EXERCISES 3.7.2012

(1) Let E be a (κ, λ) extender. Recall the objects in the ultrapower $Ult(V, E)$ are represented as pairs / equivalence classes $[a, f]$ where $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$. If $M \models ZFC^-$ we define the notion of extender over M, or M-extender in the obvious way that generalizes the notion of ultrafilter over M. In this case objects in $Ult(M, E)$ are represented by pairs $[a, f]$ as above where $f \in M$. Prove the Łoś theorem:

$$Ult(M, E) \models \varphi([a_1, f_1], \dots, [a_\ell, f_\ell])$$
$$\text{iff}$$
$$\{u \in [\kappa]^{|a_1 \cup \dots \cup a_\ell|} \mid M \models \varphi(f_1'(u), \dots, f_\ell'(u))\} \in E_{a_1 \dots a_\ell}$$

where ~~xx~~ $f_i'(u) = f_i(u_i)$ where, letting $\pi : u \longleftrightarrow a_1 \cup \dots \cup a_\ell$ is the unique order-preserving bijection, $u_i = \pi^{-1}[a_i]$.

(2) Let $j : M \longrightarrow N$ be an elementary embedding such that $cr(j) = \kappa$ and $\lambda < j(\kappa)$. Letting
$$E_a = \{z \in \mathcal{P}([\kappa]^{|a|}) \cap M \mid a \in j(z)\}$$
prove that $E = (E_a \mid a \in [\lambda]^{<\omega})$ is a (κ, λ)-extender over M.

(3) Formulate the "normality" of an extender as a property of ultrafilters E_a ~~so that~~ that implies that $\{\alpha\} = [\{\alpha\}, id]$ for all $\alpha < \lambda$.

(4) Let $j : M \to N$ be an elementary embedding such that $\kappa = cr(j)$ and $\lambda < \pi(\kappa)$. Let E be the (κ, λ) extender derived from j. Prove the following are equivalent:

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(a) $N = \mathrm{Ult}(M, E)$
(b) $N =$ the Σ_0-hull of $\lambda \cup \mathrm{rng}(j)$ in N
(c) Every element of N has the form $j(f)(a)$ for some $a \in [\lambda]^{<\omega}$ and $f \in M$ s.t. $f : [\kappa]^{|a|} \to N$
In fact, $[a, f] = j(f)(a)$

(5) Let E be a (κ, λ)-extender over M. Assume for simplicity that $\mathrm{Ult}(M, E)$ is well-founded, and let $N = \mathrm{Ult}(M, E)$. Let $\tau = \kappa^{+M}$, ~~and~~ $Q = H^M_\tau$, $Q' = \mathrm{Ult}(Q, E)$ and ~~σ(?)~~ σ be the ultrapower map. So $\sigma : Q \to Q'$ is Σ_0 and cofinal. Prove that $N = \mathrm{Ult}(M, \sigma)$ and if $\tilde{\sigma} : M \to N$ is the canonical extension of σ then $\tilde{\sigma} = j$ when $j =$ the ultrapower embedding associated with $\mathrm{Ult}(M, E)$.

(6) A (κ, λ)-extender E over M is ω-complete iff for every countable sequence $\langle x_i, a_i \mid i \in \omega \rangle$ such that $x_i \in E_{a_i}$ there is a function
$$\delta : \bigcup_{i \in \omega} a_i \longrightarrow \kappa$$
such that:
(i) δ is order-preserving
(ii) $\delta[a_i] \in x_i$ all $i \in \omega$.

Prove:
(a) If E is ω-complete then $\mathrm{Ult}(M, E)$ is well-founded
(b) If $\mathrm{Ult}(V, E)$ is well-founded, then E is ω-complete. We shall: $M = V$ here.

Hint for (b): Write down the tree T approximating the map δ: The nodes of T are pairs $\langle n, f \rangle$ s.t. $n \in \omega$ and $f : \bigcup_{i \le n} a_i \to \kappa$ satisfying (i), (ii). Using the well-foundedness of $\mathrm{Ult}(V, E)$ show that T is well-founded.

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(7) Prove that the two definitions of δ being Woodin are equivalent:

(a) δ is Woodin iff For all $A \subseteq V_\delta$ there is $\kappa < \delta$ such that κ is $<\delta$-strong, i.e. for all $\lambda < \delta$ there is an extender $E \in V_\delta$ s.t. $cr(E) = \kappa$ and $V_\lambda \subseteq \mathrm{Ult}(V_\delta, E)$ and if $(N, A') = \mathrm{Ult}(V_\delta, E)$ then $V_\lambda \cap A' = V_\lambda \cap A$.

(b) δ is Woodin iff (*) $\forall f : \delta \to \delta$ there is $\kappa < \delta$ and an extender $E \in V_\delta$ s.t.

(i) $cr(E) = \kappa$

(ii) $j(\kappa) > j(f)(\kappa)$ when $j : V_\delta \xrightarrow[E]{} N$

(iii) $V_{j(f)(\kappa)} \subseteq N$

(*) δ is strongly inaccessible

(8) Prove that every Woodin cardinal is Mahlo, but the least Woodin cardinal is not weakly compact.