(1) Prove the following version of the comparison lemma: Let M, N be coiterable premice. Let θ be regular cardinal such that $\theta > |M|, |N|$. Then the coiteration of M, N has length $< \theta$.

Hint: Assume $\mathcal{Y}^M, \mathcal{Y}^N$ is a pair of iterations coming from the coiteration of M, N and they are of length $\geq \theta$. Let $\langle M_i, \pi^M_{ij} \mid i < j \leq \theta \rangle$, $\langle N_i, \pi^N_{ij} \mid i \leq j \leq \theta \rangle$ be the corresponding premice and iteration maps where M_θ, N_θ are the direct limits at the end (they are well-founded.) Let Ω be large and $X \prec H_\Omega$ be such that

(i) $\langle M_i, \pi^M_{ij} \rangle_{ij}, \langle N_i, \pi^N_{ij} \rangle_{ij} \in X$

(ii) $X \cap \theta \in \theta$; let $\bar{\theta} = X \cap \theta$

Let H be the transitive collapse of X and $\sigma : H \to H_\Omega$ be the inverse of the collapsing isomorphism. Prove:

(a) $\sigma(\langle M_i, \pi^M_{ij} \mid i < j \leq \bar{\theta} \rangle) = \langle M_i, \pi^M_{ij} \mid i < j < \theta \rangle$ and similarly for N

(b) $\sigma \restriction M_{\bar{\theta}} = \pi^M_{\bar{\theta}\theta}$ and $\sigma \restriction N_{\bar{\theta}} = \pi^N_{\bar{\theta}\theta}$

(c) the coiteration uses compatible measures at step $\bar{\theta}$

(2) Let M be an acceptable structure, say $M = \langle J^B_\alpha, D \rangle$ and let U be an ultrafilter over M with critical point κ that is close to M. Let $n \in \omega$ be such that $\rho^{n+1}_M \leq \kappa < \rho^n_M$. Let

$$\pi : M \xrightarrow[U]{*} M'$$

be the fine ultrapower. Prove:

(a) $\mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}^{(n)}_1(M) = \mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}^{(n)}_1(M')$

(b) $\rho^{n+1}_{M'} = \rho^{n+1}_M$

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(c) If $p \in P_M^{n+1}$ then $\pi(p) \in P_{M'}^{n+1}$

(d) If M is n-solid then $\pi(p_M^{n+1}) = p_{M'}^{n+1}$

(e) M' is not sound

Also prove more general version for (a) – (d) for all $k \geq n+1$.

Hint For (a) inspect the definition of a $\Sigma_1^{(n)}(M')$ set, and use the Σ_1-amenability of U to "pull" this definition to a definition over M. Notice that only this inclusion, i.e. $\supseteq$ is non-trivial.
The case $k > n+1$ goes by induction.

(3) Prove that if $\langle M_i, \pi_{ij}^M \mid i \leq j \leq \theta \rangle$, $\langle N_i, \pi_{ij}^N \mid i \leq j \leq \theta \rangle$ is a successful coiteration of M, N then one side does not involve a truncation.

Hint Assuming the contrary, let M_{i+1}, N_{j+1} be the last truncation points on the two sides. Prove

(a) $M_i \| \beta_i^{M_i}$ projects to $\kappa_i^{M_i} = cr(E_{\nu_i}^{M_i})$; say m is s.t. $\rho^{m+1}_{M_i \| \beta_i^{M_i}} \leq \kappa_i^{M_i} < \rho^{m}_{M_i \| \beta_i^{M_i}}$

Let n play the analogous role for $N_j \| \beta_j^{N_j}$.

(b) Prove that M_θ, N_θ are not sound and $M_\theta = N_\theta$. Denote this model by R

(c) $\rho^{n+1}_{M_i \| \beta_i^{M_i}} = \rho^{n+1}_R = \rho^{n+1}_{N_j \| \beta_j^{N_j}}$; denote this ordinal by ρ^{n+1}

(d) Assume all initial segments of M, N are solid.
Prove: $\pi_{i\theta}^M(p_{M_i \| \beta_i^{M_i}} - \rho^{n+1}) = \pi_{j\theta}^N(p_{N_j \| \beta_j^{N_j}} - \rho^{n+1}) =$
$= p_R - \rho^{n+1}$.

(e) $rng(\pi_{i\theta}^M) = \tilde{h}_R^{n+1}(\rho^{n+1} \cup \{p_R - \rho^{n+1}\})$
$= rng(\pi_{j\theta}^N)$