that $\bar{p} \Vdash z = z'$, and since $p', q' \in G$, we can arrange that $\bar{p} \leq p', q'$. But this contradicts the conclusion in (1) which must hold for p'. $\square$